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Structure of Spectrum of Solvable Delay Differential Operators for First Order

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Abstract: By using the M.I. Vishik's method on the description of solvable extensions of a densely defined operator all solvable extensions of the minimal operator generated by some complete delay differential-operator expression for first order in the Hilbert space of vector-functions at finite interval are described. Later on, the structure of spectrum of these extensions is surveyed.

Keywords: Delay Differential Operators, Solvable Extension, Spectrum

1 Introduction

The first work in an area of extension of linear densely defined operator in a Hilbert space belongs to J. von Neumann. In his paper [1] all the selfadjoint extensions of the linear densely defined having equal and nonzero deficiency indexes symmetric operator in any Hilbert space have been described. But in 1949 and 1952 M.I. Vishik the boundedly (compact, regular and normal) invertible extensions of any unbounded linear operator in a Hilbert space have been established in works [2] and [3]. These results by M.O.Otelbayev, B. Kokebaev and A.N. Shynybekov have been generalized to the nonlinear operators and complete additive Hausdorff topological spaces in abstract terms in work [4,5,6,7]. A.A.Dezin [8] give a general methods for the description of regular extensions for some classes of linear differential operators in the Hilbert space of vector-functions at finite interval.

In 1985 by N.I. Pivtorak [9] and Z.I.Ismailov [10] all solvable extensions of a minimal operator generated by linear parabolic and hyperbolic type differential expressions for first order with selfadjoint operator coefficient in the Hilbert space of vector-functions at finite interval in terms of boundary values were given, respectively.

In the studies discussed above the coefficients of differential expressions have been taken for special classes of operators in corresponding functional space. Unfortunately, representation of delay type differential expression is not possible with remarkable coefficient, then mentioned above methods are not applicable to these problems. On the other hand in noted above works spectral investigations have not been done.

Note that the general theory of delay differential equations is given in many books (for example, see [11] and [12]). Applications of this theory can be found in economy, biology, control theory, electrodynamics, chemistry, ecology, epidemiology, tumor growth, neural networks and etc. (see [13,14,15]).

Let's remember that an operator $S : D(S) \subset H \to H$ in Hilbert space H is called solvable, if S is one-to-one, $SD(S)$=H and $S^{-1} \in L(H)$.

The main goal of this work is to describe all solvable extensions of the minimal operator generated by some delay differential expression for first order with operator coefficients in the Hilbert space of vector-functions at finite interval and investigate the structure of spectrum these extensions. Lastly, some applications will be given.

2 Description of Solvable Extensions

In the Hilbert space $L^2(H,(0,1))$ of vector-functions consider the following linear delay differential-operator expression for first order in the form

$$l(u) = \sum_{k=1}^{m} \alpha_k u'(\beta_k(t)) + \sum_{j=1}^{n} A_j(t)u(\gamma_j(t)), \qquad (1)$$

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where:
(1) H is a separable Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$; $\alpha_k \in \mathbb{C}, k=1,2,\dots,m$;
(2) operator-function $A_j(\cdot):[0,1]\to L(H),\ \ j=1,2,\dots,n$ is continuous on the uniformly operator topology;
(3) For $k=1,2,\dots,m,\ \ \beta_k:[0,1]\to[0,1]$ and $j=1,2,\dots,n,\ \gamma_j:[0,1]\to[0,1]$ are invertible. Moreover, let us

$$\beta_k,(\beta_k^{-1})',\gamma_j,(\gamma_j^{-1})'\in C[0,1]$$

Firstly will be considered the following differential expression

$$m(u)=u'(t) \tag{2}$$

in $L^2(H,(0,1))$.
It is clear that formally adjoint expression of (2) in $L^2(H,(0,1))$ is in form

$$m^+(v)=-v'(t) \tag{3}$$

On the dense in $L^2(H,(0,1))$ linear manifold of vector-functions D_0' :

$$D_0':=\Big\{u\in L^2(H,(0,1)):\ u(t)=\sum_{k=1}^{n}\varphi_k(t)f_k,$$
$$\varphi_k\in C_0^\infty(0,1),\ f_k\in H,k=1,2,\dots,n,n\in\mathbb{N}\Big\}$$

define a operator M_0' as:

$$M_0'u=m(u),\ u\in D_0'$$

By standard method the minimal $M_0(M_0^+)$ and maximal $M(M^+)$ operators corresponding to differential expression (2) ((3)) in $L^2(H,(0,1))$ can be defined. For any skaler function $\varphi:[0,1]\to[0,1]$ now define an operator P_φ in $L^2(H,(0,1))$ in form

$$P_\varphi u(t)=u(\varphi(t)),\ u\in L^2(H,(0,1))$$

If a function $\varphi\in C^1[0,1]$ and $\varphi'(t)>0$ for $t\in[0,1]$, then for any $u\in L^2(H,(0,1))$, it is obtained that

$$\begin{aligned}\|P_\varphi u\|^2_{L^2(H,(0,1))}&=\int_0^1\|u(\varphi(t))\|_H^2dt\\&=\int_{\varphi(0)}^{\varphi(1)}\|u(\varphi(x))\|_H^2(\varphi^{-1})'(x)dx\\&\le\left|\int_{\varphi(0)}^{\varphi(1)}\|u(x)\|_H^2|(\varphi^{-1})'(x)|dx\right|\\&\le\|(\varphi^{-1})'\|_\infty\int_0^1\|u(x)\|_H^2dx\\&=\|(\varphi^{-1})'\|_\infty\|u\|^2\end{aligned}$$

Consequently, $P_\varphi\in L(L^2(H,(0,1)))$ and $\|P_\varphi\|\le\sqrt{\|(\varphi^{-1})'\|_\infty}$. Note that in terms of the operator P_φ the differential expression $l(\cdot)$ can be written in form

$$l(u)=\sum_{k=1}^{m}(\alpha_kP_{\beta_k})u'(t)+\sum_{j=1}^{n}A_j(t)P_{\gamma_j}u(t)$$

On the other hand consider the following equation for any $f\in L^2(H,(0,1))$

$$P_\varphi u(t)=f(t),$$

i.e.

$$u(\varphi(t))=f(t)$$

From this it is obtained that

$$u(x)=\begin{cases}f(\varphi^{-1}(x))\ , if & x\in\varphi([0,1]),\\ 0 \qquad , if & x\in[0,1]\setminus\varphi([0,1])\end{cases}$$

and

$$\begin{aligned}\|u\|^2_{L^2(H,(0,1))}&=\int_{\varphi([0,1])}\|f(\varphi^{-1}(x))\|_H^2dx\\&=\int_{\varphi^{-1}(0)}^{\varphi^{-1}(1)}\|f(t)\|_H^2\varphi'(t)dt\\&\le\|\varphi'\|_\infty\left|\int_{\varphi^{-1}(0)}^{\varphi^{-1}(1)}\|f(t)\|_H^2dt\right|\\&\le\|\varphi\|_\infty\|f\|^2_{L^2(H,(0,1))}\end{aligned}$$

Hence

$$P_\varphi^{-1}u(t)=\begin{cases}u(\varphi^{-1}(t))\ , if & t\in\varphi([0,1]),\\ 0 \qquad , if & t\in[0,1]\setminus\varphi([0,1]),\end{cases}$$

$u\in L^2(H,(0,1)),\ P_\varphi^{-1}\in L(L^2(H,(0,1)))$ and

$$\|P_\varphi^{-1}\|\le\sqrt{\|(\varphi)'\|_\infty}$$

The differential expression $l(\cdot)$ can be written in the form

$$l(u)=P(\beta)u'(t)+P(A;\gamma)u(t),$$

where $P(\beta)=\sum\limits_{k=1}^{m}\alpha_kP_{\beta_k}$ and

$$P(A;\gamma)=\sum_{j=1}^{n}A_j(t)P_{\gamma_j}$$

Before of all prove the following assertion.
Lemma 2.1. If for some $q=1,2,\dots,m,\ \alpha_q\neq0$ satisfied the condition

$$\sum_{\substack{k=1\\k\neq q}}^{m}\left|\frac{\alpha_k}{\alpha_q}\right|(\|\beta_q'\|_\infty\|(\beta_k^{-1})'\|_\infty)<1,$$

then the operator $P(\beta) : L^2(H,(0,1)) \to L^2(H,(0,1))$ is solvable.

Proof. Indeed, in this case for $\alpha_q \neq 0$

$$P(\beta) = \alpha_q P_{\beta_q}\left(E + \sum_{\substack{k=1\\k\neq q}}^{m} \frac{\alpha_k}{\alpha_q} P_{\beta_q}^{-1} P_{\beta_k}\right)$$

On the other hand, since

$$\left\| \sum_{\substack{k=1\\k\neq q}}^{m} \frac{\alpha_k}{\alpha_q} P_{\beta_q}^{-1} P_{\beta_k} \right\| \leq \sum_{\substack{k=1\\k\neq q}}^{m} \left| \frac{\alpha_k}{\alpha_q} \right| \|P_{\beta_q}^{-1}\| \|P_{\beta_k}\|$$

$$\leq \sum_{\substack{k=1\\k\neq q}}^{m} \left| \frac{\alpha_k}{\alpha_q} \right| \sqrt{\|\beta_q'\|_\infty \|(\beta_k^{-1})'\|_\infty} < 1,$$

then by the important theorem on invertibility of operator theory it is implied that the operator $P(\beta)$ is boundedly invertible. $\square$.

Along of this paper it has been assumed that $\sum\limits_{k=1}^{m} |\alpha_k|^2 > 0$.

Hence now differential expression $l(\cdot)$ can be written in form again

$$l(\cdot) = P(\beta)k(\cdot),$$

where, $k(\cdot) = \frac{d}{dt} + P(A;\beta,\gamma)$ and $P(A;\beta,\gamma) = P^{-1}(\beta)P(A;\gamma)$.

In this situation the operator $P(A;\beta,\gamma) : L^2(H,(0,1)) \to L^2(H,(0,1))$ is a linear bounded operator. Throughout this work the following operators

$$K_0 := M_0 + P(A;\beta,\gamma), L_0 := P(\beta)K_0,$$

$$K_0(L_0) : \overset{o}{W}{}_2^1(H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

and

$$K := M + P(A;\beta,\gamma), L := P(\beta)K,$$

$$K(L) : W_2^1(H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

will be called the minimal and maximal operators corresponding to differential expression $k(\cdot)(l(\cdot))$ in $L^2(H,(0,1))$, respectively.

Now let $U(t,s), t,s \in [0,1]$, be a family of evolution operators corresponding to homogeneous differential equation

$$\begin{cases} U_t(t,s)f + P(A;\beta,\gamma)U(t,s)f = 0, t,s \in [0,1] \\ U(s,s)f = f, f \in H \end{cases}$$

The following assertion is true (see [16]).

Lemma 2.2. If $\widetilde{M}$, $\widetilde{K}$ and $\widetilde{L}$ are some extensions of minimal operators M_0, K_0 and L_0 in $L^2(H,(0,1))$ respectively, then

$$U^{-1}K_0U = M_0,\ M_0 \subset U^{-1}\widetilde{K}U = \widetilde{M} \subset M,\ U^{-1}KU = M,$$

$$L_0 = P(\beta)K_0,\ \widetilde{L} = P(\beta)\widetilde{K},\ L = P(\beta)K$$

In addition, if $\widetilde{M}$ is solvable extension of M_0 in $L^2(H,(0,1))$, then an extension $\widetilde{L}$ of minimal operator L_0 is solvable extension in $L^2(H,(0,1))$ and vice versa.

From this claim it is obtained that if $\widetilde{M}$ any solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$, then an operator

$$\widetilde{L} = P(\beta)(U\widetilde{M}U^{-1})$$

is the solvable extension of L_0 and contrary. In this case

$$\widetilde{M} = U^{-1}(P^{-1}(\beta)\widetilde{L})U$$

The validity of following claim is clear.

Lemma 2.3. $KerL_0 = \{0\}$ and $\overline{R(L_0)} \neq L^2(H,(0,1))$.

Using the M.I.Vishik's result in the theory of extension [3] and similarly to the works [10], [17] the following assertion can be easy to proved.

Theorem 2.4. Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,1))$ is generated by the differential-operator expression (1) and boundary condition

$$(K+E)u(0) = KU(0,1)u(1), \tag{4}$$

where $K \in L(H)$ and $E : H \to H$ is identity operator. The operator K is determined by the extension $\widetilde{L}$ uniquely, i.e. $\widetilde{L} = L_K$.

On the contrary, the restriction of the maximal operator L to the linear manifold of vector-functions satisfy the condition (4) for some bounded operator $K \in L(H)$ is a solvable extension of the minimal operator L_0 in $L^2(H,(0,1))$.

Corollary 2.5. If $\widetilde{L} = L_K$ is a solvable extension of L_0, then

$$\begin{aligned} L_K^{-1}f(t) &= exp(-P(A;\beta,\gamma))KU(0,1) \\ &\times \left(\int_0^1 exp(-\int_s^1 P(A;\beta,\gamma)dx)P^{-1}(\beta)f(s)ds\right) \\ &+ \int_0^t exp(-\int_s^t P(A;\beta,\gamma)dx)P^{-1}(\beta)f(s)ds \end{aligned}$$

Corollary 2.6. If $m = 1,\ \beta_1 = \beta,\ 0 < \beta < 1,\ n = 1, \gamma_1 = \gamma,\ 0 < \gamma < 1,\ \beta/\gamma < 1,\ A_1(t) = A$ and for any $u \in W_2^1(H,(0,1))$

$$(Au)(\gamma t) = Au(\gamma t),$$

then all solvable extension of the minimal operator L_0 are generated by following differential expression

$$l(u) = u'(\beta t) + A(t)u(\gamma t)$$

and condition

$$(K+E)u(0) = Kexp(AP(\gamma/\beta))u(1),$$

i.e.

$$(K+E)u(0)=K\left(\sum_{n=0}^{\infty}\frac{A^n}{n!}u\left((\frac{\gamma}{\beta})^n\right)\right)$$

Corollary 2.7. All solvable extensions of the minimal operator L_0 generated by following pantograph type differential expression for first order

$$l(u)=u'(\beta t)+u(\gamma_1 t)+u(\gamma_2 t)$$

$0<\beta<1,\ 0<\gamma_1,\ \gamma_2<1,\ \dfrac{\gamma_1}{\beta}<1,\ \dfrac{\gamma_2}{\beta}<1$ in $L^2(H,(0,1))$ are described by condition

$$(K+E)u(0)=K\sum_{n=0}^{\infty}\frac{\left(P(\frac{\gamma_1}{\beta})+P(\frac{\gamma_2}{\beta})\right)^n}{n!}u(1),$$

where $K\in L(H)$.

3 Spectrum of Solvable Extensions

In this section the structure of spectrum of solvable extensions of the minimal operator L_0 in $L^2(H,(0,1))$ will be investigated. Here for the simplicity of the explanation it will be considered the following differential-operator expression in form

$$l(u)=u'(\beta(t))+A(t)u(\gamma(t)) \tag{5}$$

with conditions in sec.2 (see p.2).

Firstly, prove the following assertion.

Lemma 3.1. For $f\in H$ and $\lambda\in\mathbb{C}$ it is true

$$exp(\lambda P_\beta)f=exp(\lambda)f$$

Proof. Indeed, in this case

$$exp(\lambda P_\beta)f=\sum_{n=0}^{\infty}\frac{(\lambda P_\beta)^n}{n!}f,\ f\in H.$$

On the other hand, since for any $n=1,2,\dots$

$$(P_\beta)^n f=f,$$

then

$$exp(\lambda P_\beta)f=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f=exp(\lambda)f$$

□.

Now can be proved the proposition on the spectrum of solvable extension of L_0.

Theorem 3.2. If L_K is a solvable extension of the minimal operator L_0 in the space $L^2(H,(0,1))$, then spectrum of L_K is in form

$$\sigma(L_K)=\{\lambda\in\mathbb{C}:\lambda=\ln|\frac{\mu+1}{\mu}|+i\arg(\frac{\mu+1}{\mu})+2n\pi i,\ \mu\in\sigma(K)\setminus\{0,-1\},n\in\mathbb{Z}\}$$

Proof. Consider a problem to the spectrum for a solvable extension L_K of the minimal operator L_0 generated by pantograph type delay differential-operator expression (5), that is,

$$L_K u=P_\beta u'(t)+A(t)P_\gamma u(t)=\lambda u(t)+f(t),$$

$$(K+E)u(0)=KU(0,1)u(1),\ \lambda\in\mathbb{C},\ f\in L^2(H,(0,1))$$

This problem is equivalent to following problem

$$\begin{cases} u'(t)=(\lambda P_\beta^{-1}-P_\beta^{-1}A(t)P_\gamma)u(t)+P_\beta^{-1}f(t), \\ (K+E)u(0)=KU(0,1)u(1) \end{cases}$$

It is evident that

$$\begin{aligned} u(t,\lambda)&=exp(\lambda P_\beta^{-1})U(t,0)f_0\\ &+\int_0^t exp(\lambda P_\beta^{-1}(t-s))U(t,s)P_\beta^{-1}f(s)ds,\ f_0\in H \end{aligned}$$

On the other hand from boundary condition we have

$$\begin{aligned} (K+E)f_0&=KU(0,1)\{exp(\lambda P_\beta^{-1})U(1,0)f_0\\ &+\int_0^1 exp(\lambda P_\beta^{-1}(1-s))U(1,s)P_\beta^{-1}f(s)ds\} \end{aligned}$$

From this

$$\begin{aligned} &(K(E-exp(\lambda P_\beta^{-1}))+E)f_0\\ &=\int_0^1 exp(\lambda P_\beta^{-1}(1-s))U(1,s)P_\beta^{-1}f(s)ds \end{aligned}$$

By Lemma 3.1 it is clear that

$$\begin{aligned} exp(\lambda P_\beta^{-1})f_0&=\sum_{n=0}^{\infty}\frac{(\lambda P_\beta^{-1})^n}{n!}f_0\\ &=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(P_\beta^{-1})^n f_0\\ &=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f_0\\ &=exp(\lambda)f_0. \end{aligned}$$

Consequently,

$$\begin{aligned} &(K(1-exp(\lambda))+E)f_0\\ &=\int_0^1\left[exp(\int_s^1(\lambda P_\beta^{-1}-P_\beta^{-1}A(x)P_\alpha)dx)\right]P_\beta^{-1}f(s)ds,\\ &f_0\in H,\ f\in L^2(H,(0,1)) \end{aligned}$$

If $\lambda_m=2m\pi,\ m\in\mathbb{Z}$, then from last equation the unknown element f_0 can be found uniquely and the resolvent operator L_K for such $\lambda_m,\ m\in\mathbb{Z}$ is bounded $L^2(H,(0,1))$. Now assume that, $\lambda_m\neq 2m\pi,\ m\in\mathbb{Z}$. Then from last equation

$$(K - \frac{1}{exp(\lambda)-1}) f_0$$
$$= \frac{1}{1-exp(\lambda)} \int_0^1 exp(\lambda P_\beta^{-1}(1-s))U(1,s)P_\beta^{-1} f(s)ds,$$
$$f_0 \in H, f \in L^2(H,(0,1))$$

Therefore, in order to $\lambda \in \sigma(L_K)$ if and only if $\mu = \frac{1}{exp(\lambda)-1} \in \sigma(K)$. In this case since $\mu \notin \{0,-1\}$, then $\exp(\lambda) = \frac{\mu+1}{\mu}$, $\mu \in \sigma(K)$.
Hence

$$\lambda_n = \ln|\frac{\mu+1}{\mu}| + i\arg(\frac{\mu+1}{\mu}) + 2n\pi i,\ n \in \mathbb{Z}.$$

□.

4 Applications

Example 4.1. Consider the following boundary value problems for the pantograph type delay differential equation

$$\begin{cases} u'(x) = a(x)u(qx),\ 0 < x \le T, \\ u(0) = u_0,\ 0 < q < 1,\ a \in C^1[0,T] \end{cases}$$

In order to solve this problem change the unknown function $u(t)$ by

$$y(x) = u(x) - u_0,\ 0 < t \le T$$

Hence the considered problem is transforms the following problem

$$\begin{cases} y'(x) = a(x)y(\alpha x) + u_0 a(x),\ 0 < x \le T, \\ y(0) = 0 \end{cases}$$

that is

$$\begin{cases} y'(x) - a(x)y(x) = a(x)u_0,\ 0 < x \le T, \\ y(0) = 0 \end{cases}$$

The last problem can be written in a form

$$\begin{cases} l(y) = y'(x) - a(x)y(x) - a(x)u_0, \\ y(0) = 0 \end{cases}$$

Then solution of the above Cauchy problem by Corollary 2.5 can be analytically expressed in the following form

$$y(t) = L_c^{-1}(a(x)u_0) = \int_0^t U(t,s)a(s)dsu_0$$

Note that another approach to solve this problem has been applied in [18].

Example 4.2. Now consider to solvability of problem in a form

$$\begin{cases} y'(t) = ay(t) + by(qt) + f(t),\ 0 < t \le 1, \\ y(0) = y_0,\ 0 < q < 1 \end{cases}$$

Changing the unknown function $y(t)$ by

$$u(t) = y(t) - y_0,\ 0 < t < 1$$

the last boundary value problem can be written in a form

$$\begin{cases} u'(t) = (aE + bP_q)u(t) + (ay_0 + by_0 + f(t)), \\ u(0) = 0 \end{cases}$$

The solution of last Cauchy problem can be analytically written in form

$$u(t) = \int_0^t U(t,s)(f(s) + ay_0 + by_0)ds,$$

that is,

$$y(t) = y_0 + \int_0^t U(t,s)(f(s) + ay_0 + by_0)ds$$

Another approach to solve this problem has been applied in [19] and [20].
Example 4.3. Now consider boundary value problem in a form

$$\begin{cases} u'(1-t) = a(t)u(\alpha t) + b(t),\ 0 < t \le 1, \\ u(0) = u_0,\ a \in C[0,1],\ 0 < \alpha < 1 \end{cases}$$

It is clear that this problem is equivalent to next Cauchy problem

$$\begin{cases} y'(1-t) - a(t)y(\alpha t) = b(t) + a(t)u_0, \\ y(0) = 0,\ \text{where } y(t) = u(t) - u_0, \end{cases}$$

that is,

$$\begin{cases} y'(t) + (-P_1^{-1}a(t)P_2)y(t) = P_1^{-1}(b(t) + a(t)u_0), \\ y(0) = 0,\ \text{where } P_1 y(t) = y(1-t),\ P_2 y(t) = y(\alpha t) \end{cases}$$

From this it is obtained that

$$u(t) = \int_0^t U(t,s)(b(1-s) + a(1-s)u_0)ds + u_0$$

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