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Rate of Convergence for a Fully-Discrete Reliable Scheme for a System of Nonlinear Time-Dependent Joule Heating Equations

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Abstract: An initial-boundary value problem for a system of decoupled two nonlinear time-dependent Joule heating equations is studied. Instead of well-known standard techniques, we design a reliable scheme consisting of coupling the non-standard finite difference (NSFD) method in time and finite element method (FEM) in space. We prove the rate of convergence for the fully-discrete scheme in both H^1 as well as the L^2-norms. Furthermore, we show that the above scheme preserves the properties of the exact solution. Numerical experiments are provided to confirm our theoretical analysis.

Keywords: Joule heating equations, nonlinear decoupled parabolic system, Non-standard finite difference method, finite element method, optimal rate of convergence

1 Introduction

Advances in electric and electronic technology have great technological as well as economic impact on the electric and electronic industries throughout the world. In addition, the problem of electrical heating of conductors is also a historical one. We refer in particular to [16] for more details. Here we consider 'Joule heating' phenomena which is one of the main contributors to this effect. The literature defines Joule heating as the process by which the passage of an electric current through a conductor releases heat, see for example [8] and [20] for more details. It should however be noted that, there is a vast work in the literature on finite element method for nonlinear elliptic and parabolic problem such as the work on porous media equations found in [17] and [18] which are similar to the Joule heating problem. The only difference in most cases is that, the term $\sigma(u)|\nabla\phi|^2$ is replaced by $\nabla\phi\cdot\nabla u$.

In this paper, we consider and study the rate of convergence for the following model of the time-dependent system of nonlinear Joule heating equations

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2,\ x\in\Omega,\ t\in[0,T], \tag{1}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0, \tag{2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$. The smooth boundary $\partial\Omega$ and initial conditions of the above system (1)-(2) are taken to be

$$u(x,t) = 0,\ \phi(x,t) = 0,\ \text{for},\ x\in\partial\Omega\ \text{and}\ t\in[0,T], \tag{3}$$

and

$$u(x,0) = u_0(x),\ x\in\Omega, \tag{4}$$

respectively. In summary, the above nonlinear system (1) – (4) describes the model of electric heating of a conducting body, where u represents the temperature, ϕ the electric potential and $\sigma(s)$ being the temperature-dependent electric conductivity satisfying

$$k\le\sigma(s)\le K\ \text{and}\ |\sigma'(s)|+|\sigma''(s)|\le K, \tag{5}$$

for some positive constants k and K and $\sigma\in C^2(\mathbb{R})$.

There are so many numerical methods for studying the nonlinear time-dependent Joule heating problems. Among these methods is the linearly implicit finite

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element method found in [2] for many more, we refer to [3,4,5] and [22]. For more on numerical analysis for this problem we refer to [2,5,19,30,33] and [34].

Instead of the methods stated above, with all their wonderful implementations, we exploit some of these approaches and present in this paper, a reliable technique consisting of coupling the nonstandard finite difference (NSFD) method in time and the finite element (FEM) method in the space variables. A similar approach was used for the first time in addressing the diffusion as well as the wave equation in [9] and [10] respectively. Since these two problems were all linear, then our main intention in this article is to extend the application of the technique to handle system of time-dependent Joule heating equations which is purely an example of a nonlinear system of decoupled parabolic partial differential equations. As regard the comparisons of the standard finite difference and finite element method with NSFD-FEM method we refer to [9]. For other comparisons of the standard and Nonstandard finite difference methods we refer to [26]. In this different framework, we shall use the technique to prove the optimal rate of convergence in both the H^1 as well as the L^2-norms of the solution of this problem. The reliability of our technique comes from the fact that the NSFD-FEM method replicates the properties of the solution of the decay equation. The NSFD method was initiated by Mickens [26] and major contributions to the foundation of the NSFD method could be seen in [6,7] and [28] for an overview. The method since inception has been applied to a variety of problems of physics, epidermeology, engineering and business sciences to mention a few; we refer to [24,25,27] for more details.

The rest of the paper is organized as follow: In Sect 2, we specify the notations and the spaces we shall be dealing with in this work. Followed by Sect 3, where we introduce finite element method together with the essential results that will be used in the paper. Sect 4, will be devoted to the coupling of NSFD-FEM method and furthermore, the main results with its prove. In Sect 5, we propose a numerical example and do some numerical experiments to confirm the theory presented in Sect 4. Finally, Sect 6 will be the conclusion of the paper and future remarks.

2 Preliminaries

We assemble under this section, some notations and facts about linear elliptic and parabolic finite element problems which will be needed in the paper. Throughout this paper we use the Sobolev spaces of real-valued functions defined on Ω and denoted for $r \geq 0$ by $H^r(\Omega)$. The norm on $H^r(\Omega)$ will be denoted by $\|\cdot\|_r$. See [23] for the definitions and the relevant properties of these spaces. In a particular case, where $r = 0$ the space $H^0(\Omega) = L^2(\Omega)$ and its inner product together with the norm will be stated and denoted by

$$(u,v) = \int_\Omega uv dx, \quad u,v \in L^2(\Omega),$$

and

$$\|u\|_{L^2(\Omega)} = \{(u,u)\}^{1/2}, \quad u \in L^2(\Omega).$$

We will furthermore, denote the norm in the standard Sobolev space $W_p^m \equiv W_p^m(\Omega)$ for $1 \leq p < \infty$ by

$$\|u\|_{m,p} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$$

and write $H^m = W_2^m$ when $p = 2$. Following [23], for X a Hilbert space, we will more generally use the Sobolev space $H^r[(0,T);X]$, where $r \geq 0$ and in the case where $r = 0$ we will have $H^0[(0,T);X] \equiv L^2[(0,T);X]$ with norm

$$\|v\|_{L^2[(0,T);X]} = \left(\int_0^T \|v(\cdot,t)\|_X dt \right)^{1/2}.$$

In practice, X will be the Sobolev space $H^m(\Omega)$ or $H_0^m(\Omega)$. Associated with (1) is the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla v dx, \quad u,v \in H^1(\Omega).$$

$a(\cdot,\cdot)$ will be symmetric and positive definite. i.e.

$$a(u,v) = a(v,u), \;\; a(u,u) \geq 0. \tag{6}$$

3 Finite element method

Under this section, we proceed to gather essential tools necessary to prove the main result of the paper. We begin by stating the weak formulation of (1)-(2) which is; find $u(t),\ \phi(t) \in H_0^1(\Omega)$ such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) = \left(\sigma(u)|\nabla\phi|^2, v\right), \tag{7}$$

$$(u(0),v) = (u_0,v) \tag{8}$$

and

$$(\sigma(u)\nabla\phi, \nabla v) = 0, \;\; \forall v \in H_0^1(\Omega), \;\; t \in [0,T]. \tag{9}$$

For the existence of the global weak solutions in two dimensions $\Omega \subset \mathbb{R}^2$, $u(\cdot,t)$ of (7)-(9), refer to [4,12,13,14,15] and [32]. Even though the results for the regularity of the solution of the problem are many in the literature, see [19], [29] and [31], but we will assume and adopt the regularity results in [19] for it is expressed in the form suitable to the technique we are using in our problem.

The above continuous problem leads to the next framework where we present the discrete version of (7)-(9). To this end, we let $\mathscr{T}_h$ be a regular family of triangulation of Ω consisting of compatible triangles $\mathscr{T}$ of diameter $h_{\mathscr{T}} \leq h$, see [11] for more details. For each mesh size $\mathscr{T}_h$, we associate the finite element space V_h of continuous piece-wise linear functions that are zero on the boundary

$$V_h := \left\{ v_h \in C^0(\bar{\Omega}); v_h|_{\partial\Omega} = 0,\ v_h|_{\mathscr{T}} \in P_1,\ \forall\, \mathscr{T} \in \mathscr{T}_h \right\} \quad (10)$$

where P_1 is the space of polynomial of degree less than or equal to 1 and V_h is a finite dimensional subspace of V which is contained in the Sobolev space $H_0^1(\Omega)$.

It is well known that, $\pi_h : V \to V_h$ denotes a mapping from V onto V_h where π_h denotes the Lagrangian interpolation operator. Since we are in a smooth domain, in view of [11], the Laplacian Δ with its inverse Δ^{-1} is an isomorphism from $H^2 \cap H_0^1$ onto L^2. With this, we let the map $R_h : H_0^1 \to V_h$ be defined by the equation

$$(\nabla R_h w, \nabla v) = (\nabla w, \nabla v),\ \ \forall w \in H_0^1,\ \ v \in V_h. \quad (11)$$

In view of [11], the error analysis in the linear elliptic finite element problems yield the estimate

$$\|(R_h - I)\,w\| + h\|(R_h - I)\,w\|_{1,2} \le Ch^2\|w\|_{2,2}, \quad \forall w \in H^2 \cap H_0^1. \quad (12)$$

In this way, we also denote the discrete Dirichlet Laplacian map by $\Delta_h : V_h \to V_h$ and defined it by

$$(-\Delta_h w, v) = (\nabla w, \nabla v),\ \ \forall w, v \in V_h,$$

and if we let $E_h(t) = e^{t\Delta_h}$ be the analytic semigroup generated by Δ_h and P_h a map from the space $L^2(\Omega)$ to V_h which is an orthogonal projection, then in view of [19], it is well known that $E_h(t)P_h$ satisfies the following estimate

$$\|E_h(t)P_h\rho\| + t^{1/2}\|E_h(t)P_h\rho\|_{1,2} + t\|\Delta_h E_h(t)P_h\rho\| \le C\|\rho\|,\ \ t > 0, \quad (13)$$

where $\rho \in L^2(\Omega)$ and C is independent of h and t, reflecting the inform analyticity of the evolution operator. In the same way, the discrete evolution operator $E^n_{\Delta t,h}$ defined by $E^n_{\Delta t,h} = (I - \Delta t\Delta_h)^{-n}$ and associated with the backward Euler method has the estimate

$$\|E^n_{\Delta t,h}P_h\rho\| + t_n^{1/2}\|E^n_{\Delta t,h}P_h\rho\|_{1,2} + t_n\|\Delta_h E^n_{\Delta t,h}P_h\rho\| \le C\|\rho\|,\ \ t_n > 0. \quad (14)$$

With the above discrete framework of the problem in place, we are led to state the discrete version of (7)-(9) as follows: find $u_h(t),\ \ \phi_h(t) \in V_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + (\nabla u_h, \nabla v_h) = \left(\sigma(u_h)|\nabla\phi_h|^2, v_h\right), \quad (15)$$

$$(u_h(0), v_h) = (u_h^0, v_h), \quad (16)$$

and

$$(\sigma(u_h)\nabla\phi_h, \nabla v_h) = 0,\ \ \forall v_h \in V_h,\ \ t \in [0,T], \quad (17)$$

where $u_h^0 \in V_h$ is an appropriate approximation of u^0. Under the above framework, we are led to the following discrete version of the generalized Gronwall Lemma and the Lemma that will bound $\phi(t) - \phi_h(t)$:

Lemma 1 *Assume that the sequence φ_n satisfies*

$$0 \le \varphi_n \le A_1 + A_2\Delta t \sum_{l=0}^{n-l} t_{n-l}^{-1+\beta}\varphi_l,\ \ for\ \ t_n \in [0,T]$$

where $A_1,\ A_2,\ T$ are positive numbers and $0 < \beta < 1$. Then, there is a constant $C = C(\beta, A_2, T)$ such that

$$\varphi_n \le CA_1\ \ t_n \in [0,T].$$

With the Lemma in place, we state without proof the following important result which is key to the prove of the main Theorem. For the proof of the said result, we refer to [19].

Theorem 2 *Let $u,\ \phi$ and $u_h,\ \phi_h$ be solutions of (7)-(9) and (15)-(17) respectively, with $\mathring{u}_h$ chosen so that*

$$\|\mathring{u} - \mathring{u}_h\| \le M_1 h^2. \quad (18)$$

Assume further that

$$\sup_{0<t\le T}\left(\|u(t)\|_{2,2} + t\|\frac{\partial u(t)}{\partial t}\|_{2,2}\right) \le M_2 \quad (19)$$

$$\sup_{0<t\le T}\left(\|g(t)\|_{H^2(\partial\Omega)} + \|\phi(t)\|_{2,2} + \|\phi(t)\|_{1,\infty}\right) \le M_3 \quad (20)$$

for some positive numbers T and $M_i, i = 1,2,3$. Then there is a constant
$C = C(k, K, M_1, M_2, M_3, T)$ such that

$$\|u(t) - u_h(t)\| + \|\phi(t) - \phi_h(t)\| \le Ch^2, t \in [0,T]. \quad (21)$$

Lemma 3 *Under the assumptions of Theorem 2 above, we have*

$$\|\nabla\left(\phi(t) - \phi_h(t)\right)\| \le C\left(h + \|u(t) - u_h(t)\|\right) \quad (22)$$

and

$$\|\phi(t) - \phi_h(t)\| \le C(h^2 + \|u(t) - u_h(t)\| + h^{-1/3}\|u(t) - u_h(t)\|^2). \quad (23)$$

4 Coupled Non-standard finite difference and finite element method

Unlike in other methods where the time variable was discretized using the backward Euler, we exploit and present under this section, a reliable scheme consisting of coupling the Non-standard finite difference (NSFD) method in the time step size and the finite element method (FEM) in the space step size. We show that the numerical solution obtained from this scheme NSFD-FEM attained the optimal rate of convergence in both H^1 as well as L^2-norms. To achieve this we proceed in this different framework, by letting the step size $t_{n-1} = (n-1)\Delta t$ for $n = 1,2,\cdots,N$. For a sufficiently smooth functions, we find the NSFD-FEM approximation $\{U_h^n\}$ such that $U_h^n \approx u_h^n$ at each discrete time t_{n-1}. That is, find a sequence $\{U_h^n\}_{n=0}^N$ in V_h such that for

$n = 1,2,\cdots,N-1$ the fully-discrete scheme NSFD-FEM approximate u_h^n such that $U_h^n \approx u_h^n$ at discrete time t_n. i.e, find $U_h^n, \Phi_h^n \in V_h$ such that

$$(\delta_n U_h^n, v_h) + (\nabla U_h^n, \nabla v_h) = \left(\sigma(U_h^{n-1})|\nabla \Phi_h^{n-1}|^2, v_h\right), \quad (24)$$

$$(U^0, v_h) = (u_h^0, v_h) \quad (25)$$

and

$$(\sigma(U_h^n)\nabla \Phi_h^n, \nabla v_h) = 0 \ \ \forall v_h \in V_h, \ \ t_n \in [0,T], \quad (26)$$

where

$$\delta_n U_h^n = \frac{U_h^n - U_h^{n-1}}{\psi(\Delta t)} \quad (27)$$

and $\psi(\Delta t) = \frac{e^{\lambda \Delta t}-1}{\lambda}$ is restricted between $0 < \psi(\Delta t) < 1$.

If the nonlinear function on the right hand side of (24) is zero, we will have in view of (24) the exact scheme

$$\left(\frac{U_h^n - U_h^{n-1}}{\frac{e^{\lambda\Delta t}-1}{\lambda}}, v_h\right) + (\nabla U_h^n, \nabla v_h) = 0, \quad (28)$$

which according to Mickens [26] replicates the positivity and the decay to zero, which are the main features of the exact solution of (1)-(2). The above framework leads to the following main result of the paper.

Theorem 4*Let u and ϕ be the solutions of the time-dependent Joule heating equations (7)-(9) and its respective fully-discrete NSFD-FEM solution in (24)-(26) be U_h^n and Φ_h^n given such that*

$$\|\mathring{u} - \mathring{u}_h\| \leq M_1 h^2. \quad (29)$$

Assume further that

$$\sup_{0<t\leq T}\left(\|u(t)\|_{2,2} + \left\|\frac{\partial u(t)}{\partial t}\right\| + \left\|\Delta^{-1}\frac{\partial^2 u}{\partial t^2}\right\| + t\left\|\frac{\partial u(t)}{\partial t}\right\|_{2,2} + t\left\|\frac{\partial^2 u}{\partial t^2}\right\|\right) \leq M_2, \quad (30)$$

$$\sup_{0<t\leq T}\left(\|\phi(t)\|_{2,2} + \|\phi(t)\|_{1,\infty} + \|\frac{\partial \phi}{\partial t}\|_{1,2}\right) \leq M_3, \quad (31)$$

and that $\Delta t \leq M_4 h^{1/3}$ for some positive numbers T and $M_i, i = 1,2,\cdots,4$. Then there is $C = C(k,K,M_1,M_2,M_3,M_4,T)$ such that

$$\|u(t_n) - U_h^n\| + \|\phi(t_n) - \Phi_h^n\| \leq C\left(h^2 + \Delta t\right), \quad t_n \in [0,T]. \quad (32)$$

Furthermore, the discrete solution replicates all the properties of the nonlinear equations in the limiting case of the space independent equation.

*Proof.*We begin the proof of this theorem with the error decomposition equation

$$U_h^n - u^n = U_h^n - \tilde{U}_h^n + \tilde{U}_h^n - u^n \quad (33)$$

where $\tilde{U}_h^n \in V_h$ is uniquely defined by

$$\left(\delta_n \tilde{U}_h^n, v_h\right) + \left(\nabla \tilde{U}_h^n, \nabla v_h\right) = \left(F(u^n, \phi^n), v_h\right), \quad (34)$$

$$\left(\tilde{U}_h^0, v_h\right) = \left(u^0, v_h\right), \forall v_h \in V_h, \ t_n \in [0,T] \quad (35)$$

with $u^n = u(t_n)$, $\phi^n = \phi(t_n)$ and $F(u^n,\phi^n) = \sigma(u^n)|\nabla\phi^n|^2$. Applying the well known error analysis for linear parabolic equation we obtain

$$\|\tilde{U}_h^n - u^n\| \leq C\left(h^2 + \psi(\Delta t)\right) \quad (36)$$

where C depends on M_1 and M_2 we refer to [9] for more details.

In view of the difference between (24)-(26) and (34)-(35) we have for $\xi_n = U_h^n - \tilde{U}_h^n$ that

$$\begin{aligned}&\left(\delta_n(U_h^n - \tilde{U}_h^n), v_h\right) + \left(\nabla(U_h^n - \tilde{U}_h^n), \nabla v_h\right)\\ &= \left(\sigma(U_h^{n-1})|\nabla\Phi_h^{n-1}|^2 - \sigma(u^{n-1})|\nabla\phi^{n-1}|^2, v_h\right)\end{aligned}$$

which yield

$$\begin{aligned}&\left(\delta_n(U_h^n - \tilde{U}_h^n), v_h\right) + \left(\nabla(U_h^n - \tilde{U}_h^n), \nabla v_h\right)\\ &= \left(F(U_h^{n-1}, \Phi_h^{n-1}) - F(u^{n,\phi^n}), v_h\right)\end{aligned}$$

and hence

$$\frac{\partial \xi_n}{\partial t} - \Delta_h \xi_n = Ph\left(F(U_h^{n-1}, \Phi_h^{n-1}) - F(u^n, \phi^n)\right),$$
$$\xi_n(0) = 0, \ t \in [0,T].$$

When the variation of constant formula is applied, we have using the discrete version of the Gronwall Lemma 1

$$\|\xi_n\| \leq \psi(\Delta t)\sum_{l=0}^{n-l}\|E_{\Delta t,h}^{n-l}P_h(F(U_h^l,\Phi_h^l) - F(u^{l+1},\phi^{l+1}))\|. \quad (37)$$

Since in the NSFD-FEM method, Δt was replaced by $\psi(\Delta t)$ then, in view of the discrete evolution operator denoted in the same way as $E_{\Delta t,h}^n$ and associated with the backward euler method, will surely share the same estimate as in (14). With this, we proceed to bound (37) using several bounds for the discrete evolution operator $E_{\Delta t,h}^n P_h$. This will be done thanks to [21] Lemma 5.2 for the bounds of the norm of the continuous operator $E_h(t)P_h$ considered as an operator from L^2 into L^∞, namely for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$\|E_h(t)P_h\rho\|_{0,\infty} \leq C_\varepsilon t^{\frac{-2}{4-\varepsilon}}\|\rho\|, \ \ t > 0. \quad (38)$$

By duality, we have the same bound for the norm of $E_h(t)P_h : L^1 \to L^2$, i.e.

$$\|E_h(t)P_h\rho\|_{0,\infty} \leq C_\varepsilon t^{\frac{-2}{4-\varepsilon}}\|\rho\|_{0,1}, \ \ t > 0. \quad (39)$$

In fact we have

$$\begin{aligned}\|E_h(t)P_h\rho\| &= \sup_{v\in L^2,\|v\|_{L^2}\neq 0}\frac{|(E_h(t)P_h\rho, v)|}{\|v\|_{L^2}}\\ &= \sup_{v\in L^2,\|v\|_{L^2}\neq 0}\frac{|(\rho, E_h(t)P_h v)|}{\|v\|_{L^2}}.\end{aligned} \quad (40)$$

since $E_h(t)P_h$ is self-adjoint. In view of the above continuous cases of $E_h(t)P_h$ with the discrete NSFD-FEM analogue $E^n_{\Delta t,h}P_h$ we have

$$\|E^n_{\Delta t,h}P_h\rho\| \le C_\varepsilon t_n^{\frac{-2}{4-\varepsilon}}\|\rho\|,\ t_n>0,\ \varepsilon>0 \tag{41}$$

in view of (38) and seconded by

$$\|E^n_{\Delta t,h}P_h\rho\| \le C_\varepsilon t_n^{\frac{-2}{4-\varepsilon}}\|\rho\|_{0,1},\ t_n>0,\ \varepsilon>0 \tag{42}$$

for any $C_\varepsilon>0$ in view of (39). Furthermore, in view of (40) we can expand the norm of $\|E^n_{\Delta t,h}P_h\rho\|$ as

$$\begin{aligned}\|E^n_{\Delta t,h}P_h\rho\| &= \sup_{v\in L^2,\|v\|_{L^2}\neq 0}\frac{|(E^n_{\Delta t,h}P_h\rho,v)|}{\|v\|_{L^2}}\\ &= \sup_{v\in L^2,\|v\|_{L^2}\neq 0}\frac{|(\rho,E^n_{\Delta t,h}P_h\rho v)|}{\|v\|_{L^2}}.\end{aligned} \tag{43}$$

In view of Lemma 3, we can apply it directly to the equation for Φ^n_h to have

$$\|\nabla(\Phi^n_h-\phi^n)\| \le C(h+\|U^n_h-u^n\|) \tag{44}$$

$$\|\Phi^n_h-\phi^n\| \le C\left(h^2+\|U^n_h-u^n\|+h^{-1/3}\|U^n_h-u^n\|^2\right) \tag{45}$$

Proceeding by deriving a preliminary low order estimate of $\|\xi_n\|$, we have by using the fact that $\|\nabla\Phi^n_h\|+\|\nabla\phi^n\|\le C$ and the hypothesis

$$\begin{aligned}&\|F(U^{n-1}_h,\Phi^{n-1}_h)-F(u^{n-1},\phi^{n-1})\|_{0,1}\\ &\le \|\sigma(U^{n-1}_h)|\nabla\Phi^{n-1}_h|^2-\sigma(u^{n-1})|\nabla\phi^{n-1}|^2\|_{0,1}\\ &\le \|\sigma(U^{n-1}_h)|\nabla(\Phi^{n-1}_h+\phi^{n-1})\cdot\nabla(\Phi^{n-1}_h-\phi^{n-1})\|_{0,1}\\ &+\|(\sigma(U^{n-1}_h)-\sigma(u^{n-1}))|\nabla\phi^{n-1}|^2\|_{0,1}\\ &\le C\left(\|\nabla\Phi^{n-1}_h\|+\|\nabla\phi^{n-1}\|\right)\|\nabla(\Phi^{n-1}_h-\phi^{n-1})\|\\ &+C\|U^{n-1}_h-u^{n-1}\|\|\phi^{n-1}\|_{1,\infty}\\ &\le C\left(h+\|U^{n-1}_h-u^{n-1}\|\right).\end{aligned} \tag{46}$$

If we continue in a similar manner for n, then in view of (46) we have

$$\begin{aligned}&\|F(u^{n-1},\phi^{n-1})-F(u^n,\phi^n)\|_{0,1}\\ &\le \|\sigma(u^{n-1})|\nabla\phi^{n-1}|^2-\sigma(u^n)|\nabla\phi^n|^2\|_{0,1}\\ &\le \|\sigma(u^{n-1})\nabla(\phi^{n-1}+\phi^n)\cdot\nabla(\phi^{n-1}-\phi^n)\|_{0,1}\\ &+\|\sigma(u^{n-1}-\sigma(u^n))|\nabla\phi^n|^2\|_{0,1}\\ &\le C\left(\|\nabla\phi^{n-1}\|+\|\nabla\phi^n\|\right)\|\phi^{n-1}-\phi^n\|_{1,2}\\ &+C\|u^{n-1}-u^n\|\|\phi^n\|^2_{1,\infty}\\ &\le C\left(\|\phi^{n-1}-\phi^n\|_{1,2}+\|u^{n-1}-u^n\|\right)\\ &\le C\psi(\Delta t)\sup_{0<t\le T}\left(\|\frac{\partial\phi}{\partial t}\|+\frac{\partial u}{\partial t}\right)\\ &\le C\psi(\Delta t)\quad \text{by the linear NSFD-FEM in [9]}\end{aligned} \tag{47}$$

and in view of (47) we therefore have using (46) that

$$\begin{aligned}&\|F(U^{n-1}_h,\Phi^{n-1}_h)-F(u^{n-1},\phi^{n-1})\|_{0,1}\\ &\le C\left(h+\psi(\Delta t)+\|U^{n-1}_h-u^{n-1}\|\right).\end{aligned} \tag{48}$$

In view of (36), (37) and (41) we have using the error decomposition equation

$$\begin{aligned}&\|U^n_h-u^n\|\\ &\le \|\tilde{U}^n_h-u^n\|+\|\xi_n\|\\ &\le C\left(h^2+\psi(\Delta t)\right)\\ &+C\psi(\Delta t)\sum_{l=0}^{n-1}t^{-\alpha}_{n-l}\|F(U^l_h,\Phi^l_h)-F(u^{l+1},\phi^{l+1})\|_{0,1}\\ &\le C\left(h^2+\psi(\Delta t)\right)\\ &+C\psi(\Delta t)\sum_{l=0}^{n-1}t^{-\alpha}_{n-l}\left(h+\psi(\Delta t)+\|U^{n-1}_h-u^{n-1}\|\right)\\ &\le C\left(h^2+\psi(\Delta t)\right)+C\psi(\Delta t)\sum_{l=0}^{n-1}t^{-\alpha}_{n-l}\|U^l_h-u^l\|\end{aligned} \tag{49}$$

where we have chosen $\alpha\in(3/4,1)$. Hence by Lemma 1 we have

$$\|U^n_h-u^n\| \le C\left(h^2+\psi(\Delta t)\right). \tag{50}$$

Furthermore, this leads immediately to the following equation:

$$\|\Phi^n_h-\phi^n\|_{1,2} \le C\left(h^2+\psi(\Delta t)\right), \tag{51}$$

and which in view of Lemma 3 we further deduced

$$\|\Phi^n_h-\phi^n\| \le C\left(h^2+\psi(\Delta t)+\|U^n_h-u^n\|\right). \tag{52}$$

To obtain the optimal rate of convergence of the solution of this problem, we expand more accurately $F(U^{n-1}_h,\Phi^{n-1}_h)-F(u^{n-1},\phi^{n-1})$ as follows:

$$\begin{aligned}&F(U^{n-1}_h,\Phi^{n-1}_h)-F(u^{n-1},\phi^{n-1})\\ &=\left[\sigma(U^{n-1}_h)-\sigma(u^{n-1})\right]|\nabla\Phi^{n-1}_h|^2\\ &+2\sigma(u^{n-1})\nabla\phi^{n-1}\cdot\nabla(\Phi^{n-1}_h-\phi^{n-1})\\ &+2\left[\sigma(U^{n-1}_h)-\sigma(u^{n-1})\right]\nabla\phi^{n-1}\cdot\nabla(\Phi^{n-1}_h-\phi^{n-1})\\ &+\sigma(U^{n-1}_h)|\nabla(\Phi^{n-1}_h-\phi^{n-1})|^2\\ &:=R_1+R_2+R_3+R_4.\end{aligned}$$

Using (14) and (42) we shall estimate each of the terms of the norm $\|E^n_{\Delta t,h}P_hR_i\|, i=1,2,3,4$ and later combine the result into the right hand side of (37) as follows:

$$\begin{aligned}&\|E^n_{\Delta t,h}P_hR_1\|\\ &=\|E^n_{\Delta t,h}P_h\left(\left[\sigma(U^{n-1}_h)-\sigma(u^{n-1})\right]|\nabla\Phi^{n-1}_h|^2\right)\|\\ &\le C\left(\|U^{n-1}_h-u^{n-1}\|\|\Phi^{n-1}_h\|_{1,\infty}\right)\\ &\le C\|U^{n-1}_h-u^{n-1}\|.\end{aligned} \tag{53}$$

For $R_2(s)$ we use the stability argument by (43) for $\chi \in L^2(\Omega)$ and we have

$$\begin{aligned}&\left(E^n_{\Delta t,h}P_hR_2(s),\chi\right)\\ &= 2\left(\nabla(\Phi_h^{n-1}-\phi^{n-1}),\sigma(u^{n-1})\nabla\phi^{n-1}E^n_{\Delta t,h}P_h\chi\right)\\ &= -2\left(\Phi_h^{n-1}-\phi^{n-1},\nabla\cdot\left[\sigma(u^{n-1})\nabla\phi^{n-1}E^n_{\Delta t,h}P_h\chi\right]\right)\\ &= -2\left(\Phi_h^{n-1}-\phi^{n-1},\sigma(u^{n-1})\nabla\phi^{n-1}\cdot\nabla\left[E^n_{\Delta t,h}P_h\chi\right]\right).\end{aligned}$$

Hence, in view of (14) we have

$$\begin{aligned}&\|(E^n_{\Delta t,h}P_hR_2(s),\chi)\|\\ &\le 2\|\Phi_h^{n-1}-\phi^{n-1}\|\|\phi^{n-1}\|_{1,\infty}\|E^n_{\Delta t,h}P_h\chi\|_{1,2}\\ &\le Ct_n^{-1/2}\|\Phi_h^{n-1}-\phi^{n-1}\|\|\chi\|\end{aligned}$$

and by the estimate (52) we have

$$\|E^n_{\Delta t,h}P_hR_2(s)\|\le Ct_n^{-1/2}\left(h^2+\|U_h^{n-1}-u^{n-1}\|\right). \tag{54}$$

By (42) there is $\alpha\in(3/4,1)$ such that

$$\begin{aligned}&\|E^n_{\Delta t,h}P_hR_3(s)\|\\ &\le Ct_n^{-\alpha}\|\left[\sigma(U_h^{n-1})-\sigma(u^{n-1})\right]\nabla\phi^{n-1}\cdot\nabla(\Phi_h^{n-1}-\phi^{n-1})\|_{0,1}\\ &\le Ct_n^{-\alpha}\|U_h^{n-1}-u^{n-1}\|_{0,1}\|\phi^{n-1}\|_{1,\infty}\left(\|\nabla\Phi_h^{n-1}\|+\|\phi^{n-1}\|\right)\\ &\le Ct_n^{-\alpha}\|U_h^{n-1}-u^{n-1}\|_{0,1}.\end{aligned} \tag{55}$$

since $\|\nabla\Phi_h^{n-1}\|+\|\phi^{n-1}\|\le C$.

Similarly, we have in view of (52) the next inequality

$$\begin{aligned}\|E^n_{\Delta t,h}P_hR_4(s)\| &\le Ct_n^{-\alpha}\|\sigma(U_h^{n-1})|\nabla(\Phi_h^{n-1}-\phi^{n-1})|^2\|_{0,1}\\ &\le Ct_n^{-\alpha}\|\Phi_h^{n-1}-\phi^{n-1}\|_{1,2}\\ &\le Ct_n^{-\alpha}\left(h^2+\psi(\Delta t)+\|U_h^{n-1}-u^{n-1}\|\right).\end{aligned} \tag{56}$$

Combining all these inequalities (53), (54), (55) and (56) in (37) then all these together with (36) yield

$$\|U_h^n\|\le C\left(h^2+\psi(\Delta t)\right)+C\psi(\Delta t)\sum_{l=0}^{n-1}t_n^{-\alpha}\|U_h^l-u^l\| \tag{57}$$

which finally yield

$$\|U_h^n-u^n\|\le\left(h^2+\psi(\Delta t)\right)$$

after using the discrete Gronwall Lemma 1. We also have in view of (45)

$$\|\Phi_h^n-\phi^n\|\le C\left(h^2+\psi(\Delta t)\right).$$

Since in view of Mickens [26] the above scheme was design for

$$\psi(\Delta t)=\frac{e^{\lambda\Delta t}-1}{\lambda}\approx\Delta t+0((\Delta t)^2)$$

then as $\Delta t\to 0$ this implies that $\psi(\Delta t)\approx\Delta t$ thereby yielding the first part of the result.

As for the second part of the proof which is the preserving of the properties of the exact solution of the above equations, we should first note that the convergence in both L^2 as well as H^1-norms of the pair of discrete solutions (U_h^n,Φ_h^n) to their respective exact solutions (u,ϕ) in (32) implies that there exists a pair of subsequence of (U_h^n,Φ_h^n) still denoted in the same way as (U_h^n,Φ_h^n) that converge point-wise to (u,ϕ) as $h\to 0$ and $n\to+\infty$. (See [1], Corollary 2.11) for more. In view of this, if we assume that $\Delta u=0$ near a point $a\in\Omega$ and v_h in (24) is chosen in such a way that its support containing the point a, is very small and $v_h=1$ near a, then we can use the approximation

$$\int_\Omega(\sigma(U_h^{n-1})|\nabla\Phi_h^{n-1}|^2)v_hdx=\sigma(U_h^{n-1}(a))|\nabla\Phi_h^{n-1}(a)|^2\mathbf{K}$$

where $\mathbf{K}$ is the measure of the supp(v_h). Using the above approximation in (24), it follows that the pair (U_h^n,Φ_h^n) are discrete solutions of (24)-(26) when we fix $x=a$. Of course the discrete solutions U_h^n is the solution of the exact scheme (28) if we also have

$$\sigma(U_h^{n-1}(a,t))|\nabla\Phi_h^{n-1}(a,t)|^2=0$$

and hence the proof of the second part which complete the proof of the Theorem.

5 Numerical experiments

Under this section, we present the numerical experiments on problem (1)-(2) using the NSFD-FEM method. These experiments are presented in $\Omega=(0,1)^2\times(0,T)$ where Ω is discretized into regular triangulation $\mathscr{T}_h$ of Ω. These discretization are employed using uniform $N\times N$ meshes of sizes $h=1/M$ in the space and $\Delta t=T/N$ in the time, where M denotes the number of nodes. We re-write the time-dependent system of the Joule heating equations (1)-(2) by

$$\frac{\partial u}{\partial t}-\Delta u=\sigma(u)|\nabla\phi|^2+f_1 \tag{58}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi)=f_2 \tag{59}$$

with the electric conductivity $\sigma(u)$ taken to be

$$\sigma(u)=\frac{1}{1+u},$$

where the functions f_1,f_2 the initial and boundary conditions are determined correspondingly by the exact solution

$$\begin{cases}u(x_1,x_2,t)=\exp(-t)\sin(\pi x_1)\sin(\pi x_2)\\ \phi(x_1,x_2,t)=x_1(1-x_1)\sin^2(x_1+x_2-t).\end{cases} \tag{60}$$

Setting $N=10$ and $T=1.0$ our algorithm using the above example (60) in problem (24)-(26) we have the following figures from 1 to 10. These figures illustrate pairs of exact and approximate solutions $u(x_1,x_2)$ and $\phi(x_1,x_2)$ respectively at various times $t=0.1,0.3,0.6,0.9$ and 1.0. For $t=0.1$ we have Fig 1 and 2 followed by Fig 3 and 4 for $t=0.3$ proceeded by Fig 5 and 6 for $t=0.6$ in the same manner Fig 7 and 8 for $t = 0.9$ and finally Fig 9 and 10 for $t = 1.0$.

With all these figures from Fig 1 to 10, we could exploit the data obtained from their computations to find their respective errors for $T = 1.0$ with the mesh sizes varying from $10, 15, 20$ and 25. The results from these computations are illustrated in Tables 1 and 2.

Making use of the error values of the solutions $u(x,t)$ and $\phi(x,t)$ obtained from Tables 1 and 2, we then compute for $T = 1.0$ and using still the mesh sizes stated above, the rate of convergence of $u(x,t)$ and $\phi(x,t)$ using the formulae

$$\text{Rate} = \frac{\ln(e_2/e_1)}{\ln(h_1/h_2)},$$

where h_1 and h_2 together with e_1 and e_2 are successive triangle diameters and errors respectively. Furthermore, the clarification of the convergence of the above solutions to be more specific in the L^2-norm can best be illustrated in Figure 11.

From Table 1, we can observe that the solution u has an approximate rate of almost 2 for $L^2(\Omega)$ and 1 for $H^1(\Omega)$-rates. Similarly, for ϕ in Table 2 the approximate rates for $L^2(\Omega)$ and $H^1(\Omega)$ are 2 and 1 respectively.

These results are self explanatory and we could conclude that the results as shown by these experiments exhibit the desired results as expected from our theoretical analysis.

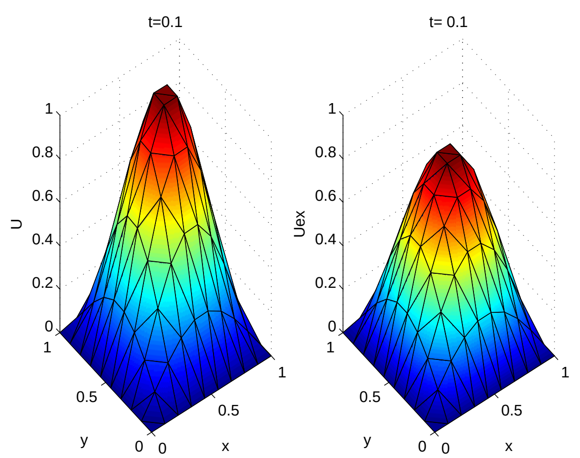

Fig. 1 Exact and Approximate Solutions of $u(x,t)$ at $t = 0.1$

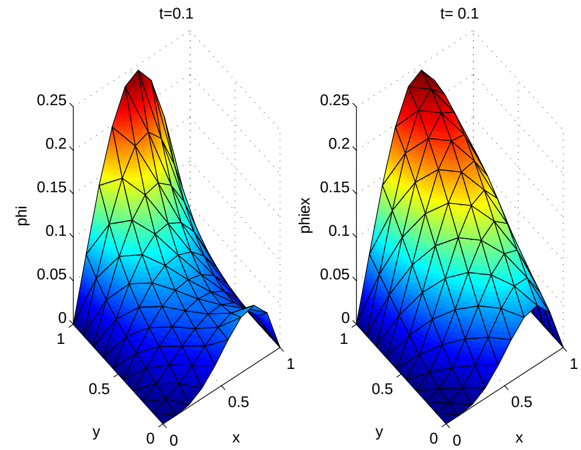

Fig. 2 Exact and Approximate Solutions of $\phi(x,t)$ at $t = 0.1$

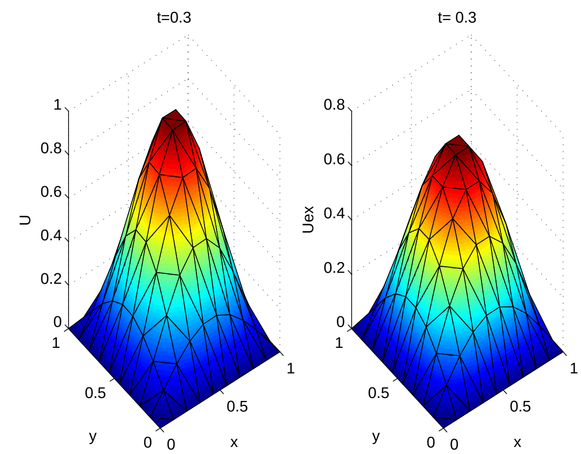

Fig. 3 Exact and Approximate Solutions of $u(x,t)$ at $t = 0.3$

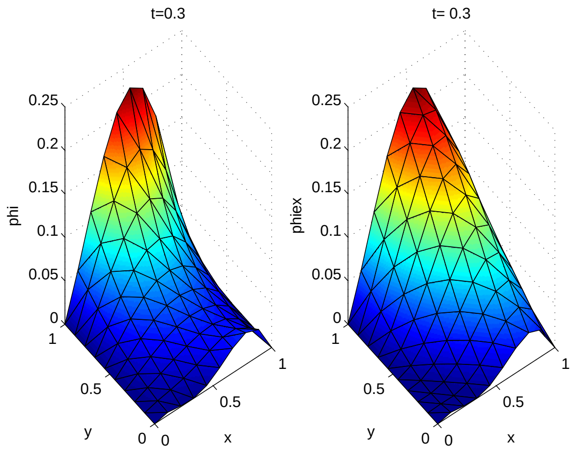

Fig. 4 Exact and Approximate Solutions of $\phi(x,t)$ at $t = 0.3$

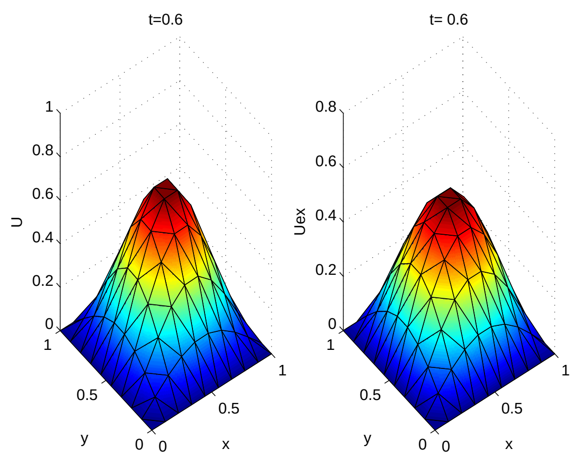

Fig. 5 Exact and Approximate Solutions of $u(x,t)$ at $t = 0.6$

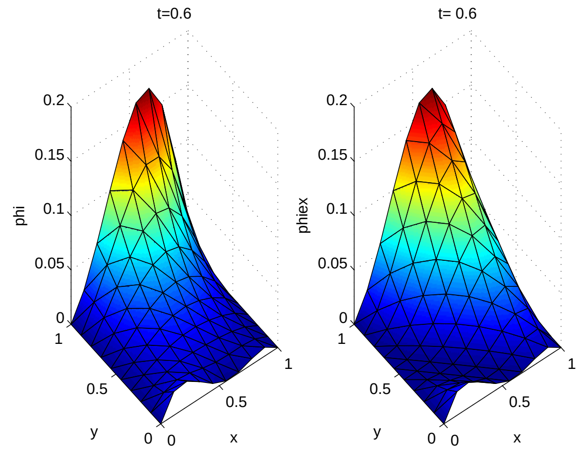

Fig. 6 Exact and Approximate Solutions of $\phi(x,t)$ at $t = 0.6$

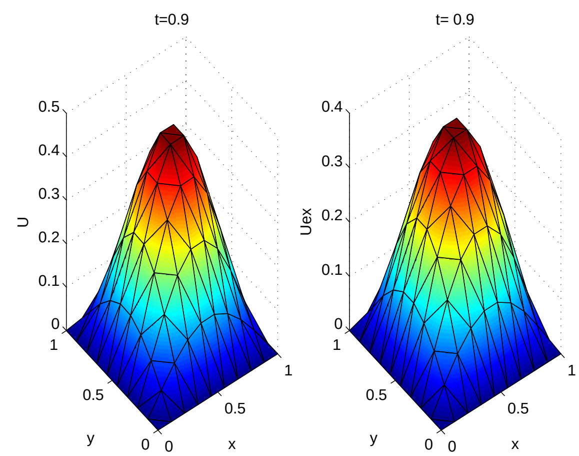

Fig. 7 Exact and Approximate Solutions of $u(x,t)$ at $t = 0.9$

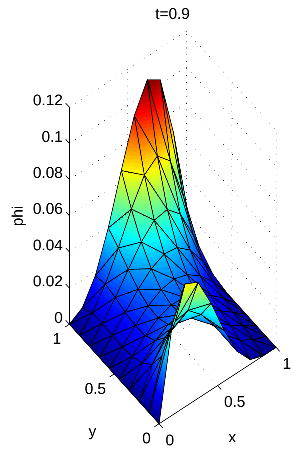

Fig. 8 Exact and Approximate Solutions of $\phi(x,t)$ at $t = 0.9$

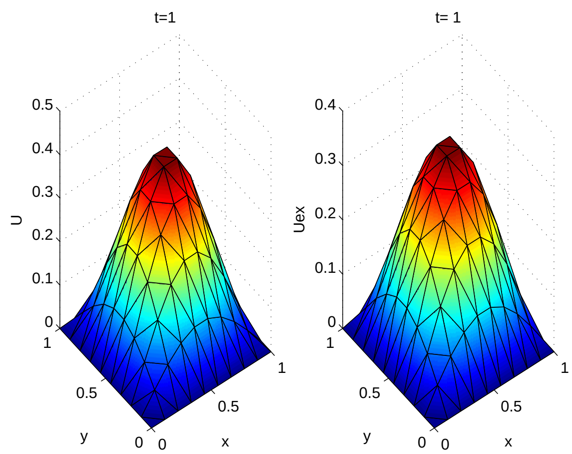

Fig. 9 Exact and Approximate Solutions of $u(x,t)$ at $t = 1.0$

Table 1 Error in L^2 and H^1-norms of u using NSFD-FEM method

M	L^2-error	L^2-Rate	H^1-error	H^1-Rate
10	3.4350E-2		2.4151E-1	
15	1.6258E-2	1.87	1.6575E-1	0.94
20	9.4743E-3	1.88	1.2611E-1	0.95
25	6.2511E-3	1.89	1.0188E-1	0.956

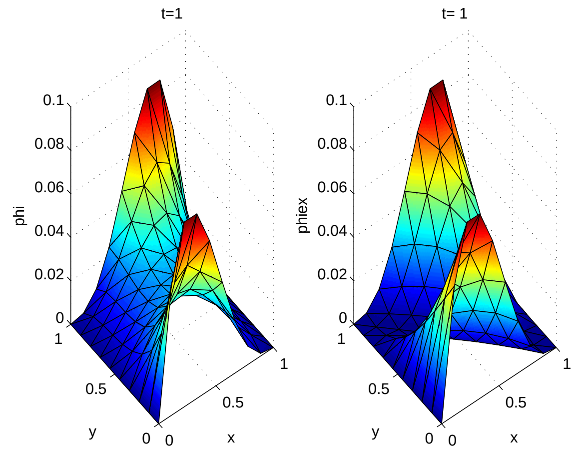

Fig. 10 Exact and Approximate Solutions of $\phi(x,t)$ at $t = 1.0$

Table 2 Error in L^2 and H^1-norms of ϕ using NSFD-FEM method

M	L^2-error	L^2-Rate	H^1-error	H^1-Rate
10	3.1794E-2		2.0130E-1	
15	1.5278E-2	1.83	1.4208E-1	0.87
20	8.9727E-3	1.85	1.1085E-1	0.88
25	5.9248E-3	1.86	9.0884E-2	0.89

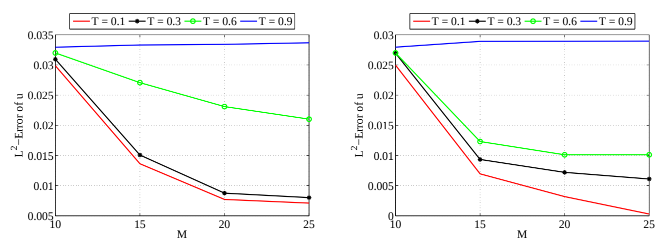

Fig. 11 2-D Unit Square

6 Conlusion

We presented a reliable scheme of the system of the time-dependent Joule heating equations consisting of coupling the non-standard finite difference in time and the finite element method in the space variable (NSFD-FEM). We proved theoretically that the numerical solution obtained from this scheme attains the optimal rate of convergence in both the H^1 as well as the L^2-norms. Furthermore, we showed that the scheme under investigation replicates the properties of the exact solution of the system of Joule heating equations. We proceeded by the help of a numerical example and showed that the optimal rate of convergence as proved theoretically is guaranteed.

The method presented in this article could be extended to other system of nonlinear equations of parabolic form defined in either smooth or non-smooth domain if at all these system of equations followed the procedure as proposed by Mickens [26]. As regard the comparison of the scheme with other schemes we will open that as the next subject.

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