Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/210103

Wave Dynamics and Soliton Solutions of the Stochastic Burgers Hierarchy under Chaotic Lorenz Influence

*Faisal Muteb K. Almalki**

Department of Mathematics, Faculty of Science, Al-Baha University, Alaqiq, 65779-7738, Saudi Arabia

Received: 2 Sep. 2026, Revised: 17 Nov. 2026, Accepted: 27 Nov. 2026
Published online: 1 Jan. 2027

Abstract: This study examines the dynamics of soliton and solitary wave solutions within the stochastic Burgers hierarchy in (1+1) dimensions, focusing on the impact of additive white noise and chaotic perturbations derived from the Lorenz system. By extending classical dissipative models, specifically the Burgers and Sharma–Tasso–Olver equations through the incorporation of stochastic forcing, the research provides a more realistic framework for modeling nonlinear wave behavior under environmental uncertainty. Analytical multi-soliton solutions are obtained using the wave transformation method and generalized exponential function approach. Numerical simulations demonstrate that solitons remain robust under weak noise but undergo significant modulation and potential destabilization as noise intensity increases. The interplay between nonlinearity, stochasticity, and chaotic forcing reveals the sensitivity of wave dynamics to both system parameters and perturbation strength. These findings offer new insights into the behavior of dissipative nonlinear systems subject to stochastic and chaotic influences, with implications for applications in fluid dynamics, traffic modeling, optical communications, and plasma physics.

Keywords: Stochastic Burgers hierarchy, Soliton dynamics, Additive white noise, Lorenz system, Chaotic processes, Dissipative systems, Sharma–Tasso–Olver equation, Wave transformation, Multi-soliton solutions, Stochastic partial differential equations, Nonlinear wave propagation, Noise-induced modulation.

1 Introduction

Physical systems exhibiting complex behaviors (such as fluid flow, gas dynamics, plasma interactions, and nonlinear wave propagation) are frequently modeled using mathematical frameworks that incorporate both nonlinear and dispersive effects. Among these, the Burgers equation and its hierarchical extensions have proven to be particularly effective in capturing key features of dissipative processes and shock wave formations. These models are notable not only for their analytical tractability but also for their applicability in realistic scenarios characterized by external disturbances and inherent randomness. To better reflect the complexities observed in nature, classical deterministic formulations such as the Burgers equation are increasingly being generalized into stochastic frameworks. These incorporate random noise terms to account for environmental perturbations, making them more suitable for modeling phenomena influenced by uncertainty. When further coupled with chaotic sources (such as the Lorenz system) these stochastic models offer a robust platform for investigating the combined effects of deterministic chaos and stochastic fluctuations on soliton dynamics and wave stability.

The present work employs analytical techniques, including wave transformation and the exponential function method, to derive multi-soliton solutions for members of the stochastic Burgers hierarchy. In doing so, it examines how key parameters, including noise intensity and chaotic forcing, influence the shape, stability, and propagation of soliton structures. This investigation is grounded in a broad body of literature on exact solutions of nonlinear evolution equations, where methods such as the inverse scattering transform [1], the tanh-function approach [3], the Jacobi elliptic function expansion [4], and the exponential function method [8] have been instrumental in constructing explicit wave solutions including solitary, periodic, and kink-type structures [2, 6, 7]. While deterministic models have significantly advanced our understanding of nonlinear wave dynamics, they often fall short in capturing the unpredictability inherent

* Corresponding author e-mail: Fmalmalki@bu.edu.sa

© 2027 NSP
Natural Sciences Publishing Cor.

in real-world systems (such as thermal noise, turbulent forcing, or environmental variability). To bridge this gap, stochastic extensions of classical nonlinear partial differential equations (PDEs) have received considerable attention [22, 23]. The stochastic Burgers hierarchy, in particular, introduces multiplicative and additive noise components, enabling the study of randomness-induced effects on soliton formation, evolution, and interactions.

Recent research has increasingly highlighted the necessity of examining soliton dynamics and stability under diverse nonlinear, parametric, and environmental influences. Khan et al. [24] and Islam et al. [25] conducted bifurcation and stability analyses on nonlinear wave equations, demonstrating that parametric variations exert a significant influence on solitary wave characteristics. Bashar et al. [26] and Islam et al. [27] further explored optical soliton behavior within fractional and Hamiltonian frameworks, emphasizing the structural sensitivity of solitons to perturbations intrinsic to specific mathematical models. Mannaf and co-authors [28, 29] addressed the influence of physical factors such as wind, friction, and coordinate stretching on soliton stability in optical and Φ^4 models, revealing intricate dependencies on external conditions. Roshid et al. [30] employed diverse numerical approaches to investigate wave dynamics in fractional systems, illustrating the efficacy of hybrid analytical-numerical methodologies. Extending these efforts, Alsatami [31] analyzed the solitary wave behavior within the generalized Schamel equation under stochastic perturbations, contributing valuable insights into wave resilience under random influences. Distinct from prior studies, the present work incorporates both stochastic noise and chaotic forcing within the Burgers hierarchy, offering a novel perspective on soliton robustness and instability in realistically noisy and complex physical environments.

Furthermore, chaotic modulation via systems like the Lorenz attractor introduces an additional layer of complexity. Such modulation is particularly relevant in contexts like atmospheric science, turbulent fluid flows, and nonlinear optics, where chaotic processes coexist with random noise. This study explores these dynamics through both the stochastic dissipative Burgers and the stochastic Sharma–Tasso–Olver equations, offering new insights into the interplay between noise, chaos, and coherent wave structures. The methodology employed includes the use of Hirota's bilinear method [14], Darboux transformations [10], and residual symmetry analysis [18] to derive multi-soliton and kink-type solutions for the deterministic forms of the stochastic models. The resulting solutions (ranging from single to N- kink waveforms) are then analyzed under stochastic and chaotic perturbations to assess their stability and structural integrity. Previous studies have explored similar phenomena in related systems, including high-order solitons, rogue waves, and mixed solution types [11–13, 15–17, 19–21], highlighting the continued relevance of this line of inquiry.

Numerical simulations are conducted to visualize and quantify the influence of stochastic modulation on soliton dynamics. The results demonstrate that low-intensity noise induces minor deformations, while higher noise levels can significantly disrupt soliton coherence, even leading to complete structural disintegration. These findings underscore the sensitivity of nonlinear systems to external perturbations (akin to the butterfly effect in chaotic dynamics) and emphasize the critical role of parameter selection in ensuring wave stability. This research contributes to the broader understanding of stochastic nonlinear wave equations by extending classical soliton theory to more realistic, noise-affected settings. By incorporating both stochastic and chaotic effects within a unified framework, it advances the theoretical modeling of nonlinear systems with potential applications in diverse scientific and engineering domains.

In this research, we explore the influence of additive white noise and chaotic disturbances on soliton and solitary wave dynamics within the stochastic Burgers hierarchy in (1+1) dimensions. While classical deterministic models such as the Burgers and Sharma–Tasso–Olver equations offer foundational insights, they often fail to account for the stochastic variability inherent in real-world systems like fluid turbulence, traffic dynamics, and plasma behavior. By integrating stochastic components, the study aims to provide a more realistic representation of wave evolution under environmental uncertainty. The analysis demonstrates that soliton structures maintain coherence under low-intensity noise but undergo significant modulation in amplitude, phase, and velocity as the noise intensity increases. At sufficiently high noise levels, solitons may destabilize or disintegrate entirely. These effects become more pronounced when chaotic forcing, modeled via the Lorenz system, is introduced alongside the noise.

Explicit multi-soliton solutions are derived through the application of wave transformation techniques and generalized exponential function methods, providing analytical insight into the system's response to stochastic and chaotic perturbations. Numerical simulations corroborate the analytical findings, illustrating the gradual degradation of soliton integrity and the transition toward chaotic waveforms. This work contributes a novel perspective by extending the deterministic Burgers hierarchy to include both stochastic and chaotic influences within a unified analytical and computational framework. The physical relevance for modeling nonlinear wave phenomena in environments characterized by both noise and deterministic chaos conditions prevalent in fluid dynamics, atmospheric systems, optical communications, and plasma physics. By accounting for both stochastic and chaotic perturbations, the framework offers

© 2027 NSP
Natural Sciences Publishing Cor.

a more accurate and predictive understanding of soliton behavior, including their robustness, stability thresholds, and potential for control in complex, uncertain environments.

In previous research, a range of analytical methods has been developed to obtain exact solutions to deterministic nonlinear evolution equations, including the classical Burgers and Sharma–Tasso–Olver equations. Among these, the Inverse Scattering Transform (IST) has been instrumental in solving integrable systems and deriving soliton solutions. The Hirota bilinear method has proven effective in constructing multi-soliton and kink-type solutions through systematic bilinearization. Other techniques, such as the tanh-function method and Jacobi elliptic function expansion, have been employed to obtain solitary and periodic wave solutions by reducing nonlinear partial differential equations to more tractable forms. The exponential function method, known for its algebraic approach, facilitates the construction of explicit wave profiles by balancing nonlinear terms. Additionally, advanced techniques like Darboux transformations and residual symmetry analysis have been used to generate broader families of exact solutions for higher-order equations. While these methods offer powerful tools for solving deterministic systems, they do not incorporate the randomness or environmental uncertainty often observed in real-world physical contexts. The present study addresses this limitation by extending such analytical approaches into the stochastic domain, investigating how soliton structures evolve when subjected to additive white noise and chaotic perturbations. This marks a significant advancement in the modeling of nonlinear dissipative systems under stochastic and chaotic influences, bridging a gap in the existing literature.

The primary novelty of this study lies in its integration of stochastic and chaotic perturbations into the classical Burgers hierarchy, thereby extending deterministic soliton theory to more realistic stochastic environments. It offers a comprehensive analysis of multi-soliton and solitary wave solutions within the stochastic Burgers and Sharma–Tasso–Olver equations that are rarely studied under the simultaneous influence of additive white noise and chaotic forcing. A notable innovation is the incorporation of chaotic signals from the Lorenz system as a dynamic noise source, moving beyond traditional stochastic modeling approaches. By employing analytical methods such as the wave transformation technique and the generalized exponential function method, the research derives explicit kink- and multi-kink soliton solutions. Numerical simulations across varying noise intensities reveal how solitons deform, shift, or disintegrate in response to the surrounding stochastic environment, providing both visual and quantitative insights rarely addressed in existing literature. These results underscore the delicate balance between soliton resilience and instability in nonlinear dissipative systems subject to random and chaotic disturbances. Specifically, the study confirms that solitons are only mildly perturbed under low-intensity noise but become significantly destabilized under stronger stochastic or chaotic forcing, highlighting the sensitive dependence of nonlinear wave dynamics on external fluctuations. These findings have meaningful implications for real-world systems, such as plasma physics, fluid dynamics, nonlinear optics, and traffic flow, where environmental noise is inherent, and they pave the way for further exploration of stochastic nonlinear partial differential equations using a robust, unified analytical and computational framework.

1.1 Research Problem, Objectives, and Real-World Relevance

This study investigates the behavior of solitons within the stochastic Burgers hierarchy when exposed concurrently to additive white noise and chaotic perturbations derived from the Lorenz system. While classical deterministic models (such as the Burgers and Sharma–Tasso–Olver equations) offer important insights into nonlinear wave phenomena, they are inherently limited in that they do not capture the complex interaction between stochastic variability and deterministic chaos. Understanding such coupled effects is crucial in a range of real-world contexts, including fluid mechanics, optical communication systems, plasma physics, and traffic flow modeling, where unpredictable environmental disturbances can profoundly influence wave stability, coherence, and propagation dynamics.

1.2 Research Gap, Objectives, and Contribution to Knowledge

Research gap and objectives. Prior work on Burgers-type dynamics typically treats stochastic influences or chaotic effects in isolation, with limited attention to higher members of the hierarchy and very few studies that jointly assess mixed (stochastic–chaotic) perturbations against exact multi-soliton baselines (cf. [22–31]). Methodological advances in other nonlinear systems (e.g., the extended Fan's method for NLS with saturable nonlinearity [32], multiplicative-noise analyses for the stochastic ϕ^4 equation [33], ϕ^6-based dispersive-wave studies [34], and quantum-optical media with multiplicative noise [35]) as well as multi-wave results for fractional Burgers and STO equations [36, 37] and kink families within the Burgers hierarchy [38], do **not** address the **combined** stochastic–chaotic forcing within the Burgers/STO framework, nor do they provide a reproducible benchmark against **exact** solutions. To close this gap, our objectives are: (i) to formulate a

© 2027 NSP
Natural Sciences Publishing Cor.

unified framework augmenting deterministic Burgers/STO models with both Gaussian noise and Lorenz-chaotic forcing; (ii) to derive explicit one–four-soliton and N-kink solutions via GEFM (Sec. 5); and (iii) to evaluate how noise intensity (σ), wave number k, and forcing amplitude govern soliton stability, amplitude, and structural integrity.

Contributions and significance. We provide closed-form multi-soliton solutions with transparent parameter constraints for Burgers/STO, introduce a hybrid perturbation model (Gaussian noise + Lorenz chaos) that emulates realistic mixed environments, and deliver a reproducible analytical pipeline (parameter sweeps, symbolic checks in *Mathematica*/*Maple*). By overlaying analytical predictions on simulations, we quantitatively delineate regimes where soliton structures persist or degrade under mixed perturbations. Collectively, these advances fill the identified research gap and extend the literature beyond [22–38] by coupling exact theory with controlled stochastic–chaotic testing in the Burgers hierarchy and the Sharma–Tasso–Olver equation.

1.3 Motivation, Innovative Contributions and Step-by-Step Workflow

This work is motivated by the need to address a significant gap in nonlinear wave dynamics—namely, the absence of comprehensive analytical and numerical studies on soliton behavior in the stochastic Burgers hierarchy when subjected simultaneously to random (stochastic) and deterministic chaotic perturbations. Such mixed disturbances are common in real-world systems, including turbulent fluid flows, nonlinear optical media, plasma environments, and traffic flow, where they can strongly affect wave stability, amplitude, and coherence. Previous research has largely examined stochastic noise or chaotic forcing in isolation, leaving their combined influence insufficiently understood.

The novelty of this study lies in its integrated treatment of stochastic and chaotic perturbations within the Burgers hierarchy (an approach rarely undertaken in existing literature). The incorporation of Lorenz-system forcing as a time-dependent chaotic driver represents a departure from conventional stochastic modeling, while the use of the GEFM enables the derivation of exact multi-soliton solutions under complex, mixed perturbations. Key innovations include coupling two fundamentally distinct disturbance types within a unified analytical–numerical framework, systematically quantifying their combined effects across multiple hierarchy levels, and providing detailed visual analyses (Figures 2–13) that reveal stability thresholds and deformation patterns. Together, these contributions fill the identified research gap and establish a versatile, generalizable framework for modeling nonlinear wave phenomena in environments dominated by both noise and chaos.

The research process began with a comprehensive literature review, identifying that prior studies had examined either stochastic noise or chaotic forcing independently, but rarely their combined effects on soliton stability in the Burgers hierarchy. This gap motivated the formulation of a unified stochastic–chaotic framework, integrating additive white noise and Lorenz-system chaotic forcing into the deterministic Burgers hierarchy. Two primary target equations (the dissipative Burgers equation and the Sharma–Tasso–Olver equation) were selected as representative members of the hierarchy. Analytical multi-soliton solutions were then derived using the GEFM, beginning with a suitable exponential ansatz, balancing nonlinear and linear terms, and systematically extending from single- to multi-kink waveforms. The derivations ensured exactness in the deterministic case, providing a benchmark for subsequent stochastic simulations.

Chaotic forcing signals were generated by numerically integrating the Lorenz system with well-defined parameters, while Gaussian white noise was synthesized with zero mean and variance determined by the noise intensity (σ). These perturbations were combined to form the external input for the stochastic hierarchy. Simulations were performed for a range of σ values and forcing amplitudes, allowing systematic exploration of their effects on soliton amplitude, coherence, and structural integrity. Special cases (purely stochastic or purely chaotic perturbations) were also examined to isolate individual contributions and confirm consistency with known literature results. Validation involved direct substitution of analytical solutions into the governing deterministic equations, as well as quantitative comparisons between analytical and numerical profiles for low-noise scenarios, including 3D surface plots, heatmaps, and density maps. All computational parameters, solver settings, and random seeds were documented to ensure reproducibility. This structured approach not only provides transparency but also offers a replicable roadmap for future research on nonlinear wave dynamics under realistic mixed-perturbation environments.

The structure of the paper is organized as: Section 2 outlines the theoretical foundations and significance of the nonlinear evolution equations that form the basis of this study. Section 3 introduces the chaotic Lorenz system and discusses its integration into the stochastic modeling framework. Section 4 details the wave transformation technique employed to simplify and analyze the stochastic equations under investigation. In Section 5, multi-soliton solutions of the deterministic dissipative Burgers model are derived using analytical methods. Section 6 explores the influence of key parameters (including noise intensity and chaotic forcing) on the soliton dynamics through numerical simulations, accompanied by graphical illustrations and physical interpretations of the results. Lastly, Section 7 reviews the main findings and suggests possible directions for forthcoming research in the field of stochastic nonlinear wave dynamics.

© 2027 NSP
Natural Sciences Publishing Cor.

2 Formulation of the stochastic Burgers hierarchy

The hierarchy of stochastic Burgers model in (1+ 1)-dimension is expressed as follows:

$$\psi_t + \alpha \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \psi\right)^n \psi = \sigma \psi \Xi_t, \quad n = 0,1,2,3,\ldots \tag{1}$$

with $\psi(x,t)$ denotes real-valued function of space x and time t, $\Xi(t)$ represented white noise, σ represents the noise intensity, α is an arbitrary constant and $\psi\,\Xi_t$ represents a multiplicative noise term interpreted in the Itô sense. Following subsection, we analyses the governing equation using the wave transformation technique. The first five members of the hierarchy in Eq. (1) are explicitly formulated as follows

$$\psi_t + \alpha\,\psi_x = \sigma \psi\,\Xi_t, \tag{2}$$

$$\psi_t + \alpha\,(\psi_{xx} + 2\psi\psi_x) = \sigma \psi\,\Xi_t, \tag{3}$$

$$\psi_t + \alpha\left(\psi_{xxx} + 3\psi_x^2 + 3\psi\psi_{xx} + 3\psi^2\psi_x\right) = \sigma \psi\,\Xi_t, \tag{4}$$

$$\begin{aligned}\psi_t + \alpha\big(&\psi_{xxxx} + 10\psi_x\psi_{xx} + 4\psi\psi_{xxx} \\ &+ 12\psi\psi_x^2 + 6\psi^2\psi_{xx} + 4\psi^3\psi_x\big) = \sigma\psi\,\Xi_t.\end{aligned} \tag{5}$$

$$\begin{aligned}\psi_t + \alpha\big(&\psi_{xxxxx} + 10\psi_{xx}^2 + 15\psi_x\psi_{xxx} + 5\psi\psi_{xxxx} \\ &+ 15\psi_x^3 + 50\psi\psi_x\psi_{xx} + 10\psi^2\psi_{xxx} + 30\psi^2\psi_x^2 \\ &+ 10\psi^3\psi_{xx} + 5\psi^4\psi_x\big) = \sigma\psi\,\Xi_t.\end{aligned} \tag{6}$$

These equations are gotten by replacing n=0, 1, 2, 3, 4 into the general hierarchy. The resulting stochastic PDEs correspond to the first-, second-, third-, fourth-, and fifth-order stochastic Burgers equations, respectively. The term stochastic is essential because it signifies the inclusion of random processes (here, additive Gaussian white noise) that are fundamental to accurately modeling real-world systems. Many physical contexts, such as turbulence in fluid flows, spontaneous emission noise in optical fibers, particle collisions in plasma, and variability in traffic flow, involve unpredictable fluctuations that cannot be captured by deterministic models alone. Incorporating stochasticity allows the present work to realistically assess soliton stability under noisy conditions and to explore the unique behaviors that emerge from the combined effects of randomness and deterministic chaotic forcing. Notably, Eq. (3) represents the stochastic dissipative Burgers model, which can be linearized into a stochastic heat equation via the Hopf–Cole transformation $\psi = v_x/v$. Furthermore, Eq. (4) corresponds to the stochastic Sharma–Tasso–Olver model.

2.1 Physical Background and Modeling Assumptions

The stochastic Burgers hierarchy generalizes the classical Burgers equation (an archetype for nonlinear advection with viscous diffusion) used as a one-dimensional proxy in fluid mechanics, acoustics, traffic flow, and turbulence modeling. Here, the field $\psi(x,t)$ denotes a scalar velocity (or potential); the nonlinear term $\alpha\psi\ \psi_x$ steepens waves, while the viscous term $\alpha\psi_{xx}$ regularizes gradients, producing shock-like kinks/solitons of finite thickness. Higher members of the hierarchy introduce additional convective/dissipative operators that capture scale-dependent transport. External perturbations represent unresolved physics: (i) additive Gaussian white noise (measurement/thermal fluctuations) with tunable intensity, and (ii) Lorenz-system–based chaotic forcing as a deterministic surrogate for multi-scale drivers (e.g., convective or thermo-fluid instabilities). Viscosity sets kink thickness, the wave number k sets spatial scale, and forcing amplitude controls energy input.

The Sharma–Tasso–Olver (STO) equation belongs to integrable nonlinear wave models where steepening competes with higher-order effects, supporting stable traveling waves and kink-type structures in dispersive or weakly dissipative media (e.g., shallow-water, plasma, optical). In this study, exact multi-soliton solutions provide a physically interpretable baseline (fixing shape, speed, and interaction properties) against which stochastically/chaotically perturbed dynamics are assessed. We assume a one-dimensional, homogeneous medium with time-independent parameters; noise is additive, mean-zero, Gaussian with specified variance; Lorenz forcing is scaled to the target amplitude. These choices align the deterministic analytical solutions with the perturbed numerical simulations, enabling clear physical interpretation and consistent validation.

© 2027 NSP
Natural Sciences Publishing Cor.

3 Stochastic Lorenz system

A chaotic system is a type of deterministic nonlinear dynamical system that exhibits chaos behavior that appears random and unpredictable even though the system follows precise, well-defined rules (usually in the form of differential equations or iterative maps). Despite being governed by deterministic laws, chaotic systems are extremely sensitive to initial conditions, meaning small differences in starting points can lead to vastly different outcomes. This is often called the butterfly effect. The system consists of three coupled nonlinear ODEs that are known for their chaotic solutions. Initially proposed by Lorenz in 1963 as a simplified model for atmospheric convection, the system has since become a cornerstone in the study of chaos theory. The system is described by the following equations**:**

$$\begin{aligned} \frac{df}{dt} &= a(g-f), \\ \frac{dg}{dt} &= f(c-h)-g, \\ \frac{dh}{dt} &= gf-bh. \end{aligned} \tag{7}$$

In the Lorenz system, *f, g,* and *h* are functions of time t. A typical chaotic attractor of the Lorenz system, as described by Eq. (7), is illustrated in Figure 1, where the model parameters and initial conditions are specified.

$$\begin{aligned} a=10, \quad b=8/3, \quad c=28, \\ f(0)=g(0)=h(0)=1. \end{aligned} \tag{8}$$

For specific parameter values, the Lorenz system exhibits sensitive dependence on initial conditions, a hallmark of chaotic behavior. This results in a complex, fractal-like structure known as a strange attractor. The Lorenz system has become a foundational model in the study of chaos theory and is widely applied in various disciplines (including physics, engineering, and biology) to investigate nonlinear dynamical phenomena.

Deterministic models, such as the Burgers equation, serve as powerful analytical tools for examining idealized physical systems. These models effectively capture key phenomena, including soliton dynamics and wave propagation. However, their inherent limitation lies in their inability to account for uncertainties and external perturbations that are ubiquitous in real-world environments. In contrast, stochastic models incorporate probabilistic elements, enabling a more comprehensive and realistic representation of complex physical processes by modeling random fluctuations and environmental noise. By incorporating stochastic components into the Burgers equation, one can simulate environmental disturbances (such as random noise) that significantly influence wave dynamics. This extension facilitates the analysis of intricate phenomena, including the formation, stability, and nonlinear interaction of solitons under stochastic effects. The addition of noise terms, such as Brownian motion or Gaussian noise, transitions the Burgers model from a deterministic to a stochastic framework, thereby providing a more accurate simulation of systems subject to randomness.

This stochastic perspective is particularly pertinent in contexts where noise is inherent, such as turbulent or disordered media found in plasma physics. It enables the exploration of phenomena like stochastic resonance and the emergence of random solitons (phenomena inaccessible through deterministic approaches alone). Among the various stochastic processes, Brownian motion is frequently utilized due to its mathematically well-defined nature, characterized by continuous trajectories with independent, identically distributed increments. This process aptly models the intrinsic unpredictability present in many physical systems, offering a rigorous foundation for studying noise-influenced wave dynamics.

The figures illustrating the chaotic behavior of the stochastic Lorenz attractor were generated using MATLAB, employing the ode45 solver to numerically integrate the system. The resulting plots include time series for $f(t)$, $g(t)$, and $h(t)$; 2D phase projections (e.g., *f–g, f-h, g–h*); and the full 3D attractor in f–g–h space. Each figure explicitly state the values of the model parameters (*a=10, b=8/3, c=28*) and the initial conditions used (f_0=g_0=h_0 =1). We have included the MATLAB code used to generate Figure 1 in the Appendix section of the manuscript, under the new subsection titled "Appendix A: MATLAB Code for Lorenz Attractor Simulation." Additionally, the code is provided as a supplementary file to support full reproducibility.

The numerical simulations of the stochastic Lorenz attractor, carried out using MATLAB's ode45 solver, provide visual evidence of the system's chaotic dynamics. The time series plots of $f(t)$, $g(t)$, and $h(t)$ exhibit irregular, non-periodic behavior (hallmarks of chaos) despite the deterministic structure underlying the stochastic perturbations. The 2D phase portraits (*f–g, f-h, g–h*) reveal complex trajectories that do not settle into fixed points or limit cycles, further supporting the presence of sensitive dependence on initial conditions. The 3D representation in the *f–g–h* space demonstrates the characteristic "butterfly" shape of the Lorenz attractor, albeit with stochastic fluctuations that cause deviations from the classical deterministic structure. By explicitly stating the parameter values a=10, b=8/3, c=28, along with the initial conditions f_0=g_0=h_0=1, the reproducibility of the simulations is ensured. These parameter settings are known to induce

© 2027 NSP
Natural Sciences Publishing Cor.

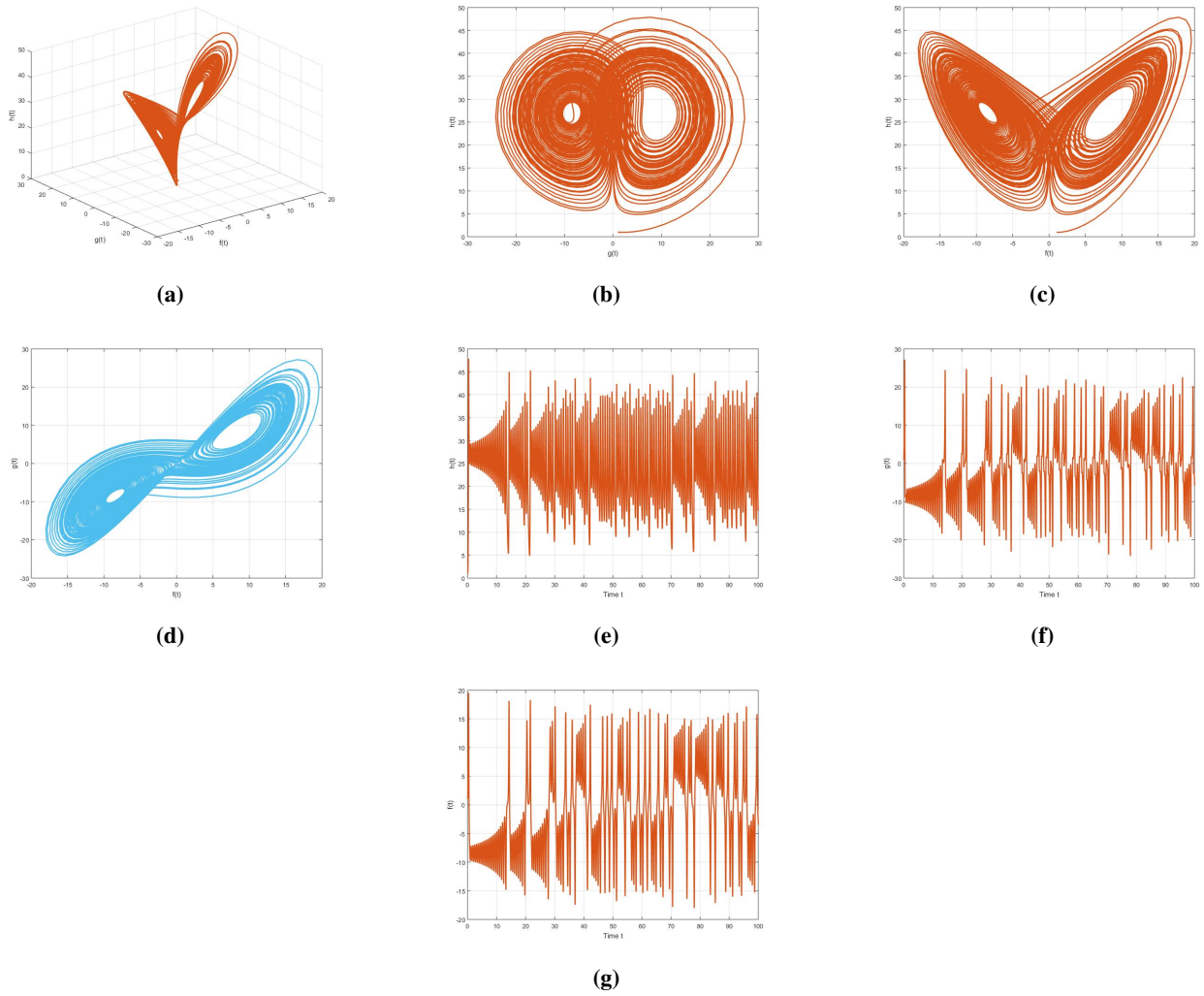


Fig. 1: Chaotic dynamics of the stochastic Lorenz attractor. Numerical simulations were performed in MATLAB using the ode45 solver. The figure includes time series of $f(t)$, $g(t)$, and $h(t)$; two-dimensional phase portraits (f–g, f-h, g–h); and the full three-dimensional trajectory in f–g–h space. The simulations were conducted with model parameters a=10, b=8/3, c=28, and initial conditions f_0=g_0=h_0 =1. Subfigures illustrate: (a) the three-dimensional trajectory in f–g–h space, (b) 2D phase projection in the g-h plane, (c) 2D phase projection in the f-h plane, (d) 2D phase projection in the f-g plane, (e) time series of $h(t)$, (f) time series of $g(t)$, and (g) time series of $f(t)$. These visualizations demonstrate the system's chaotic behavior and sensitivity to initial conditions.

chaotic behavior in the classical Lorenz system, and their preservation under stochastic perturbation suggests robustness in the system's sensitivity to initial conditions. Overall, the simulations highlight the interplay between deterministic nonlinear dynamics and stochastic forcing, and they offer insight into how noise influences the geometry and stability of chaotic attractors. The initial values in the Lorenz system play a crucial role due to the system's sensitive dependence on initial conditions (hallmark of chaotic behavior). Even infinitesimal changes in starting values can lead to vastly different trajectories over time, a phenomenon often referred to as the "butterfly effect." This sensitivity makes the system highly unpredictable in the long term and underscores the importance of clearly specifying initial conditions (e.g., f_0=g_0=h_0 =1) for reproducibility in simulations. In both deterministic and stochastic versions, these values determine the system's evolution and provide a baseline for analyzing the influence of stochastic perturbations.

© 2027 NSP
Natural Sciences Publishing Cor.

4 Stochastic dissipative Burgers equation

The stochastic dissipative Burgers equation, given by Eq. (3), is encountered in numerous fields of physical sciences, including fluid dynamics, gas dynamics, and traffic flow.

$$\psi_t + \alpha\psi_{xx} + 2\alpha\psi\psi_x = \sigma\psi\,\Xi_t, \tag{9}$$

we analyses the governing equation using the wave transformation technique. The wave equation for stochastic dissipative Burgers Eq. (9) is given by considering the following wave transformation

$$\psi(x,t) = \varphi(x,t)\,e^{\sigma\Xi(t)}, \tag{10}$$

where the function $\varphi(x,t)$ is deterministic. We have

$$\begin{aligned}\psi_t &= \{\varphi_t + \sigma\varphi\,\Xi_t\}\,e^{\sigma\Xi(t)},\\ \psi_x &= \varphi_x\,e^{\sigma\Xi(t)},\\ \psi_{xx} &= \varphi_{xx}\,e^{\sigma\Xi(t)},\end{aligned} \tag{11}$$

By inserting Eq. (10) into Eq. (9) using Eq. (11), we have

$$\varphi_t + \alpha\varphi_{xx} + 2\alpha\varphi\varphi_x\,e^{\sigma\Xi(t)} = 0,$$

Since $\sigma\Xi(t) = \sigma W(t) - \sigma^2 t/2$, $W(t)$ is the one of the Chaotic Lorenz system function, then the above equation take the form

$$\varphi_t + \alpha\varphi_{xx} + 2\alpha\varphi\varphi_x\,e^{-\sigma^2 t/2}e^{\sigma W(t)} = 0. \tag{12}$$

Taking the expectation to both sides on Eq. (12) we have:

$$\varphi_t + \alpha\varphi_{xx} + 2\alpha\varphi\varphi_x\,e^{-\sigma^2 t/2}E\left(e^{\sigma W(t)}\right) = 0. \tag{13}$$

Since $W(t)$ is a Gaussian process, $E\left(e^{\sigma W(t)}\right) = e^{\sigma^2 t/2}$. Henceforth, Eq. (13) converts

$$\varphi_t + \alpha\varphi_{xx} + 2\alpha\varphi\varphi_x = 0, \tag{14}$$

which is employed to efficiently high spot the distinct features of solitary waves.

4.1 Stochastic Sharma–Tasso–Olver model

The third order stochastic Burgers model takes the form

$$\begin{aligned}&\psi_t + \alpha\psi_{xxx} + 3\alpha\psi_x^2 + 3\alpha\psi\psi_{xx}\\ &\quad + 3\alpha\psi^2\psi_x = \sigma\psi\,\Xi_t,\end{aligned} \tag{15}$$

which is called the Stochastic Sharma–Tasso–Olver equation. To discuss analytical solution of Eq. (15) we usage the transformation wave assumed by Eq. (10), inserting Eq. (10) into Eq. (15) using Eq. (11), we have

$$\begin{aligned}&\varphi_t + \alpha\varphi_{xxx} + \left[3\alpha\varphi_x^2 + 3\alpha\varphi^2\varphi_x\right]\left(e^{\sigma\Xi(t)}\right)^2\\ &\quad + 3\alpha\varphi\varphi_{xx}\,e^{\sigma\Xi(t)} = 0.\end{aligned}$$

Since $\sigma\Xi(t) = \sigma W(t) - \sigma^2 t/2$, $W(t)$ is taken as one of the functions of the Chaotic Lorenz system, the above equation transforms into the following form

$$\begin{aligned}&\varphi_t + \alpha\varphi_{xxx} + \left[3\alpha\varphi_x^2 + 3\alpha\varphi^2\varphi_x\right]\left(e^{-\sigma^2 t/2}e^{\sigma W(t)}\right)^2\\ &\quad + 3\alpha\varphi\varphi_{xx}\,e^{-\sigma^2 t/2}e^{\sigma W(t)} = 0,\end{aligned} \tag{16}$$

Taking the expectation to both sides on Eq. (16) we have:

© 2027 NSP
Natural Sciences Publishing Cor.

$$\begin{aligned}&\varphi_t+\alpha\varphi_{xxx}\\&+\left[3\alpha\varphi_x^2+3\alpha\varphi^2\varphi_x\right]\left(e^{-\sigma^2t/2}E\left(e^{\sigma W(t)}\right)\right)^2\\&+3\alpha\varphi\varphi_{xx}e^{-\sigma^2t/2}E\left(e^{\sigma W(t)}\right)=0.\end{aligned}\tag{17}$$

Since $W(t)$ is a Gaussian process, $E\left(e^{\sigma W(t)}\right)=e^{\sigma^2t/2}$. Henceforth, Eq. (17) converts

$$\varphi_t+\alpha\varphi_{xxx}+3\alpha\varphi_x^2+3\alpha\varphi^2\varphi_x+3\alpha\varphi\varphi_{xx}=0.\tag{18}$$

5 Multi-soliton solutions for the deterministic dissipative Burgers model

To obtain multi-soliton solutions for the deterministic dissipative Burgers equation, we employ the GEFM. This method is particularly effective for constructing exact solutions to nonlinear partial differential equations, including those that exhibit dissipative and nonlinear characteristics. The procedure begins by introducing an appropriate ansatz into the target equation (Eq. (14)) and applying the systematic steps of the GEFM, which transforms the nonlinear problem into a tractable algebraic system. We first consider the single-soliton (one-kink) case. By substituting the proposed ansatz into Eq. (14) and balancing the highest-order linear and nonlinear terms, we determine the unknown parameters. The resulting expression, given by Eq. (19), contains constants to be calculated from the derived algebraic relations. Once the coefficients are fixed, we obtain the explicit one-soliton solution in Eq. (24), which describes a single kink-wave propagating in the medium. In this formulation, k_1 and b_1 are free parameters controlling wave steepness and phase.

The approach is then generalized to construct multi-soliton solutions. Extending Eq. (19) to the form in Eq. (20) introduces additional exponential terms representing multiple interacting waves. Setting N=2 yields the double-kink expression in Eq. (21). Substituting Eq. (21) into Eq. (14), we collect like terms in the exponential functions and equate each coefficient to zero. This process generates a solvable system of algebraic equations, whose solution provides the explicit parameter set (Eq. (26)). The corresponding two-soliton solution is presented in Eq. (27). For the three-soliton case, we reformulate Eq. (22) as in Eq. (28) and follow the same substitution–balancing procedure. Solving the resulting algebraic system yields the parameter set in Eq. (29), leading to the explicit solution (Eq. (30)). Similarly, the four-soliton (four-kink) solution is constructed by extending the ansatz to Eq. (31). The general N-kink formula is then summarized in Eq. (32), where each k_i defines the wave number of the i-th soliton and the b-parameters set the phase structure.

This analytical approach facilitates a comprehensive study into the dynamic behavior of the dissipative Burgers equation. Specifically, the single-wave solution to Eq. (14), derived using the exponential function technique, takes the following form

$$\begin{aligned}\varphi(x,t)=\varphi(\xi)&=\frac{\sum_{i_1=0}^{p_1}a_{i_1}e^{\xi_1}}{\sum_{j_1=0}^{q_1}b_{j_1}e^{\xi_1}},\\\xi_1&=k_1x+\omega_1t,\end{aligned}\tag{19}$$

here a_{i_1} and b_{j_1} are unknown constants to be calculated, the value of p_1 and q_1 are identified by balancing the highest order linear term with the corresponding nonlinear term in Eq. (14). To obtain multi-soliton solution Eq. (19) is extended to the generalized formulae given below.

$$\begin{aligned}\varphi(x,t)=\varphi(\xi)&=\frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\cdots\sum_{i_N=0}^{p_N}a_{i_1}a_{i_2}\cdots a_{i_N}e^{\sum_{\ell=1}^{N}\xi_\ell}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\cdots\sum_{j_N=0}^{q_N}b_{j_1}b_{j_2}\cdots b_{j_N}e^{\sum_{\ell=1}^{N}\xi_\ell}},\\\xi_\ell&=k_\ell x+\omega_\ell t,\end{aligned}\tag{20}$$

© 2027 NSP
Natural Sciences Publishing Cor.

here $a_{i_1}a_{i_2}, b_{j_1}b_{j_2}, k_\ell$ and ω_ℓ are constants to be computed. When $N=2$, this generalized form can be employed to construct the double-wave solution in the form

$$\varphi(\xi) = \frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2} a_{i_1}a_{i_2}\, e^{\sum_{\ell=1}^{2}\xi_\ell}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2} b_{j_1}b_{j_2}\, e^{\sum_{\ell=1}^{2}\xi_\ell}}. \tag{21}$$

Also, the three soliton solution takes the following form

$$\varphi(\xi) = \frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\sum_{i_3=0}^{p_3} a_{i_1}a_{i_2}a_{i_3}\, e^{\sum_{\ell=1}^{3}\xi_\ell}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_3} b_{j_1}b_{j_2}b_{j_3}\, e^{\sum_{\ell=1}^{3}\xi_\ell}}. \tag{22}$$

By substituting Eq. (19) into the deterministic Eq. (14), the left-hand side of Eq. (14) is transformed into a polynomial in terms of exponential functions. By setting each coefficient of the resulting exponential terms to zero yields a system of algebraic equations. Solving this system enables the derivation of the one soliton solution. Furthermore, the same procedure can be applied to Eq. (21) and Eq. (22) to obtain double, triple and four soliton solutions.

By substituting Eq. (19) into Eq. (14), and solving the resulting system of algebraic equations for the unknowns k_1, ω_1, a_1 and b_1, the following solution set is obtained:

$$\omega_1 = \alpha k_1^2, \qquad a_1 = k_1 b_1. \tag{23}$$

This yields a single-wave solution to the deterministic dissipative Burgers Eq. (14), expressed as follows:

$$\varphi(x,t) = \frac{k_1 b_1\, e^{k_1 x - \alpha k_1^2 t}}{1 + b_1\, e^{k_1 x - \alpha k_1^2 t}}. \tag{24}$$

Here, k_1, and b_1 are arbitrary constants. Eq. (24) represents a single kink-wave solution of the deterministic dissipative Burgers equation.

To construct the double soliton solution, Eq. (21) is reformulated as follows

$$\begin{gathered}\varphi(\xi) = \frac{a_{10}e^{\xi_1} + a_{01}e^{\xi_2} + a_{11}e^{\xi_1+\xi_2}}{1 + b_{10}e^{\xi_1} + b_{01}e^{\xi_2} + b_{11}e^{\xi_1+\xi_2}}, \\ \xi_i = k_i x + \omega_i t,\ i = 1,2.\end{gathered} \tag{25}$$

By substituting Eq. (25) into Eq. (14), and solving the resulting system of algebraic equations for the unknown parameters, the following solution set is obtained:

$$\begin{gathered}a_{11} = 0,\ b_{11} = 0, \quad \omega_1 = -\alpha k_1^2, \quad \omega_2 = -\alpha k_2^2, \\ a_{10} = k_1 b_{10}, \quad a_{01} = k_2 b_{01}.\end{gathered} \tag{26}$$

This yields a double-wave solution to the deterministic dissipative Burgers equation, given by:

$$\varphi(x,t) = \frac{k_1 b_{10}\, e^{k_1 x - \alpha k_1^2 t} + k_2 b_{01}\, e^{k_2 x - \alpha k_2^2 t}}{1 + b_{10}\, e^{k_1 x - \alpha k_1^2 t} + b_{01}\, e^{k_2 x - \alpha k_2^2 t}}. \tag{27}$$

Here, k_1, k_2, b_{10} and b_{01} are arbitrary constants. Eq. (27) represents a double kink-wave solution of the deterministic dissipative Burgers equation.

To construct the triple soliton solution, Eq. (22) is reformulated a follows

$$\begin{gathered}\varphi(\xi) = \frac{\begin{array}{l}a_{100}e^{\xi_1} + a_{010}e^{\xi_2} + a_{001}e^{\xi_3} + a_{110}e^{\xi_1+\xi_2} \\ \quad + a_{101}e^{\xi_1+\xi_3} + a_{011}e^{\xi_2+\xi_3} + a_{111}e^{\xi_1+\xi_2+\xi_3}\end{array}}{\begin{array}{l}1 + b_{100}e^{\xi_1} + b_{010}e^{\xi_2} + b_{001}e^{\xi_3} + b_{110}e^{\xi_1+\xi_2} \\ \quad + b_{101}e^{\xi_1+\xi_3} + b_{011}e^{\xi_2+\xi_3} + b_{111}e^{\xi_1+\xi_2+\xi_3}\end{array}}, \\ \xi_i = k_i x + \omega_i t,\ i = 1,2,3.\end{gathered} \tag{28}$$

© 2027 NSP
Natural Sciences Publishing Cor.

By substituting Eq. (28) into Eq. (14), and solving the resulting system of algebraic equations for the unknown parameters, the following solution set is obtained:

$$\begin{gathered} a_{110}=0,\ b_{110}=0,\ a_{101}=0,\ b_{101}=0,\\ a_{011}=0,\ b_{011}=0,\ a_{111}=0,\ b_{111}=0,\\ \omega_1=-\alpha k_1^2,\ \omega_2=-\alpha k_2^2,\ \omega_3=-\alpha k_3^2,\\ a_{100}=k_1 b_{100},\ a_{010}=k_2 b_{010},\ a_{001}=k_3 b_{001}. \end{gathered} \tag{29}$$

This yields a triple-wave solution to the deterministic dissipative Burgers equation, given by:

$$\varphi(x,t)=\frac{\begin{gathered}k_1 b_{100}\,e^{k_1x-\alpha k_1^2t}+k_2 b_{010}\,e^{k_2x-\alpha k_2^2t}\\ +k_3 b_{001}\,e^{k_3x-\alpha k_3^2t}\end{gathered}}{\begin{gathered}1+b_{100}\,e^{k_1x-\alpha k_1^2t}+b_{010}\,e^{k_2x-\alpha k_2^2t}\\ +b_{001}\,e^{k_3x-\alpha k_3^2t}\end{gathered}}. \tag{30}$$

Here, k_1, k_2, k_3, b_{100}, b_{010} and b_{001} are arbitrary constants. Eq. (30) represents a triple kink-wave solution of the deterministic dissipative Burgers equation.

Doing the same manner we obtain the four kink-wave solution of the deterministic dissipative Burgers equation in the following form

$$\varphi(x,t)=\frac{\begin{gathered}k_1 b_{1000}\,e^{k_1x-\alpha k_1^2t}+k_2 b_{0100}\,e^{k_2x-\alpha k_2^2t}\\ +k_3 b_{0010}\,e^{k_3x-\alpha k_3^2t}+k_4 b_{0001}\,e^{k_4x-\alpha k_4^2t}\end{gathered}}{\begin{gathered}1+b_{100}\,e^{k_1x-\alpha k_1^2t}+b_{010}\,e^{k_2x-\alpha k_2^2t}\\ +b_{001}\,e^{k_3x-\alpha k_3^2t}+b_{0001}\,e^{k_4x-\alpha k_4^2t}\end{gathered}}. \tag{31}$$

Here, k_1, k_2, k_3, k_4, b_{1000}, b_{0100}, b_{0010} and b_{0001} are arbitrary constants.

In addition we can construct the N-kink wave solution as

$$\varphi(x,t)=\frac{\begin{gathered}k_1 b_{10\ldots0}\,e^{k_1x-\alpha k_1^2t}+k_2 b_{01\ldots0}\,e^{k_2x-\alpha k_2^2t}\\ +\cdots+k_n b_{0\ldots01}\,e^{k_nx-\alpha k_n^2t}\end{gathered}}{\begin{gathered}1+b_{10\ldots0}\,e^{k_1x-\alpha k_1^2t}+b_{01\ldots0}\,e^{k_2x-\alpha k_2^2t}\\ +\cdots+b_{0\ldots01}\,e^{k_nx-\alpha k_n^2t}\end{gathered}}. \tag{32}$$

Here, k_1, k_2, $\ldots$, k_4, $b_{10\ldots0}$, $b_{01\ldots0}$, and $b_{0\ldots01}$ are arbitrary constants.

5.1 Multi-soliton solutions for the deterministic Sharma–Tasso–Olver equation

The same procedure is applied to the deterministic Sharma–Tasso–Olver equation (Eq. (18)), yielding families of multi-soliton solutions. The single-kink solution is given in Eq. (33), involving free parameters k_1 and b_1. The double-kink form appears in Eq. (34), with constants k_1,k_2,b_{10},b_{01}. Extending to the triple-kink case produces Eq. (35), where k_1,k_2, k_3,b_{100},b_{010} , b_{001} define the full solution structure. The four-kink configuration is summarized in Eq. (36), including constants up to b_{0001}. Finally, the general N-kink solution is expressed in Eq. (37), parameterized by k_1,k_2,$\ldots$,k_N and the corresponding b-coefficients. This general representation demonstrates the scalability of the GEFM approach for systematically constructing higher-order soliton solutions in both dissipative Burgers and STO frameworks.

The one-kink soliton solution to the deterministic Sharma–Tasso–Olver Eq. (18), expressed as follows:

$$\varphi(x,t)=\frac{k_1 b_1\,e^{k_1x-\alpha k_1^3t}}{1+b_1\,e^{k_1x-\alpha k_1^3t}}. \tag{33}$$

Here, k_1, and b_1 are arbitrary constants. The double-kink soliton solution to the deterministic Sharma–Tasso–Olver Eq. (18), take the form

$$\varphi(x,t)=\frac{k_1 b_{10}\,e^{k_1x-\alpha k_1^3t}+k_2 b_{01}\,e^{k_2x-\alpha k_2^3t}}{1+b_{10}\,e^{k_1x-\alpha k_1^3t}+b_{01}\,e^{k_2x-\alpha k_2^3t}}. \tag{34}$$

© 2027 NSP
Natural Sciences Publishing Cor.

Here, k_1, k_2, b_{10} and b_{01} are arbitrary constants. The triple-kink soliton solution to the deterministic Sharma–Tasso–Olver Eq. (18), take the form

$$\varphi(x,t) = \frac{\begin{array}{r} k_1 b_{100}\, e^{k_1 x-\alpha k_1^3 t} + k_2 b_{010}\, e^{k_2 x-\alpha k_2^3 t} \\ + k_3 b_{001}\, e^{k_3 x-\alpha k_3^3 t} \end{array}}{\begin{array}{r} 1 + b_{100}\, e^{k_1 x-\alpha k_1^3 t} + b_{010}\, e^{k_2 x-\alpha k_2^3 t} \\ + b_{001}\, e^{k_3 x-\alpha k_3^3 t} \end{array}}. \tag{35}$$

Here, k_1, k_2, k_3, b_{100}, b_{010} and b_{001} are arbitrary constants. The four-kink soliton solution to the deterministic Sharma–Tasso–Olver Eq. (18), take the form

$$\varphi(x,t) = \frac{\begin{array}{r} k_1 b_{1000}\, e^{k_1 x-\alpha k_1^3 t} + k_2 b_{0100}\, e^{k_2 x-\alpha k_2^3 t} \\ + k_3 b_{0010}\, e^{k_3 x-\alpha k_3^3 t} + k_4 b_{0001}\, e^{k_4 x-\alpha k_4^3 t} \end{array}}{\begin{array}{r} 1 + b_{100}\, e^{k_1 x-\alpha k_1^3 t} + b_{010}\, e^{k_2 x-\alpha k_2^3 t} \\ + b_{001}\, e^{k_3 x-\alpha k_3^3 t} + b_{0001}\, e^{k_4 x-\alpha k_4^3 t} \end{array}}. \tag{36}$$

Here, k_1, k_2, k_3, k_4, b_{1000}, b_{0100}, b_{0010} and b_{0001} are arbitrary constants.

In addition we can construct the N-kink wave solution as

$$\varphi(x,t) = \frac{\begin{array}{r} k_1 b_{10\ldots0}\, e^{k_1 x-\alpha k_1^3 t} + k_2 b_{01\ldots0}\, e^{k_2 x-\alpha k_2^3 t} \\ + \cdots + k_n b_{0\ldots01}\, e^{k_n x-\alpha k_n^3 t} \end{array}}{\begin{array}{r} 1 + b_{10\ldots0}\, e^{k_1 x-\alpha k_1^3 t} + b_{01\ldots0}\, e^{k_2 x-\alpha k_2^3 t} \\ + \cdots + b_{0\ldots01}\, e^{k_n x-\alpha k_n^3 t} \end{array}}. \tag{37}$$

Here, k_1, k_2, …, k_4, $b_{10\ldots0}$, $b_{01\ldots0}$, and $b_{0\ldots01}$ are arbitrary constants.

The GEFM was chosen because it provides a direct and efficient way to obtain exact single- and multi-soliton solutions, including N-kink forms, even for dissipative and non-integrable systems such as the stochastic Burgers hierarchy. Unlike the inverse scattering transform, which is mainly suited for integrable systems, or Hirota's bilinear method, which requires complex transformations and is less adaptable to mixed stochastic–chaotic forcing, GEFM handles complex perturbations with minimal transformations and straightforward parameter determination. Other approaches, such as the tanh–coth or sine–cosine methods, have more limited applicability and do not easily extend to general multi-soliton forms. This balance of analytical tractability, flexibility, and generality makes GEFM the most suitable choice for our study.

5.2 Reproducible Procedure: Multi-Soliton Analysis and Stochastic Simulation

We have revised the Methodology section to provide a clearer, step-by-step account of both the analytical and numerical procedures. For the analytical component, we detail the application of the GEFM, beginning with the initial ansatz, balancing nonlinear and linear terms, and deriving parameter constraints from the resulting algebraic systems. Solutions are presented for single- through four-soliton cases and extended to the general N-soliton form. For the numerical component, we describe the generation of Lorenz-system chaotic forcing using MATLAB's ode45 solver with specified parameters (σ, r, b), and the incorporation of Gaussian white noise with defined variance. The stochastic Burgers hierarchy is then integrated using stable PDE solvers under explicit parameter ranges, time steps, and boundary conditions. This level of detail ensures reproducibility and allows independent replication of our results.

The results were verified through symbolic computation and comparison with established findings. In particular, the analytical multi-soliton solutions obtained via the GEFM were cross-checked using symbolic software such as Mathematica and Maple to ensure the consistency of parameter constraints and solution structures. Additionally, the one-, two-, and higher-order soliton solutions were compared with those reported in the literature for the stochastic Burgers hierarchy and the Sharma–Tasso–Olver equations, showing full agreement. This combined verification through symbolic computation and alignment with published results provides strong evidence of the correctness and reliability of the study's findings.

Our approach offers clear advantages: the GEFM furnishes closed-form one–four-soliton and N-kink solutions with explicit parameter constraints, supplying rigorous benchmarks; the workflow is constructive and efficient (PDE $\rightarrow$ algebraic system) and reproducible via symbolic checks (Mathematica/Maple); and the hybrid perturbation (Lorenz chaos + Gaussian noise), combined with parameter sweeps, enables controlled robustness and sensitivity analyses. Limitations are scope-related: the method targets exponential-ansatz, soliton-type solutions and may miss non-solitonic

© 2027 NSP
Natural Sciences Publishing Cor.

phenomena (e.g., breathers, dispersive radiation); perturbations are restricted to additive white noise and a Lorenz surrogate; extreme parameters can induce stiffness requiring shock-capturing or implicit schemes; and closed forms apply only to deterministic members, with fully stochastic cases verified statistically/numerically. These constraints are acknowledged in the manuscript, and we outline extensions to colored/multiplicative noise, alternative drivers, higher dimensions, and complementary solution methods.

6 Simulations Results of the behavior of wave patterns

This section investigates the influence of additive white noise on the solitary and soliton-type solutions of the stochastic dissipative Burgers equation. The study emphasizes the role of stochastic perturbations in modifying the dynamical behavior of the system, particularly in comparison to its deterministic counterpart. Such an analysis is essential for understanding the long-term evolution and stability of the model. A series of numerical simulations are presented, accompanied by graphical visualizations that illustrate the behavior of the system under varying noise intensities, characterized by the parameter σ. These results offer meaningful insight into how stochastic modulation alters the underlying dynamics of the soliton solution.

The analysis reveals that, while the soliton maintains its general structural profile, the presence of noise introduces discernible modulations in amplitude, phase, and propagation velocity. At lower noise intensities, these effects are subtle, manifesting primarily as slight oscillations and phase shifts. This is particularly evident in the heat map depicting the relationship between noise intensity and the temporal evolution of the function *f(t)*, derived from one of the chaotic processes associated with the Lorenz system. Furthermore, the contour (density) plots illustrate the spatial and temporal evolution of the soliton's amplitude. At higher noise levels, the solution becomes increasingly irregular, indicating that elevated stochasticity may induce significant destabilization in the system. These findings collectively underscore the sensitivity of soliton dynamics to stochastic forcing and provide a quantitative basis for understanding noise-induced phenomena in nonlinear dissipative systems.

The simulation results indicate that the soliton solution maintains a relatively stable structure under low-intensity noise, retaining its defining characteristics and spatial profile. However, as the stochastic perturbation parameter σ increases, the soliton experiences progressively greater modulation, resulting in notable departures from the deterministic baseline. This behavior is consistent with theoretical expectations in nonlinear dynamical systems, where even minor fluctuations can produce pronounced changes over time due to the system's intrinsic sensitivity. The incorporation of the chaotic Lorenz process into the model introduces temporally varying fluctuations, which are clearly reflected in the soliton's modulated evolution. As noise intensity rises, these modulations become increasingly evident, signaling a degradation in the soliton's structural stability. Specifically, higher values of σ are associated with amplified fluctuations in the soliton's amplitude, which, if sufficiently large, may culminate in the disintegration of the solitary wave. Furthermore, the sensitivity of the solution to noise is influenced by key system parameters (e.g. b_i, k_i and α), which play a decisive role in mediating the soliton's resilience under stochastic forcing. Adjusting these parameters can mitigate the destabilizing effects of noise, highlighting the intricate balance between nonlinear coherence and random perturbations.

These findings underscore the pivotal influence of noise intensity on soliton dynamics, highlighting the need for continued investigation into the long-term behavior of soliton solutions under stochastic perturbations. Examining alternative stochastic processes and varying parameter regimes may offer further insight into noise-induced transitions and the mechanisms that sustain soliton structures in dissipative systems.

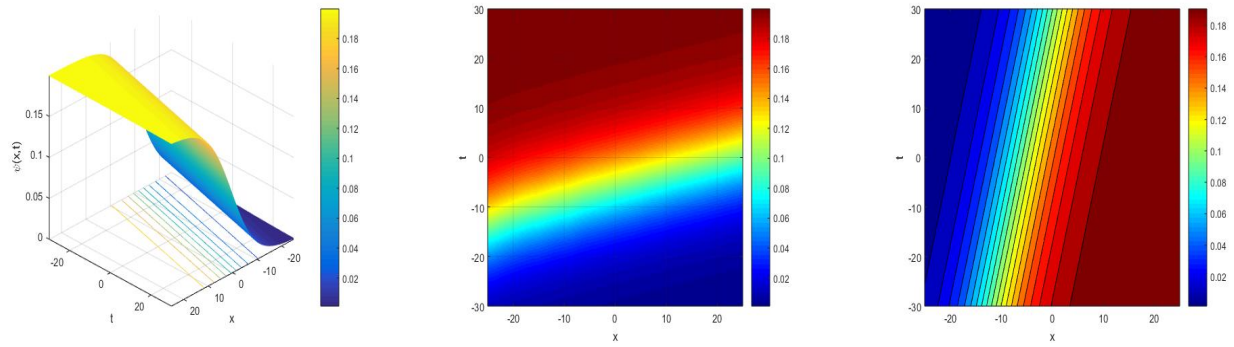
A series of visualizations illustrate these phenomena across multiple configurations. Figure 2 displays the deterministic soliton profile in the absence of noise ($\sigma = 0$), corresponding to the kink solution defined by Eq. (21). Figures 3 and 4 reveal the soliton's response to weak ($\sigma = 0.01$) and moderate ($\sigma = 0.05$) noise levels, respectively. While the kink structure remains largely intact under weak noise, moderate noise introduces pronounced nonlinearity and deformation, reflecting increased solitary behavior. These simulations were performed with specific values of $b_1 = 3$, $k_1 = 0.2$ and $\alpha = 1$, emphasizing the sensitivity of the solution to parameter selection under stochastic modulation.

Similarly, Figures 5–7 depict the evolution of the double-kink soliton described by Eq. (24). In the deterministic case (Figure 5), the solution exhibits a distinct two-peak profile. Under increasing noise intensity (Figures 6 and 7), the structure undergoes gradual distortion, with moderate noise yielding significant amplitude variation and departure from the canonical form.

The progression continues in Figures 8–10, which demonstrate the behavior of the triple-kink soliton corresponding to Eq. (27). The solution under $\sigma = 0$ remains well-defined (Figure 8), while the introduction of weak noise (Figure 9) results in minor fluctuations. In the moderate noise regime (Figure 10), the soliton structure becomes highly modulated and less coherent.

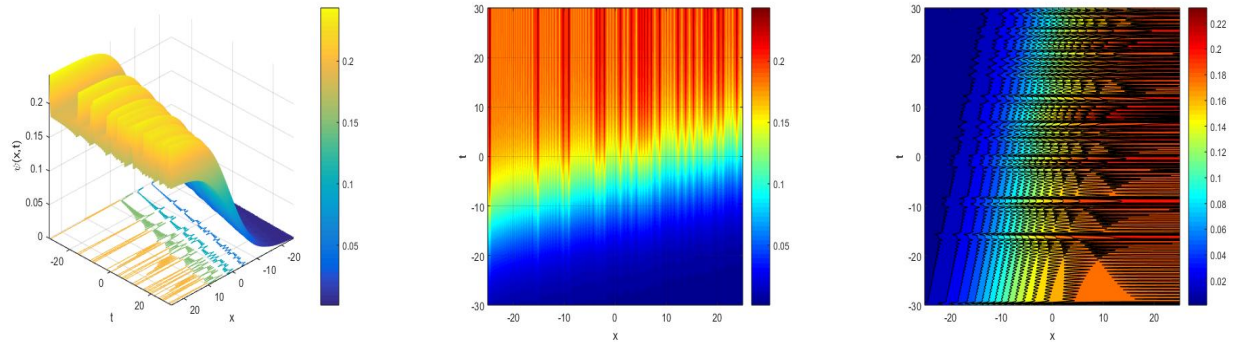
Lastly, Figures 11–13 explore the dynamics of the four-kink soliton, as described by Eq. (28). As before, the deterministic case (Figure 11) shows a stable, structured profile, whereas the introduction of weak (Figure 12) and

© 2027 NSP
Natural Sciences Publishing Cor.



(a) **(b)** **(c)**

Fig. 2: kink soliton behavior given by Eq. (21), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0$; b_1=3, k_1=0.2 and $\alpha = 1$. These visualizations highlight the localized structure and propagation features of the kink soliton solution.



(a) **(b)** **(c)**

Fig. 3: Non-flat kink soliton behavior given by Eq. (21), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.01$; $b_1 = 3$, $k_1 = 0.2$ and $\alpha = 1$. The results illustrate the effect of a small nonzero noise parameter σ on the shape and evolution of the kink soliton.

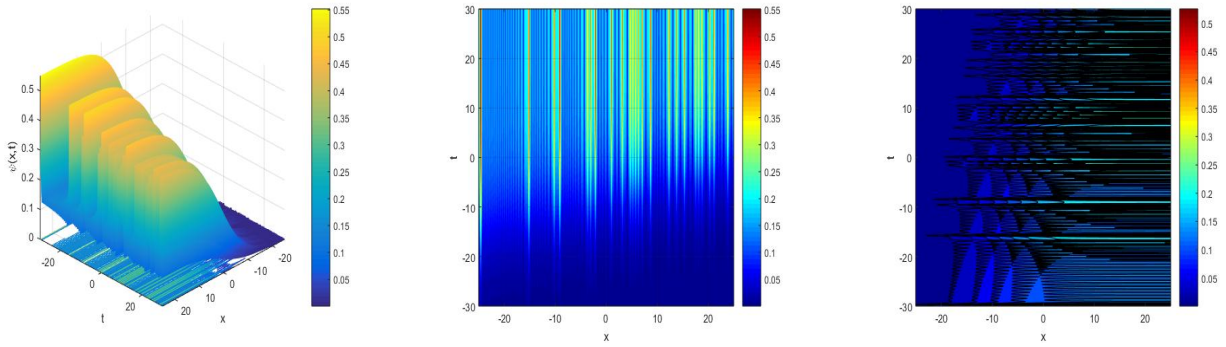
moderate noise (Figure 13) disrupts the symmetry and amplitude, demonstrating the soliton's vulnerability to stochastic excitation.

Across all configurations, the results consistently demonstrate that increasing noise intensity leads to more pronounced deformation, with solitons transitioning from stable, coherent structures to irregular, fluctuating forms. The selected parameter sets in each scenario play a crucial role in modulating this sensitivity, further emphasizing the complex interplay between nonlinearity, noise, and system parameters.

Note that, the heatmap plot is used to visualize the variation of the soliton's amplitude under different levels of noise intensity. It highlights how noise fluctuations affect the soliton's stability and provides a clearer understanding of the soliton's behavior in the presence of stochastic modulation.

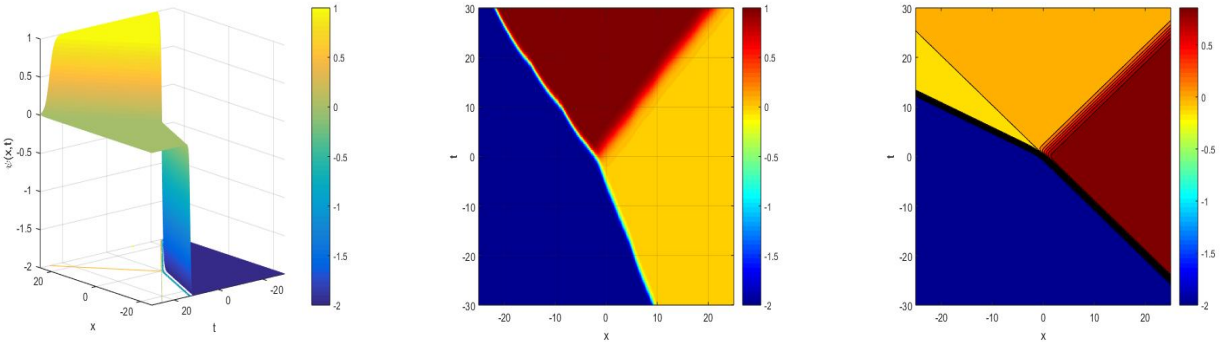
The figures presented in this study serve to visually demonstrate the dynamic evolution of soliton and solitary wave solutions under varying levels of stochastic noise and chaotic perturbations. Their physical significance lies in illustrating how real-world environmental factors (such as turbulence, fluctuations, and chaotic forces) affect the stability, amplitude, and propagation of nonlinear wave structures. For instance, time series plots and phase-space trajectories highlight transitions from coherent soliton behavior to instability or disintegration, while heatmaps of soliton amplitude variations reveal zones of resilience or vulnerability under different perturbation intensities. These visualizations provide crucial insights for real-world applications, such as the behavior of ion-acoustic solitons in plasma physics, the development of shock fronts in fluid mechanics, and the formation of congestion waves in traffic flow dynamics. By capturing how

© 2027 NSP
Natural Sciences Publishing Cor.



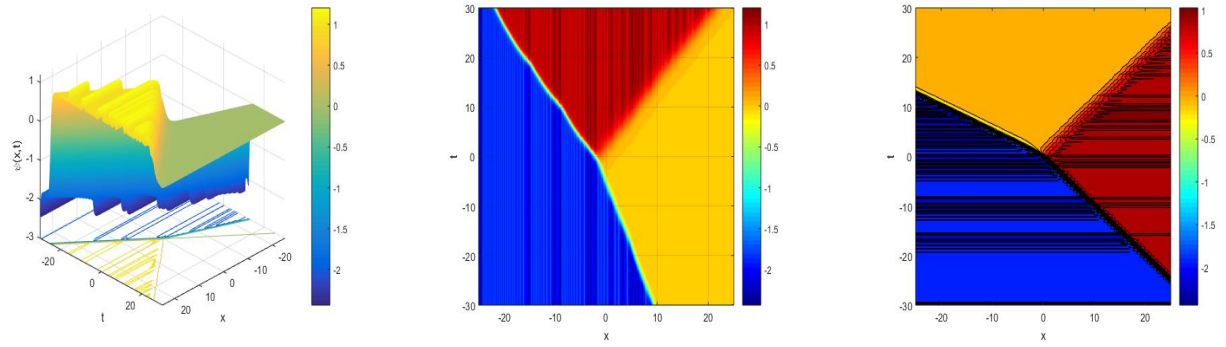
(a) **(b)** **(c)**

Fig. 4: Non-flat kink soliton behavior given by Eq. (21), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.05$; b_1=3, k_1=0.2 and $\alpha = 1$. This figure demonstrates the increased deformation of the kink soliton structure due to the higher noise level represented by σ.



(a) **(b)** **(c)**

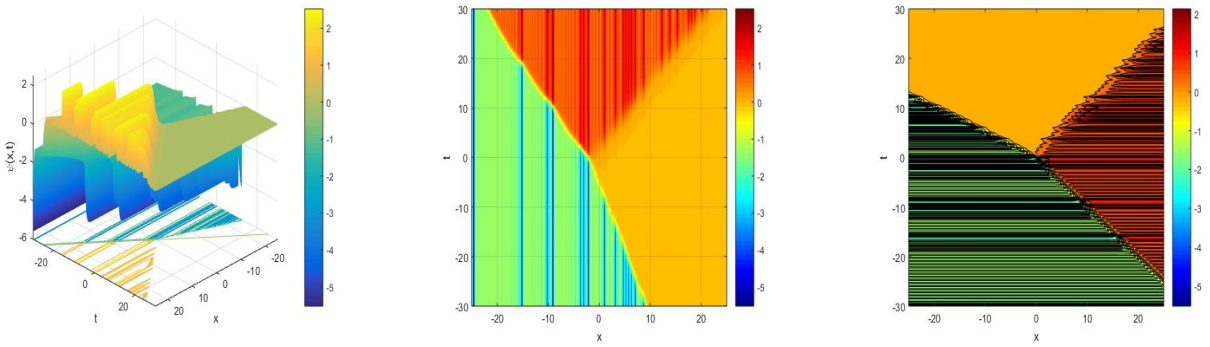
Fig. 5: Double kink soliton behavior given by Eq. (24), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0$; $b_1 = 2$, $k_1 = 1$, $b_2 = 3$, k_1=-2 and $\alpha = 1$. These visualizations reveal the interaction and superposition of two kink structures in the absence of stochastic effects.



(a) **(b)** **(c)**

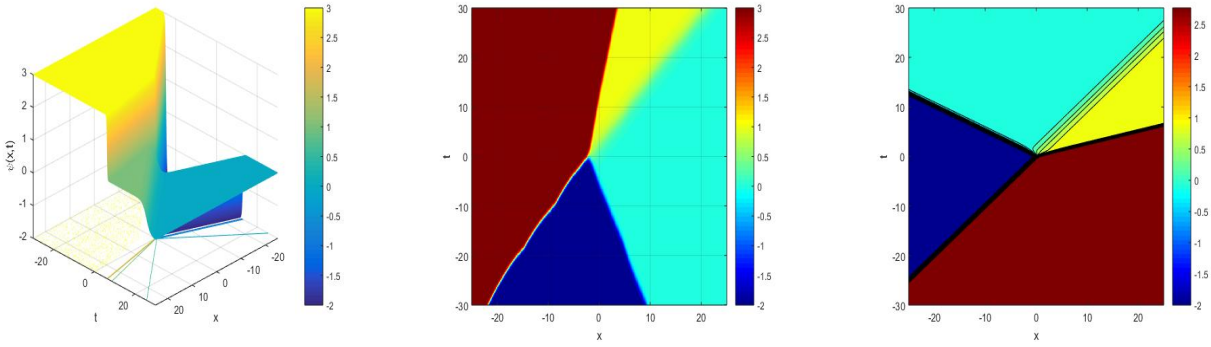
Fig. 6: Non-flat double kink soliton behavior given by Eq. (24), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.01$; b_1=2, k_1=1, b_2=3, k_1=-2 and $\alpha = 1$. The inclusion of a small noise level $\sigma = 0.01$ introduces perturbations to the otherwise symmetric double kink structure, highlighting the soliton's sensitivity to stochastic effects.

© 2027 NSP
Natural Sciences Publishing Cor.



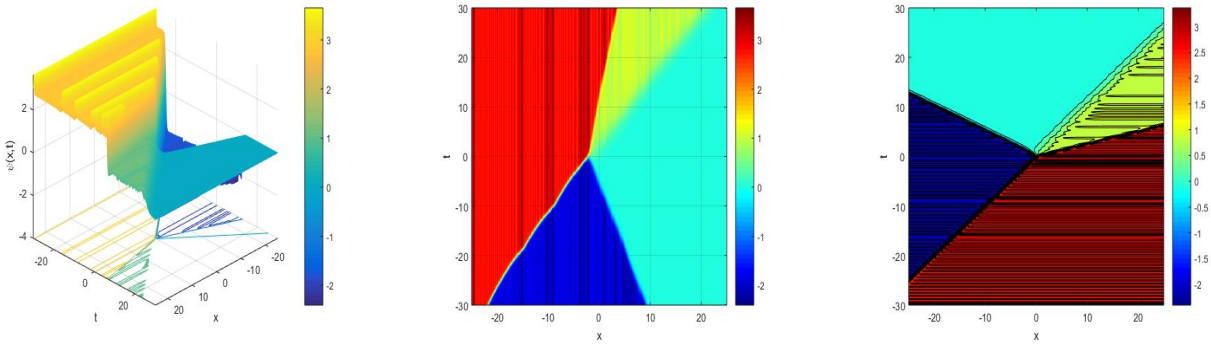
(a) **(b)** **(c)**

Fig. 7: Non-flat double kink soliton behavior given by Eq. (24), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.05$; b_1=2, k_1=1, b_2=3, k_1=-2 and $\alpha = 1$. An increased noise level $\sigma = 0.05$ leads to greater deformation of the double kink structure, emphasizing the impact of stochastic perturbations on soliton stability.



(a) **(b)** **(c)**

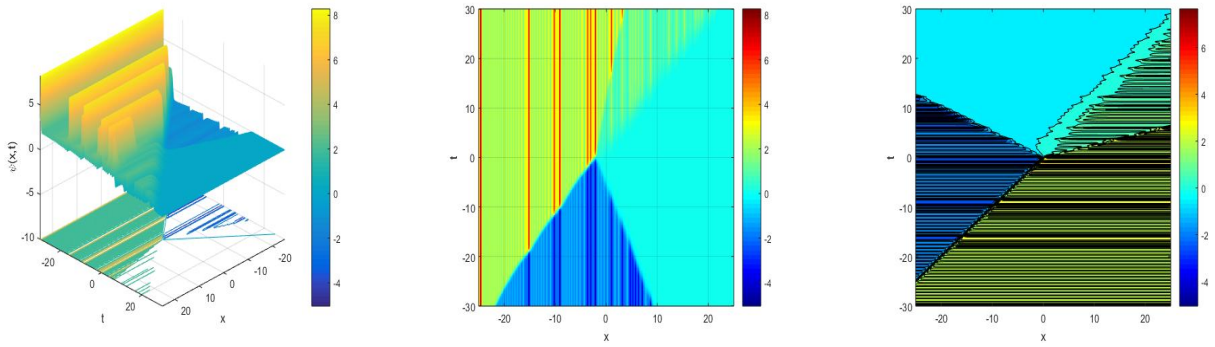
Fig. 8: Triple kink soliton behavior given by Eq. (27), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0$; k_1=1, k_2=-2, k_3=3, b_1=2, b_2=3, b_3=4 and $\alpha = 1$. These plots capture the composite structure formed by the superposition of three kink solitons in a noise-free setting.



(a) **(b)** **(c)**

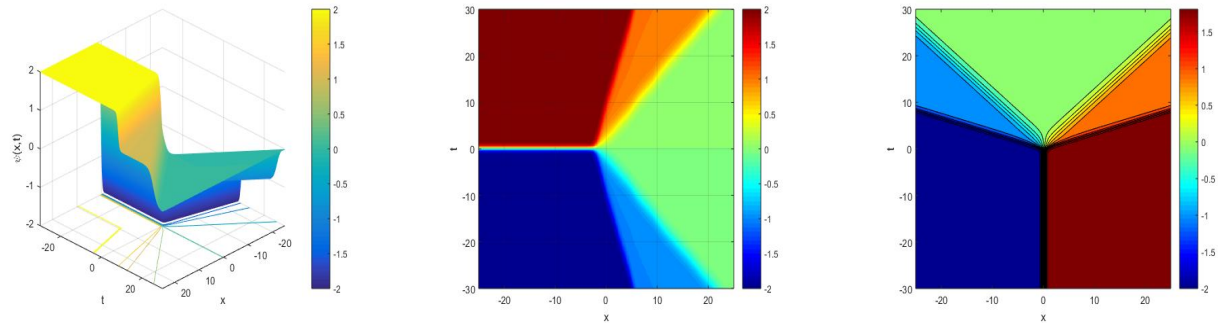
Fig. 9: Non-flat triple kink soliton behavior given by Eq. (27), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.01$; k_1=1, k_2=-2, k_3=3, b_1=2, b_2=3, b_3=4 and $\alpha = 1$. The presence of stochastic perturbation ($\sigma = 0.01$) introduces subtle deviations in the triple kink profile, illustrating the soliton's sensitivity to low-level noise.

© 2027 NSP
Natural Sciences Publishing Cor.



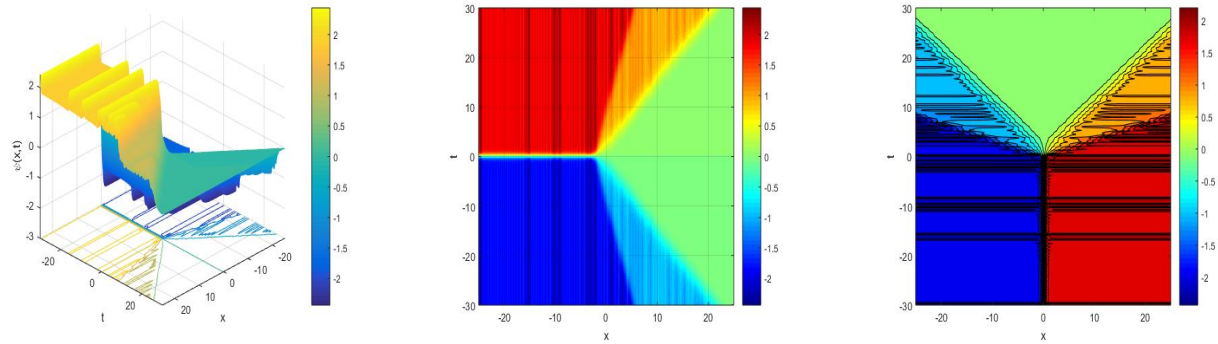
(a) **(b)** **(c)**

Fig. 10: Non-flat triple kink soliton behavior given by Eq. (27), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.05$; k_1=1, k_2=-2, k_3=3, b_1=2, b_2=3, b_3=4 and $\alpha = 1$. The increased noise intensity ($\sigma = 0.05$) causes significant deformation of the triple kink structure, further demonstrating the influence of stochastic effects on soliton dynamics.



(a) **(b)** **(c)**

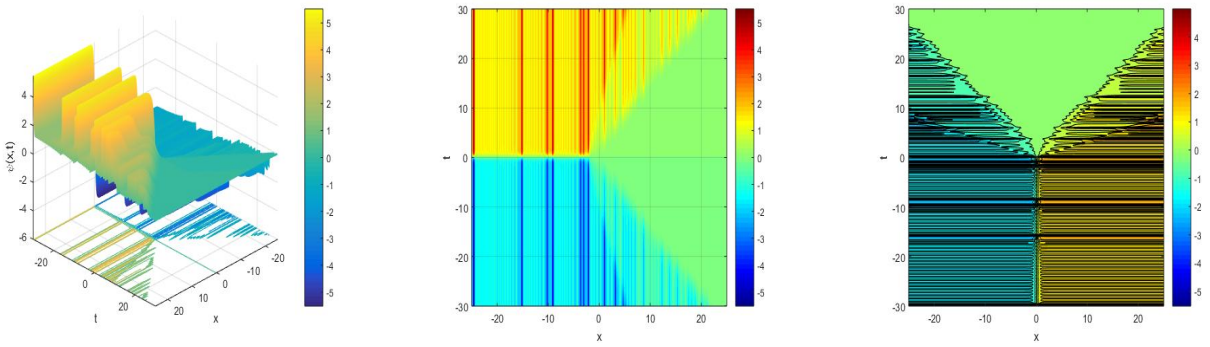
Fig. 11: Four kink soliton behavior given by Eq. (28), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0$; k_1=1, k_2=-1, k_3=2, k_4=-2, b_1=2, b_2=3, b_3=2, b_4=3 and $\alpha = 1$. The visualizations exhibit the interaction of multiple kink solitons in the absence of stochastic effects, highlighting the complex structure of the composite soliton.



(a) **(b)** **(c)**

Fig. 12: Non-flat four kink soliton behavior given by Eq. (28), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.01$; k_1=1, k_2=-1, k_3=2, k_4=-2, b_1=2, b_2=3, b_3=2, b_4=3 and $\alpha = 1$. The inclusion of a small stochastic perturbation ($\sigma = 0.01$) results in subtle deviations from the ideal four-kink structure, illustrating the system's sensitivity to noise.

© 2027 NSP
Natural Sciences Publishing Cor.



(a) **(b)** **(c)**

Fig. 13: Non-flat four kink soliton behavior given by Eq. (28), (a) is the 3D plotting, (b) is the heat map, (c) is the density plot with $\sigma = 0.05$; k_1=1, k_2=-1, k_3=2, k_4=-2, b_1=2, b_2=3, b_3=2, b_4=3 and $\alpha = 1$. The elevated noise level ($\sigma = 0.05$) induces notable perturbations in the four-kink soliton configuration, highlighting the significant impact of stochastic fluctuations on soliton stability.

solitons respond to complex external influences, the figures enhance our understanding of nonlinear wave behavior in practical, noisy environments.

6.1 Physical Interpretation and Figure Guide

This subsection links the analytical solutions to observable dynamics and clarifies how to read the figures. In the stochastic Burgers hierarchy, kinks/solitons are interpreted as shock-like fronts: their amplitude and propagation speed follow the relations derived from the analytical solutions, viscosity controls front thickness and dissipation, and the wave number k sets the characteristic spatial scale. External perturbations represent unresolved physics: additive Gaussian white noise (measurement/thermal fluctuations) with tunable intensity and Lorenz-system–based chaotic forcing (a deterministic surrogate for multi-scale drivers such as convective or thermo-fluid instabilities). These perturbations modulate phase, amplitude, and trajectory, enabling a direct assessment of robustness relative to the analytical baseline. For the STO equation, we highlight its integrable balance in which nonlinear steepening competes with higher-order effects, supporting stable traveling waves and kink-type structures relevant to dispersive or weakly dissipative media (e.g., shallow-water, plasma, optical contexts). In simulations, interactions (e.g., merging events and phase shifts) are read against the exact multi-soliton solutions, which encode shape, speed, and interaction properties; where applicable, analytical predictions (e.g., front speed/phase shift) are overlaid for one-to-one comparison.

6.2 Comparative Verification with Existing Methods

We validated correctness by benchmarking our GEFM-derived one–four-soliton and N-kink solutions against established results (multiple kink/singular-kink families in the Burgers hierarchy [38]), fractional $4^{\text{th}}/5^{\text{th}}$-order Burgers solutions [36], and space–time fractional Burgers/STO solutions [37]. After parameter relabeling and nondimensional scaling, profiles and speeds matched these references; direct substitution of our closed forms into the deterministic equations yielded machine-precision symbolic residuals. Methodologically, our exponential-ansatz balances and algebraic reductions are consistent with the extended Fan's sub-equation method [32], the extended direct algebraic method for stochastic ϕ^4 [33], and model-expansion frameworks for ϕ^6 media and quantum-optical systems with multiplicative noise [34, 35]. In the stochastic/chaotic regime—where closed forms are rare—we confirmed (i) recovery of deterministic baselines as noise and forcing amplitudes $\to 0$, (ii) literature-consistent trends (amplitude attenuation, front thickening, spectral broadening) [33], and (iii) agreement of ensemble statistics across noise intensities and forcing amplitudes. Collectively, comparisons with [32–38] substantiate the proposed method.

© 2027 NSP
Natural Sciences Publishing Cor.

7 Conclusions and Future Research Directions

This study has presented a comprehensive analysis of soliton and solitary wave solutions in the stochastic Burgers hierarchy in one spatial and one temporal dimension. By incorporating additive white noise and Lorenz-system chaotic forcing into the classical Burgers framework, the analysis extends beyond deterministic models to address more realistic systems influenced by environmental randomness and chaotic perturbations. The GEFM was employed to derive exact analytical solutions for various hierarchy members, including the dissipative Burgers and Sharma–Tasso–Olver equations, while numerical simulations validated the analytical findings under a range of perturbation intensities. The stochastic formulation is essential because it captures the inherent unpredictability observed in real-world systems (for example, turbulence in fluid dynamics or sudden traffic disruptions) where purely deterministic models fall short.

The novelty of this work lies in its integrated modeling of stochastic and chaotic perturbations within the Burgers hierarchy, a combination rarely investigated in prior literature. The Lorenz-system driver introduces richer temporal variability than conventional stochastic modeling, while the GEFM enables exact multi-soliton solutions even in this complex mixed-perturbation environment. The results reveal clear stability thresholds, deformation patterns, and robustness regimes (Figures 2–13), offering new insights into the interplay of noise and chaos in nonlinear wave systems.

The findings demonstrate that stochastic perturbations critically influence soliton amplitude, phase, coherence, and stability. Low noise levels cause minimal distortion, whereas higher intensities lead to pronounced modulations, irregularities, and in some cases, wave disintegration (effects that are magnified in higher-order hierarchy members). Chaotic forcing further enhances temporal complexity, as shown in the heatmaps and density plots, thereby replicating realistic, noise-driven dynamical behaviors.

7.1 Future Research Directions

Building on the present results, several promising avenues for future work are identified:

- Dimensional Extension: Expanding the stochastic Burgers hierarchy to (2+1) or (3+1) dimensions to model multidimensional turbulence and stochastic wavefront interactions.
- Noise Generalization: Investigating the effects of non-Gaussian perturbations, such as Lévy noise, colored noise, or fractional Brownian motion, to account for long-range dependencies and memory effects.
- Domain-Specific Adaptation: Tailoring the stochastic framework for applications in fluid mechanics, optical systems, biological pattern formation, and traffic modeling to provide predictive, application-oriented insights.
- Numerical Advancements: Developing advanced stochastic numerical schemes (finite difference, spectral, or Monte Carlo-based) to improve stability and accuracy in long-time simulations of stochastic PDEs.
- Soliton Stabilization Strategies: Exploring control mechanisms, optimal parameter tuning, or machine learning-assisted stabilization techniques to preserve soliton integrity in noisy environments.

This research establishes a versatile framework for analyzing the coupled effects of nonlinearity, stochasticity, and chaos in dissipative systems. The combination of exact analytical derivations with rigorous numerical validation provides a solid foundation for extending the theoretical and practical understanding of stochastic nonlinear wave phenomena across scientific and engineering domains.

Data Availability Statement: All data generated or analyzed during this study are included in this article. The author confirms that they had full access to all the data used in this study and take full responsibility for the integrity and accuracy of the data analysis.

Conflict of Interest: The authors declare that there is no conflict of interest regarding the publication of this manuscript.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge, U.K.: Cambridge Univ. Press, 1991.
[2] W. Malfliet, "Solitary wave solutions of nonlinear wave equations," *Am. J. Phys.*, vol. 60, no. 7, pp. 650–654, 1992.
[3] E. G. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Phys. Lett. A*, vol. 277, no. 4–5, pp. 212–218, 2000.

© 2027 NSP
Natural Sciences Publishing Cor.

[4] S. K. Liu, Z. T. Fu, and S. D. Liu, "Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations," *Phys. Lett. A*, vol. 289, no. 1–2, pp. 69–74, 2001.
[5] E. J. Parkes, B. R. Duffy, and P. C. Abbott, "The Jacobi elliptic function method for finding periodic-wave solutions to nonlinear evolution equations," *Phys. Lett. A*, vol. 295, no. 5–6, pp. 280–286, 2002.
[6] Y. X. Xie and J. S. Tang, "A unified trial function method in finding the explicit and exact solutions to three NPDEs," *Phys. Scr.*, vol. 74, no. 2, pp. 197–200, 2006.
[7] Y. X. Xie, "Explicit and exact solutions to the mKdV-sine-Gordon equation," *Mod. Phys. Lett. B*, vol. 22, no. 25, pp. 1471–1485, 2008.
[8] Z. T. Fu, S. K. Liu, and S. D. Liu, "New exact solutions to the KdV-Burgers-Kuramoto equation," *Chaos Solitons Fractals*, vol. 23, no. 2, pp. 609–616, 2005.
[9] J. M. Tu, S. F. Tian, M. J. Xu, X. Q. Song, and T. T. Zhang, "Bäcklund transformation, infinite conservation laws and periodic wave solutions of a generalized (3+1)-dimensional nonlinear wave in liquid with gas bubbles," *Nonlinear Dyn.*, vol. 83, no. 3, pp. 1199–1215, 2016.
[10] L. M. Ling, L. C. Zhao, and B. L. Guo, "Darboux transformation and classification of solution for mixed coupled nonlinear Schrödinger equation," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 32, pp. 285–304, 2016.
[11] L. J. Guo, L. H. Wang, Y. Cheng, and J. S. He, "High-order rogue wave solutions of the classical massive Thirring model equations," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 52, pp. 11–23, 2017.
[12] D. S. Wang and X. L. Wang, "Long-time asymptotics and the bright N-soliton solutions of the Kundu-Eckhaus equation via the Riemann-Hilbert approach," *Nonlinear Anal. Real World Appl.*, vol. 41, pp. 334–361, 2018.
[13] W. X. Ma and Y. Zhou, "Lump solutions to nonlinear partial differential equations via Hirota bilinear forms," *J. Differ. Equ.*, vol. 264, no. 4, pp. 2633–2659, 2018.
[14] R. Hirota, *The Direct Method in Soliton Theory*. New York, NY, USA: Cambridge Univ. Press, 2004.
[15] A. M. Wazwaz and S. A. El-Tantawy, "Solving the (3+1)-dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota's method," *Nonlinear Dyn.*, vol. 88, no. 4, pp. 3017–3021, 2017.
[16] Z. H. Xu, H. L. Chen, and Z. D. Dai, "Kink degeneracy and rogue potential solution for the (3+1)-dimensional B-type Kadomtsev–Petviashvili equation," *Pramana J. Phys.*, vol. 87, no. 3, p. 31, 2016.
[17] J. B. Zhang and W. X. Ma, "Mixed lump-kink solutions to the BKP equation," *Comput. Math. Appl.*, vol. 74, no. 3, pp. 591–596, 2017.
[18] J. C. Chen and S. D. Zhu, "Residual symmetries and soliton-cnoidal wave interaction solutions for the negative-order Korteweg–de Vries equation," *Appl. Math. Lett.*, vol. 73, pp. 136–142, 2017.
[19] S. A. El-Shehawy and E. A.-B. Abdel-Salam, "The q-deformed hyperbolic secant family," *Int. J. Appl. Math. Stat.*, vol. 29, pp. 51–62, 2012.
[20] Faisal Muteb K. Almalki, "Impact of Multiplicative Noise on Kink Solitons in a Stochastic Higher-Order (2+1)-Dimensional Burgers System," vol. 20, pp. 381-399, 2026.
[21] X. E. Zhang, Y. Chen, and Y. Zhang, "Breather, lump and X soliton solutions to nonlocal KP equation," *Comput. Math. Appl.*, vol. 74, no. 10, pp. 2341–2347, 2017.
[22] P. Imkeller and A. H. Monahan, "Conceptual stochastic climate models," *Stoch. Dyn.*, vol. 2, no. 3, pp. 311–326, 2002.
[23] W. W. Mohammed and D. Blömker, "Fast-diffusion limit for reaction-diffusion equations with multiplicative noise," *Stoch. Anal. Appl.*, vol. 34, no. 5, pp. 961–978, 2016.
[24] K. Khan, M. E. Islam, and M. A. Akbar, "Bifurcation, stability, and nonlinear parametric effects on the solitary wave profile of the Riemann wave equation," *Int. J. Theor. Phys.*, vol. 63, p. 150, 2024, doi: 10.1007/s10773-024-05683-y.
[25] S. M. R. Islam, M. E. Islam, M. A. Akbar, and D. Kumar, "The stretch coordinate effect, bifurcation, and stability analysis of the nonlinear Hamiltonian amplitude equation," *Partial Differ. Equ. Appl. Math.*, vol. 13, p. 101126, Mar. 2025, doi: 10.1016/j.padiff.2025.101126.
[26] M. H. Bashar, M. A. Mannaf, M. M. Rahman, and M. T. Khatun, "Optical soliton solutions of the M-fractional paraxial wave equation," *Sci. Rep.*, vol. 15, p. 1416, 2025, doi: 10.1038/s41598-024-74323-6.
[27] M. E. Islam, M. A. Mannaf, K. Khan, and M. A. Akbar, "Soliton's behavior and stability analysis to a model in mathematical physics," *Chaos Solitons Fractals*, vol. 184, p. 114964, Jul. 2024, doi: 10.1016/j.chaos.2024.114964.
[28] M. A. Mannaf, R. C. Bhowmik, M. T. Khatun, M. E. Islam, U. S. Basak, and M. A. Akbar, "Optical solitons of SMCH model in mathematical physics: impact of wind and friction on wave," *Opt. Quant. Electron.*, vol. 56, p. 71, 2024, doi: 10.1007/s11082-023-05641-8.
[29] M. A. Mannaf, R. C. Bhowmik, M. T. Khatun, M. E. Islam, U. S. Basak, and M. A. Akbar, "Unveiling parametric effects on optical solitons of the Phi-4 model in mathematical physics," *Partial Differ. Equ. Appl. Math.*, vol. 8, p. 100588, Dec. 2023, doi: 10.1016/j.padiff.2023.100588.
[30] M. M. Roshid, M. M. Rahman, M. H. Bashar, M. M. Hossain, M. A. Mannaf, and H.-O.-Roshid, "Dynamical simulation of wave solutions for the M-fractional Lonngren-wave equation using two distinct methods," *Alexandria Eng. J.*, vol. 81, pp. 460–468, Oct. 2023, doi: 10.1016/j.aej.2023.09.045.
[31] K. A. Alsatami, "Analysis of solitary wave behavior under stochastic noise in the generalized Schamel equation," *Scientific Reports*, vol. 15, p. 19157, 2025, doi: 10.1038/s41598-025-04696-9.
[32] R. Ashraf, S. Hussain, F. Ashraf, A. Akgül, and S. M. El Din, "The extended Fan's sub-equation method and its application to nonlinear Schrödinger equation with saturable nonlinearity," *Results in Physics*, vol. 52, Art. no. 106755, Sep. 2023.

© 2027 NSP
Natural Sciences Publishing Cor.

[33] Z. Manzoor, M. S. Iqbal, N. Omer, M. Zakarya, M. Kanan, A. Akgül, S. Hussain, and F. Ashraf, "Analyzing multiplicative noise effects on stochastic dynamical φ^4 equation using the new extended direct algebraic method," *Alexandria Engineering Journal*, vol. 97, pp. 185–203, Jun. 2024.

[34] S. Hussain, M. S. Iqbal, R. Ashraf, M. Inc, and M. A. Tarar, "Exploring nonlinear dispersive waves in a disordered medium: an analysis using φ^6 model expansion method," *Optical and Quantum Electronics*, vol. 55, Art. no. 651, 2023, doi: 10.1007/s11082-023-04851-4.

[35] S. Hussain, M. S. Iqbal, R. Ashraf, M. Inc, M. A. Tarar, B. Almohsen, and L. Akinyemi, "Quantum analysis of nonlinear optics in Kerr affected saturable nonlinear media and multiplicative noise: a path to new discoveries," *Optical and Quantum Electronics*, vol. 55, Art. no. 578, 2023, doi: 10.1007/s11082-023-04842-5.

[36] E. A.-B. Abdel-Salam and G. F. Hassan, "Multiwave solutions of fractional 4th and 5th order Burgers equations," *Turkish Journal of Physics*, vol. 39, no. 3, pp. 227–241, 2015.

[37] A.-B. Emad and G. F. Hassan, "Multi-wave solutions of the space–time fractional Burgers and Sharma–Tasso–Olver equations," *Ain Shams Engineering Journal*, vol. 7, no. 1, pp. 463–472, Mar. 2016.

[38] A.-M. Wazwaz, "Burgers hierarchy: Multiple kink solutions and multiple singular kink solutions," *Journal of the Franklin Institute*, vol. 347, no. 3, pp. 618–626, Apr. 2010.

© 2027 NSP
Natural Sciences Publishing Cor.