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Generalized Fractional Calculus of the Incomplete Generalized Wright Function

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Abstract: Saigo operators of generalized fractional calculus associated with incomplete hypergeometric function, as the kernel, are presented here. These operators are general in nature and provided extension of the well–known operators of factional calculus which includes, the Riemann-Liouville, Weyl operators. Certain properties of these new operators associated with the incomplete generalized Wright function are established.

Keywords: Incomplete generalized Wright function, Wright function, fractional integrals and derivatives.

1 Introduction

Let $\alpha,\beta,\eta \in C$ and $x \in \Re_+$; then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function due to Saigo [1,2] are defined as follows

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)dt; \quad (\Re(\alpha)>0); \tag{1}$$

$$= \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n} f\right)(x); \quad (\Re(\alpha)\le 0; n=[\Re(-\alpha)]+1); \tag{2}$$

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t)dt; \quad (\Re(\alpha)>0); \tag{3}$$

$$= (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{\alpha+n,\beta-n,\eta} f\right)(x); \quad (\Re(\alpha)\le 0; \quad n=[\Re(-\alpha)]+1) \tag{4}$$

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+n} f\right)(x) = \frac{d^n}{dx^n}\left(I_{0,+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x); \quad (\Re(\alpha)>0; n=[\Re(\alpha)]+1) \tag{5}$$

$$\left(D_{0-}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+n} f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x); \quad (\Re(\alpha)>0; n=[\Re(\alpha)]+1). \tag{6}$$

Riemann-Liouville, Weyl and Erdelyi-Kober fractional Calculus operator follows as special cases of the operators $I_{0,+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ as shown below

$$\left(R_{0,x}^{\alpha} f\right)(x) = (I_{+}^{\alpha,-\alpha,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad (\Re(\alpha)>0); \tag{7}$$

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$$= \frac{d^n}{dx^n}\left(R_{0,x}^{\alpha+n} f\right)(x), \quad (0 < \Re(x) + n \leq 1; n = 1,2,...); \tag{8}$$

$$\left(w_{x,\infty}^{\alpha} f\right)(x) = \left(I_{-}^{\alpha,-\alpha,n} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)dt; \quad (\Re(\alpha) > 0); \tag{9}$$

$$= (-1)^n \frac{d^n}{dx^n}\left(w_{x,\infty}^{\alpha+n} f\right)(x); \quad (0 < \Re(\alpha) + n \leq 1; n = 1,2,...); \tag{10}$$

Now here the generalized hypergeometric Wright function ${}_p\psi_q(z)$ defined for $z \in C$, complex $a_i, b_j \in C$ and real $\alpha_i, \beta_j \in \Re = (-\infty,\infty)$ $(\alpha_i, \beta_j \neq 0; i = 1,2,...,p; j = 1,2,...q)$ by the series

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{i,p}\\((b_j,\beta_j)_{i,q}\end{array}\middle| z\right] = \sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}\Gamma(a_i+\alpha_i k)z^k}{\Pi_{j=1}^{q}\Gamma(b_j+\beta_j k)k!}. \tag{11}$$

This function was introduced by Wright [3] and is called the generalized hpergeometric Wright function conditions for the existence of (11) together with its representation in terms of the Mellin-Barnes integral and of the H-function were established in (12). In particular, ${}_p\psi_q(z)$ is an entire function if there holds the condition

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1. \tag{12}$$

The special case of the function (11) in the form

$$\phi(\delta,b;z) \equiv {}_0\psi_1\left[\begin{array}{c}\\(b,\delta)\end{array}\middle| z\right] = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\delta k+b)}\frac{z^k}{k!} \tag{13}$$

with complex $z, b \in C$ and real $\delta \in \Re$, known as the Wright function [[4], Section 18.1], was introduced by Wright in [5] when $b = v+1$ and z is replaced by $-z$, the function

$$\phi(\delta,v+1;-z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\delta k+v+1)}\frac{(-z)^k}{k!}$$

and such a function is known as the Bassel-Maitland function, or the Wright generalized Bessel function [6,7,8]. Now with the help of incomplete Pochhammer symbol which is introduced by Srivastava et al. [9] and depends as follows

$$(\lambda;x)_v = \frac{\gamma(\lambda+v;x)}{\Gamma(\lambda)}, \quad (\lambda, v \in C; x \geq 0)$$

$$[\lambda;x]_v = \frac{\Gamma(\lambda+v;x)}{\Gamma(\lambda)}, \quad (\lambda, v \in C; x \geq 0)$$

and these incomplete Pochhammer symbol satisfy the following decomposition relation

$$(\lambda;x)_v + [\lambda;x]_v = (\lambda)_v; \quad (\lambda, v \in C; x \geq 0) \tag{14}$$

and where incomplete gamma function $\gamma(s,x)$ and $\Gamma(s,x)$ defined by

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt; \quad (\Re(s) > 0; x \geq 0) \tag{15}$$

and

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt; \quad (\Re(s) > 0; x \geq 0). \tag{16}$$

D.K. Singh and S. Porwal [10] was introduced incomplete generalized hypergeometric Wright function

$${}_p\bar{\psi}_q(z) = {}_p\bar{\psi}_q\left[\begin{array}{c}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{array}\middle| z\right] = \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)\Gamma(a_2+\alpha_2 k)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)\Gamma(b_2+\beta_2 k)...\Gamma(b_q+\beta_q k)}\frac{z^k}{k!} \tag{17}$$

and

$$ {}_p\underline{\Psi}_q(z) = {}_p\underline{\Psi}_q\left[\begin{matrix}(a_1,\alpha_1;x)...(a_p,\alpha_p)\\(b_1,\beta_1;x)...(b_q,\beta_q)\end{matrix}\middle|z\right] = \sum_{k=0}^{\infty}\frac{\gamma(a_1+\alpha_1 k;x)\Gamma(a_2+\alpha_2 k)...\Gamma(a_p+\alpha_p k)}{\gamma(b_1+\beta_1 k;x)\Gamma(b_2+\beta_2 k)...\Gamma(b_q+\beta_q k)}\frac{z^k}{k!}. \tag{18} $$

Provided that the defining the infinite series in each case is absolutely convergent if satisfy the condition

$$ |(\lambda;x)_n| \le |(\lambda)_n| \text{ and } |[\lambda;x]_n| \le |(\lambda)_n|. \tag{19} $$

By decomposition formula (14), eqn (17) and (18) can be written in terms of generalized hypergeometric wright function (11).

2 Fractional integration of the incomplete generalized hypergeometric Wright function

Theorem 1. *Let $\alpha,\beta,\eta,\in C$ be such that $\Re(\alpha)>0,\Re(\beta)>0,\Re(\eta)>0$, $\Re(\gamma)>0$ and condition (19) is satisfied and let a and $b\in C$ and $\mu>0$. Then for $x>0$ there holds the formula*

$$ I_{0+}^{\alpha,\beta,\eta}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle|at^{\mu}\right]\right)(x) $$

$$ = x^{\gamma-\beta-1}{}_{p+2}\bar{\Psi}_{q+2}\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu),(\gamma+\eta-\beta,\mu)\\ [b_1,\beta_1;x]...b_q,\beta_q),(\gamma+\alpha+\eta,\mu),(\gamma-\beta,\mu)\end{matrix}\middle|ax^{\mu}\right]. \tag{20} $$

Proof. With (1) and (17), we have

$$ I_{0+}^{\alpha,\beta,\eta}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,(\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle|at^{\mu}\right]\right)(x) $$

$$ = x^{\gamma-\alpha-\beta}\int_0^x (x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-t/x)t^{\gamma-1} $$

$$ \times\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)}\frac{a^k t^{\mu k}}{k!}dt $$

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)}\frac{a^k}{k!}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} $$

$$ \times{}_2F_1(\alpha+\beta,-\eta;\alpha;1-t/x)t^{\mu k+\gamma-1}dt $$

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)}\frac{a^k}{k!}(I_{0+}^{\alpha,\beta,n}t^{\mu k+\gamma-1})(x). $$

Since $\Re(\mu k+\gamma)\ge\Re(\gamma)>0$ for any $k=0,1,2,...$ then using formula [1]

$$ (I_{0+}^{\alpha,\beta,\eta}t^{\gamma-1})(x) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta+\eta)}{\Gamma(\gamma+\beta)\Gamma(\gamma+\alpha+\eta)} \tag{21} $$

with γ being replaced by $\gamma+\mu k$, we obtain

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)}\frac{a^k}{k!}\times $$

$$ \times\frac{\Gamma(\mu k+\gamma)\Gamma(\gamma+\mu k-\beta+\eta)}{\Gamma(\gamma+\mu k-\beta)\Gamma(\gamma+\mu k+\alpha+\eta)}x^{\gamma+\mu k-\beta-1} $$

$$ = x^{\gamma-\beta-1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma b_1+\beta_1 K;x)...\Gamma(b_q+\beta_q k)}\frac{\Gamma(\gamma+\mu k)\Gamma(\gamma+\eta-\beta+\mu k)}{\Gamma(\gamma-\beta+\mu k)\Gamma(\gamma+\alpha+\eta+\mu k)}\frac{(ax^{\mu})^k}{k!} $$

$$ = x^{\gamma-\beta-1}{}_{p+2}\bar{\Psi}_{q+2}\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu),(\gamma+\eta-\beta,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma+\alpha+\eta,\mu),(\gamma-\beta,\mu)\end{matrix}\middle|ax^{\mu}\right]. $$

Thus the theorem is proved.

Corollary 1. *Let $\alpha,\gamma \in C$ be such that $\Re(\alpha)>0$, $\Re(\gamma)>0$, and let a and $b\in C$, and condition (19) is satisfied. Then for $\mu>0$ and $x>0$ there holds the formula*

$$\left(I_{0+}^{\alpha}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\right| at^{\mu}\right]\right)\right)(x).$$

$$= x^{\gamma+\alpha-1}{}_{p+1}\bar{\Psi}_{q+1}\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma+\alpha,\mu)\end{matrix}\right| ax^{\mu}\right]. \tag{22}$$

Proof. With (7) and (17), we have

$$\left(I_{0+}^{\alpha}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\right| at^{\mu}\right]\right)\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^kt^{\mu k+\gamma-1}}{k!}dt$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\mu k+\gamma-1}dt$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}(I_{0+}^{\alpha,-\alpha,\eta}t^{\mu k+\gamma-1})(x).$$

Since $\Re(\mu k+\gamma)\geq\Re(\gamma)>0$ for any $k=0,1,2,...$, then using the (21), we obtain

$$= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\frac{\Gamma(\gamma+\mu k)}{\Gamma(\gamma+\mu k+\alpha)}x^{\gamma+\mu k+\alpha-1}$$

$$= x^{\gamma+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)}{\Gamma(\gamma+\mu k+\alpha)}\frac{(ax^{\mu})^k}{k!}$$

$$= x^{\gamma+\alpha-1}{}_{p+1}\bar{\Psi}_{q+1}\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma+\alpha,\mu)\end{matrix}\right| ax^{\mu}\right].$$

Theorem 2. *Let $\alpha,\beta,\eta,\in C$ be such that $\Re(\alpha)>0,\Re(\beta)>0,\Re(\eta)>0$, $\Re(\gamma)>0$ and condition (19) is satisfied and let a and $b\in C$ and $\mu>0$. Then for $x>0$ there holds the formula*

$$I_{-}^{\alpha,\beta,\eta}\left(t^{-\gamma}{}_p\bar{\Psi}_q\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\right| at^{-\mu}\right]\right)(x)$$

$$= x^{-\gamma-\beta}{}_{p+2}\bar{\Psi}_{q+2}\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma+\beta,\mu),(\gamma+\eta,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\alpha+\eta+\gamma+\beta,\mu),(\gamma,\mu)\end{matrix}\right| at^{-\mu}\right]. \tag{23}$$

Proof. From (3) and (17), we get

$$I_{-}^{\alpha,\beta,\eta}\left(t^{-\gamma}{}_p\bar{\Psi}_q\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\right| at^{-\mu}\right]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\tfrac{x}{t})dt$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}$$

$$\times {}_2F_1(\alpha+\beta,\eta;\alpha;1-\tfrac{x}{t})t^{-\mu k-\gamma-\alpha-\beta}dt$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{a^k}{k!} (I_-^{\alpha,\beta,\eta} t^{-\gamma-\mu k})(x).$$

Since $\Re(\mu k+\gamma) \geq \Re(\gamma) > 0$ for any $k = 0,1,2,...$ then we apply the formula [1]

$$(I_-^{\alpha,\beta,\eta} t^{-\gamma})(x) = \frac{\Gamma(\beta+\gamma)\Gamma(\eta+\gamma)}{\Gamma(\gamma)\Gamma(\alpha+\beta+\eta+\gamma)} x^{-\beta-\gamma} \tag{24}$$

with γ being replaced by $\gamma+\mu k$, we obtain

$$= x^{-\gamma-\beta} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{\Gamma(\beta+\gamma+\mu k)}{\Gamma(\gamma+\mu k)}$$

$$\times \frac{\Gamma(\eta+\gamma+\mu k)}{\Gamma(\alpha+\beta+\eta+\gamma+\mu k)} \frac{(ax^{-\mu})^k}{k!}$$

$$= x^{-\gamma-\beta} {}_{p+2}\bar{\Psi}_{q+2} \left[\begin{array}{c} [a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma+\beta,\mu),(\gamma+\eta,\mu) \\ [b_1,\beta_1;x]...(b_q,\beta_q),(\alpha+\eta+\gamma+\beta,\mu),(\gamma,\mu) \end{array} \middle| at^{-\mu} \right].$$

Corollary 2.*Let $\alpha,\gamma \in C$ be such that $\Re(\alpha) > 0$, $\Re(\gamma) > 0$, and let a and $b \in C$,and condition (19) is satisfied. Then for $\mu > 0$ and $x > 0$ there holds the formula*

$$I_-^{\alpha}\left(t^{-\gamma}{}_p\bar{\Psi}_q \left[\begin{array}{c} [a_1,\alpha_1;x]...(a_p,\alpha_p) \\ [b_1,\beta_1;x]...(b_q,\beta_q) \end{array} \middle| at^{-\mu} \right]\right)(x)$$
$$= x^{\alpha-\gamma} {}_{p+1}\bar{\Psi}_{q+1} \left[\begin{array}{c} [a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma-\alpha,\mu) \\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma,\mu) \end{array} \middle| at^{-\mu} \right] (x>0). \tag{25}$$

*Proof.*With (9) and (17), we have

$$I_-^{\alpha}\left(t^{-\gamma}{}_p\bar{\Psi}_q \left[\begin{array}{c} [a_1,\alpha_1;x]...(a_p,\alpha_p) \\ [b_1,\beta_1;x]...(b_q,\beta_q) \end{array} \middle| at^{-\mu} \right]\right)(x)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{a^k}{k!} \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\mu k-\gamma} dt$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{a^k}{k!} (I_-^{\alpha,-\alpha,\eta} t^{-\mu k-\gamma})(x)$$

Since $\Re(\mu k-\gamma) \geq \Re(\mu k+\gamma) > 0$ for any $k = 0,1,2,...$, then by using (24), we achieved

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{a^k}{k!} \frac{\Gamma(-\alpha+\gamma+\mu k)\Gamma(\eta+\gamma+\mu k)}{\Gamma(\gamma+\mu k)\Gamma(\alpha-\alpha+\eta+\gamma+\mu k)} x^{\alpha-\gamma-\mu k}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{a^k}{k!} \frac{\Gamma(\gamma-\alpha+\mu k)}{\Gamma(\gamma+\mu k)} x^{\alpha-\gamma-\mu k}$$

$$= x^{\alpha-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{\Gamma(\gamma-\alpha+\mu k)}{\Gamma(\gamma+\mu k)} \frac{(ax^{-\mu})^k}{k!}$$

$$= x^{\alpha-\gamma} {}_{p+1}\bar{\Psi}_{q+1} \left[\begin{array}{c} [a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma-\alpha,\mu) \\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma,\mu \end{array} \middle| \lambda t^{-\mu} \right] (x>0).$$

which is required result.

3 Fractional differentiation of incomplete generalized hypergeometric Wright function

Theorem 3. *Let $\alpha,\beta,\eta,\in C$ be such that $\Re(\alpha)\geq 0$, $\Re(\gamma)>-\min[0,\Re(\alpha+\beta+\eta)]$, and condition (19) is satisfied and let a and $b\in C$ and $\mu>0$. Then for $x>0$ there holds the formula*

$$\left(D_{0+}^{\alpha,\beta,\eta}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)\right)(x)$$
$$=x^{\gamma+\beta-1}{}_{p+2}\bar{\Psi}_{q+2}\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu),(\alpha+\beta+\gamma+\eta,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma+\eta,\mu),(\gamma+\beta,\mu)\end{matrix}\middle| ax^{\mu}\right]. \quad (26)$$

Proof. With the help of (5), we get

$$\left(D_{0+}^{\alpha,\beta,\eta}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)\right)(x).$$
$$=\left(\frac{d}{dx}\right)^n\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)(x)$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\left(\frac{d}{dx}\right)^n\left(I^{-\alpha+n,-\beta-n,\alpha+\eta-n}t^{\mu k+\gamma-1}\right)(x).$$

Using formula (21), we obtain

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\left(\frac{d}{dx}\right)^n x^{\gamma+\beta+n+\mu k-1}\times$$
$$\times\frac{\Gamma(\gamma+\mu k)\Gamma(\alpha+\beta+\gamma+\eta+\mu k)}{\Gamma(\gamma+\beta+n+\mu k)\Gamma(\mu k+\gamma+\eta)}$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)\Gamma(\alpha+\beta+\gamma+\eta+\mu k)}{\Gamma(\gamma+\eta+\mu k)\Gamma(\gamma+\beta+n+\mu k)}\times$$
$$\times\frac{a^k}{k!}\left(\frac{d}{dx}\right)^n x^{\gamma+\beta+n+\mu k-1}$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)\Gamma(\alpha+\beta+\gamma+\eta+\mu k)}{\Gamma(\gamma+\eta+\mu k)\Gamma(\gamma+\beta+n+\mu k)}\frac{a^k}{k!}$$
$$\times\frac{\Gamma(\gamma+\beta+n+\mu k)}{\Gamma(\gamma+\beta+\mu k)}x^{\gamma+\beta+\mu k-1}$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)\Gamma(\alpha+\beta+\gamma+\eta+\mu k)}{\Gamma(\gamma+\eta+\mu k)}\times$$
$$\times\frac{1}{\Gamma(\gamma+\beta+\mu k)}x^{\gamma+\beta+\mu k-1}$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)...\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)...\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)\Gamma(\alpha+\beta+\gamma+\eta+\mu k)}{\Gamma(\gamma+\eta+\mu k)\Gamma(\gamma+\beta+\mu k)}\frac{a^k}{k!}x^{\gamma+\beta+\mu k-1}$$
$$=x^{\gamma+\beta-1}{}_{p+2}\bar{\Psi}_{q+2}\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu),(\alpha+\beta+\gamma+\eta,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma+\eta,\mu),(\gamma+\beta,\mu)\end{matrix}\middle| ax^{\mu}\right].$$

Corollary 3. *Let $\alpha,\gamma\in C$ be such that $\Re(\alpha)>0$, $\Re(\gamma)>0$, and let a and $b\in C$, and condition (19) is satisfied. Then for $\mu>0$ and $x>0$ there holds the formula*

$$\left(D_{0+}^{\alpha}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)\right)(x)$$
$$=x^{\gamma-\alpha-1}{}_{p+1}\bar{\Psi}_{q+1}\left[\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma-\alpha,\mu)\end{matrix}\middle| ax^{\mu}\right]. \quad (27)$$

Proof. Using (2) and (17), we get

$$\left(D_{0+}^{\alpha}\left(t^{\gamma-1}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)\right)(x)$$

$$D_{0+}^{\alpha,-\alpha,\eta}\left[t^{\gamma-1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^kt^{\mu k}}{k!}\right]$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\left(\frac{d}{dx}\right)^n I_{0+}^{-\alpha+n,\alpha-n,\alpha+\eta-n}t^{\mu k+\gamma-1}.$$

Using (21), we achieved

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\frac{\Gamma(\gamma+\mu k)\Gamma(\gamma+\mu k-\eta)}{\Gamma(\gamma+\mu k-\alpha+n)\Gamma(\gamma+\mu k+\eta)}$$

$$\times\left(\frac{d}{dx}\right)^n x^{\gamma+\mu k-\alpha+n-1}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\frac{\Gamma(\gamma+\mu k)\Gamma(\gamma+\mu k-\alpha+n)}{\Gamma(\gamma-\alpha+n+\mu k)\Gamma(\gamma+\mu k-\alpha)}x^{\gamma+\mu k-\alpha-1}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{\Gamma(\gamma+\mu k)}{\Gamma(\gamma-\alpha+\mu k)}x^{\gamma-\alpha+\mu k-1}\frac{a^k}{k!}$$

$$=x^{\gamma-\alpha-1}{}_{p+1}\bar{\Psi}_{q+1}\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p),(\gamma,\mu)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q),(\gamma-\alpha,\mu)\end{matrix}\middle| ax^{\mu}\right](x>0).$$

Theorem 4. *Let $\alpha,\beta,\eta,\in C$ be such that $\Re(\alpha)\geq 0$, $\Re(\gamma)>-\min[\Re(-\beta-n),\Re(\alpha+\eta)]$, and condition (19) is satisfied and let a and $b\in C$ and $\mu>0$. Then for $x>0$ there holds the formula*

$$\left(D_{-}^{\alpha,\beta,\eta}\left(t^{-\gamma}{}_p\Psi_q\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q)\end{matrix}\middle| at^{\mu}\right]\right)\right)(x)$$
$$=x^{\beta-\gamma}{}_{p+2}\bar{\Psi}_{q+2}\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p),(\alpha+\gamma+\eta,\mu),(\gamma-\beta,\mu)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q),(\gamma-\beta+\eta,\mu),(\gamma,\mu)\end{matrix}\middle| ax^{-\mu}\right]. \quad (28)$$

Proof. Using (6), we get

$$\left(D_{-}^{\alpha,\beta,\eta}\left(t^{-\gamma}{}_p\Psi_q\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q)\end{matrix}\middle| at^{-\mu}\right]\right)\right)(x)$$

$$\equiv\left(\frac{-d}{dx}\right)^n\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}{}_p\bar{\Psi}_q\left[\begin{matrix}[a_1,\alpha_1;x]\ldots(a_p,\alpha_p)\\ [b_1,\beta_1;x]\ldots(b_q,\beta_q)\end{matrix}\middle| at^{-\mu}\right]\right)(x).$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\left(\frac{-d}{dx}\right)^n(I^{-\alpha+n,-\beta-n,\alpha+\eta}t^{-\gamma-\mu k})(x).$$

Using formula (24), we achieved

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{a^k}{k!}\left(\frac{-d}{dx}\right)^n x^{\gamma-\beta+n-\mu k}$$

$$\times\frac{\Gamma(\beta-\gamma-n+\mu k)\Gamma(\alpha+\gamma+\eta+\mu k)}{\Gamma(\gamma+\mu k)\Gamma(\gamma-\beta+\eta+\mu k)}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1k;x)\ldots\Gamma(a_p+\alpha_pk)}{\Gamma(b_1+\beta_1k;x)\ldots\Gamma(b_q+\beta_qk)}\frac{\Gamma(\beta-\gamma-n+\mu k)\Gamma(\alpha+\gamma+\eta+\mu k)}{\Gamma(\gamma-\beta+\eta+\mu k)\Gamma(\gamma+\mu k)}$$

$$\times(-1)^n\frac{a^k}{k!}\left(\frac{d}{dx}\right)^n x^{\gamma-\beta+n-\mu k}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{\Gamma(\beta-\gamma-n+\mu k)\Gamma(\alpha+\gamma+\eta+\mu k)}{\Gamma(\gamma-\beta+\eta+\mu k)\Gamma(\gamma+\mu k)}$$
$$\times \frac{(-1)^n a^k}{k!} \frac{\Gamma(\beta-\gamma+n-\mu k+1)}{\Gamma(\beta-\gamma)} x^{\beta-\gamma-\mu k}$$
$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)} \frac{\Gamma(\beta-\gamma-n+\mu k)\Gamma(\alpha+\gamma+\eta+\mu k)}{\Gamma(\gamma-\beta-\eta+\mu k)\Gamma(\gamma+\mu k)}$$
$$\times \frac{(-1)^n a^k}{k!} \frac{\Gamma(\beta-\gamma+n-\mu k+1)}{\Gamma(\beta-\gamma-\mu k+1)} x^{\beta-\gamma-\mu k}. \tag{29}$$

By the reflection formula for the gamma function

$$\Gamma(\gamma-\beta-n+\mu k)\Gamma(\beta-\gamma+n-\mu k+1) = \frac{\pi}{\sin[(\gamma-\beta-n+\mu k)\pi]}$$
$$= \frac{\pi}{\sin(\gamma-\beta-n+\mu k)\pi\cos n\pi} = \frac{(-1)^n\pi}{\sin[(\gamma-\beta+\mu k)\pi]}$$
$$= (-1)\Gamma(\gamma-\beta+\mu k)\Gamma(1-\gamma+\beta-\mu k). \tag{30}$$

From (29) and (30), we obtain

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k;x)...\Gamma(a_p+\alpha_p k)\Gamma(\alpha+\gamma+\eta+\mu k)\Gamma(\gamma-\beta+\mu k)}{\Gamma(b_1+\beta_1 k;x)...\Gamma(b_q+\beta_q k)\Gamma(\gamma-\beta+\eta+\mu k)\Gamma(\gamma+\mu k)} \frac{a^k}{k!} x^{\beta-\gamma-\mu k}$$
$$= x^{\beta-\gamma} {}_{p+2}\bar{\Psi}_{q+2}\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\alpha+\gamma+\eta,\mu),(\gamma-\beta,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma-\beta+\eta,\mu),(\gamma,\mu)\end{matrix}\right| ax^{-\mu}\right].$$

Corollary 4. *Let $\alpha,\gamma\in C$ be such that $\Re(\alpha)>0$, $\Re(\gamma)>0$, and let a and $b\in C$, and condition (19) is satisfied. Then for $\mu>0$ and $x>0$ there holds the formula*

$$\left(D_{-}^{\alpha}\left(t^{-\gamma}{}_p\psi_q\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p)\\ [b_1,\beta_1;x]...(b_q,\beta_q)\end{matrix}\right| at^{\mu}\right]\right)\right)(x) = x^{-\alpha-\gamma}{}_{p+2}\bar{\Psi}_{q+2}\left[\left.\begin{matrix}[a_1,\alpha_1;x]...(a_p,\alpha_p),(\gamma+\alpha,\mu)\\ [b_1,\beta_1;x]...(b_q,\beta_q),(\gamma,\mu)\end{matrix}\right| ax^{-\mu}\right]. \tag{31}$$

Proof. Put $\beta=-\alpha$ in (28), we achieved our required result (31).

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