Appl. Math. Inf. Sci. **8**, No. 4, 1575-1582 (2014)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080411

Numerical Solution and Exponential Decay to Von Kármán System with Frictional Damping

D. C. Pereira[1], *C. A. Raposo*[2] *and J. A. J. Avila*[2,*]

[1] Department of Mathematics, Universidade do Estado do Pará, 66050-540, Belém, PA, Brazil
[2] Department of Mathematics and Statistics, Universidade Federal de São João del Rei, 36307-904, São João del Rei, MG, Brazil

Received: 12 Jul. 2013, Revised: 14 Oct. 2013, Accepted: 15 Oct. 2013
Published online: 1 Jul. 2014

Abstract: In this work we consider the Von Kármán system with frictional damping acting on the displacement and using the Method of Nakao we prove the exponential decay of the solution. The numerical scheme is presented for calculate the solution and to verify the long-time decay energy.

Keywords: Von Kármán system, Method of Nakao, Decay of solutions, Numerical solution, Finite differences method.

1 Introduction

For several years the system of Theodor von Kármán [19] was studied in different situations and methods.

The exponential decay of the energy to the von Kármán equations with memory in noncylindrical domains was studied by Park and Kang [17] in 2009 using the same method as in [18].

To the models of von Kármán taking into account for rotational forces, Bradley and Lasiecka [6] in 1994 showed the uniform decay rates for the solutions in cylindrical domain

The uniform decay of the solution was considered for frictional dissipative at the boundary, for example, in the works of Horn and Lasiecka [8] in 1994, Horn and Lasiecka [9] in 1995, and Horn, Favini, Lasiecka and Tataru [7] in 1996.

Applying multipliers method Avalos and Lasiecka [3] in 1987 showed the uniform decay for two-dimensional linear thermoelastic plates and Avalos and Lasiecka [2] in 1998 the one-dimensional thermoelastic von Kármán model was studied.

For thermal damping Menzala and Zuazua [10] in 1998 proved the exponential decay by the semigroup properties.

For Viscoelastic plates with memory, using energy method, we cite Rivera and Menzala [15] in 1999, and the work of Rivera, Oquendo and Santos [16] in 2005 where was proved that the energy decays uniformly, exponentially or algebraically with the same rate of decay of the relaxation function.

Based on multipliers method, the exponential decay of solution for the full von Kármán System of Dynamic Thermoelasticity was proved by Benabdallah and Lasiecka [4] in 2000.

For von Kármán System with memory Raposo and Santos [13] in 2011 obtained the General Decay of solution using the idea of Messaoudi [11] in the study of the asymptotic behavior of viscoelastic equations.

For the numerical scheme we mention for example Reinhart [14] in 1982 where was studied the approximation of the von Kármán equations stationary by the mixed finite element. The work of Yosibash, Kirby and Gottlieb [20] in 2004 where was studied the von Kármán system over rectangular domains and numerically solved using both the Chebyshev-collocation and Legendre-collocation methods for the spacial discretization and the implicit Newmark-β scheme combined with a non-linear fixed point algorithm for the temporal discretization, and Bilbao [5] in 2007 used numerical stability for numerical methods for the von Kármán system, through the use of energy-conserving methods.

What distinguishes this paper from other related works is that we apply the Method of Nakao in the von Kármán system to prove the exponential decay of the solution and we present an numerical scheme by finite

* Corresponding author e-mail: avila_jaj@ufsj.edu.br

differences method to numerical solution and the long-time decay energy, in this sense, there is few result in the literature.

The remainder of this paper is organized as follows. In section 2 we present the result of existence of weak solution, in the section 3 we prove the exponential decay of the solution, in the section 4 we applied the Finite Difference Method in the von Kármán system and finally in the section 5 we give the conclusion.

2 Existence of solution

We use the standard Lebesgue space and Sobolev Space with their usual properties as in Adams (1975) [1] and in this sense $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the inner product in L^2 and H_0^1 respectively. By $|\cdot|$ we denote the usual norm in L^2. Let $\Omega\subset\mathbb{R}^2$ be a bounded domain of the plane with regular boundary Γ. For a real number $T>0$ we denote $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. Here $u=u(x,y,t)$ is the displacement, $v=v(x,y,t)$ the Airy stress function and η is the unit normal external in Ω and $u'=u_t$. With this notation we have the following system

$$u''-\Delta^2u-[u,v]+u'=f \quad \text{in} \quad Q, \tag{1}$$
$$\Delta^2v+[u,u]=0 \quad \text{in} \quad Q, \tag{2}$$

and

$$u(0)=u_0, \quad u'(0)=u_1 \quad \text{in} \quad \Omega, \tag{3}$$
$$u=v=0, \quad \frac{\partial u}{\partial\eta}=\frac{\partial v}{\partial\eta}=0 \quad \text{on} \quad \Sigma, \tag{4}$$

where

$$[u,v]=\frac{\partial^2u}{\partial x^2}\frac{\partial^2v}{\partial y^2}-2\frac{\partial^2u}{\partial x\partial y}\frac{\partial^2v}{\partial x\partial y}+\frac{\partial^2u}{\partial y^2}\frac{\partial^2v}{\partial x^2}$$

Now using the same idea as in [10] we have the following result of existence of solution.

Theorem 1. *For $u_0\in H_0^2(\Omega)$, $u_1\in L^2(\Omega)$ and $f\in L^2_{loc}(\mathbb{R}^+;L^2(\Omega))$ there exists $u,v:Q\to\mathbb{R}$ such that*

$$u,v\in L^\infty(0,T;H_0^2(\Omega)), \quad u'\in L^\infty(0,T;L^2(\Omega)),$$

and u, v weak solution (1)-(4).

3 Asymptotic behavior

In this section, we will use the Method of Nakao (1978) [12] to prove the exponential decay of the solution. First we define

$$E(t)=|u'(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2 \tag{5}$$

Lemma 1. *The functional of energy $E(t)$ is limited.*

Proof. Multiplying (1) by u' and integrating in Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left[|u'(t)|^2+|\Delta u(t)|^2\right]-\langle[u(t),v(t)],u'(t)\rangle + |u'(t)|^2=(f(t),u'(t))$$

Using (2) we obtain

$$\begin{aligned}\langle[u(t),v(t)],u'(t)\rangle &= \langle[u(t),u'(t)],v(t)\rangle\\ &=\frac{1}{2}\langle\frac{d}{dt}[u(t),u(t)],v(t)\rangle\\ &\quad-\frac{1}{2}\langle\Delta^2v'(t),v(t)\rangle\\ &=\frac{1}{4}\frac{d}{dt}|\Delta v(t)|^2\end{aligned}$$

from where follows

$$\frac{d}{dt}\left[|u'(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2\right]+ 2|u'(t)|^2=2(f(t),u'(t)) \tag{6}$$

Performing integration from 0 to t follows by Cauchy-Schwarz inequality we obtain

$$|u'(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2+2\int_0^t|u'(s)|^2ds\leq \int_0^t|u'(s)|^2ds+\int_0^t|f(s)|^2ds+|u_1|^2+|\Delta u_0|^2+\frac{1}{2}|\Delta v_0|^2$$

then

$$E(t)+\int_0^t|u'(s)|^2ds\leq E(0)+\int_0^t|f(s)|^2ds$$

from where follows $E(t)\leq C$ with C constant independently of t.

Now we introduce a new functional.

Lemma 2. *The functional*

$$F^2(t)=E(t)-E(t+1)+\int_t^{t+1}|f(s)|^2ds$$

satisfies

$$\int_t^{t+1}|u'(s)|^2ds\leq F^2(t)$$

Proof. Integrating (6) from τ_1 to τ_2 with $0<\tau_1<\tau_2$, we obtain

$$E(\tau_2)+2\int_{\tau_1}^{\tau_2}|u'(s)|^2ds=E(\tau_1) + 2\int_{\tau_1}^{\tau_2}(f(s),u'(s))ds \tag{7}$$

and for all $t>0$

$$\begin{aligned}E(t+1)+2\int_t^{t+1}|u'(s)|^2ds&=E(t)+2\int_t^{t+1}(f(s),u'(s))ds\\ &\leq E(t)+\int_t^{t+1}|f(s)|^2ds+\int_t^{t+1}|u'(s)|^2ds\end{aligned}$$

then

$$\int_t^{t+1} |u'(s)|^2 ds \le E(t) - E(t+1) + \int_t^{t+1} |f(s)|^2 ds = F^2(t) \qquad (8)$$

Lemma 3. *The functional*

$$G^2(t) = 8C \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta u(s)| F(t) + 2(1+C^2)\int_{t_1}^{t_2} |u'(t)|^2 dt + 2C^2 \int_{t_1}^{t_2} |f(t)|^2 dt$$

satisfies

$$\int_{t_1}^{t_2} \left(|\Delta u(t)|^2 + \frac{1}{2}|\Delta v(t)|^2 \right) dt \le G^2(t)$$

Proof. First we note that

$$\langle [u(t), v(t)], u(t)\rangle = \langle [u(t), u(t)], v(t)\rangle = -\langle \Delta^2 v(t), v(t)\rangle = -|\Delta v(t)|^2 \qquad (9)$$

From (8) there exists $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$|u'(t_i)| \le 2F(t), \quad i = 1,2 \qquad (10)$$

Multiplying (1) by u and integrating in Ω, we have

$$\frac{d}{dt}(u'(t), u(t)) - |u'(t)|^2 + |\Delta u(t)|^2 - \langle [u(t), v(t)], u(t)\rangle + (u'(t), u(t)) = (f(t), u(t))$$

Performing integration from t_1 to t_2 and using (9) we have

$$\int_{t_1}^{t_2} \left(|\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt = (u'(t_1), u(t_1)) - (u'(t_2), u(t_2)) + \int_{t_1}^{t_2} \left(|u'(t)|^2 - (u'(t), u(t)) \right) dt + \int_{t_1}^{t_2} (f(t), u(t)) dt$$

Now, choosing C such that $|u| \le C|\Delta u|$ and applying Cauchy-Schwarz inequality we get

$$\int_{t_1}^{t_2} \left(\frac{1}{2}|\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt \le C \operatorname*{ess\,sup}_{s\in[t,t+1]} \left(|\Delta u(s)| (|u'(t_1)| + |u'(t_2)|) \right) + (1+C^2)\int_{t_1}^{t_2} |u'(t)|^2 dt + C^2 \int_{t_1}^{t_2} |f(t)|^2 dt,$$

using (10),

$$\int_{t_1}^{t_2} \left(\frac{1}{2}|\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt \le 8C \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta u(s)| F(t) + 2(1+C^2)\int_{t_1}^{t_2} |u'(t)|^2 dt + 2C^2 \int_{t_1}^{t_2} |f(t)|^2 dt,$$

and then

$$\int_{t_1}^{t_2} \left(|\Delta u(t)|^2 + \frac{1}{2}|\Delta v(t)|^2 \right) dt \le G^2(t) \qquad (11)$$

Theorem 2. *For $f \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$ with $\int_0^t |f(s)| ds \le \alpha_1 e^{-\alpha_2 t}$, for all $t \ge 1$ and $\alpha_1, \alpha_2 > 0$, then the solution (u, v) satisfies*

$$|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2}|\Delta v(t)|^2 + \int_t^{t+1} |u'(s)|^2 ds \le k_1 e^{-k_2 t}, \qquad (12)$$

for almost every $t \ge 1$, with $k_1, k_2 > 0$, constants independently from t.

Proof. From (8) and (11) we concludes

$$\int_{t_1}^{t_2} \left(|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2}|\Delta v(t)|^2 \right) dt \le F^2(t) + G^2(t)$$

There is $t^* \in [t_1, t_2]$ such that

$$E(t^*) = |u'(t^*)|^2 + |\Delta u(t^*)|^2 + \frac{1}{2}|\Delta v(t^*)|^2 \le 2(F^2(t) + G^2(t)) \qquad (13)$$

From (7) we get

$$E(t_1) = E(t^*) + 2\int_{t_1}^{t^*} |u'(s)|^2 ds - 2\int_{t_1}^{t^*} (f(s), u'(s)) ds$$

Then

$$E(t) = E(t^*) + 2\int_t^{t^*} |u'(s)|^2 ds - 2\int_t^{t^*} (f(s), u'(s)) ds \le E(t^*) + 3\int_t^{t+1} |u'(s)|^2 ds + \int_t^{t+1} |f(s)|^2 ds,$$

and

$$\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) \le E(t^*) + 3\int_t^{t+1} |u'(s)|^2 ds + \int_t^{t+1} |f(s)|^2 ds$$

Now using (8) and (13) we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) &\le 2(F^2(t) + G^2(t)) + 3F^2(t) + \int_t^{t+1} |f(s)|^2 ds \\
&\le 5F^2(t) + 16C \operatorname*{ess\,sup}_{s\in[t,t+1]} |\Delta u(s)| F(t) \\
&\quad + 4(1+C^2)\int_t^{t+1} |u'(s)|^2 ds \\
&\quad + (1+4C^2)\int_t^{t+1} |f(s)|^2 ds \\
&\le (9+4C^2)F^2(t) + \frac{1}{2}\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) \\
&\quad + 128C^2F^2(t) + (1+4C^2)\int_t^{t+1} |f(s)|^2 ds,
\end{aligned}$$

from where follows

$$\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) \leq (274+8C^2)F^2(t)+(2+8C^2)\int_t^{t+1}|f(s)|^2ds,$$

and then

$$E(t) \leq C_1(E(t)-E(t+1))+C_2\int_t^{t+1}|f(s)|^2ds,$$

where C_i, $i=1,2$, constants independently from t.

Without lost of generality, we can suppose $C_1>1$ and for $0<\beta=\frac{1}{C_1}<1$ we, have

$$E(t+1) \leq (1-\beta)E(t)+\beta C_2\int_t^{t+1}|f(s)|^2ds$$

For $t\geq 1$ and $n\in\mathbf{N}$ such that $n\leq t\leq n+1$

$$\begin{aligned} E(t) &\leq (1-\beta)E(t-1)+\beta C_2\int_{t-1}^{t}|f(s)|^2ds \\ &\leq (1-\beta)^nE(t-n)+\beta C_2\int_{t-n}^{t}|f(s)|^2ds \end{aligned}$$

Now

$$(1-\beta)^{n+1}<(1-\beta)^t \quad \text{implies} \quad (1-\beta)^n<(1-\beta)^{t-1}$$

Then

$$\begin{aligned} E(t) &\leq (1-\beta)^{t-1}\operatorname*{ess\,sup}_{s\in[0,1]}E(s)+\beta C_2\int_0^t|f(s)|^2ds \\ &= \frac{m_0}{1-\beta}(1-\beta)^t+\beta C_2\int_0^t|f(s)|^2ds, \end{aligned}$$

with $m_0=\operatorname*{ess\,sup}_{s\in[0,1]}E(s)<\infty$.

Now we have

$$\begin{aligned} E(t) &< \frac{m_0}{1-\beta}e^{t\ln(1-\beta)}+\beta C_2\int_0^t|f(s)|^2ds \\ &< \frac{m_0}{1-\beta}e^{-\beta_1 t}+\beta C_2\alpha_1e^{-\alpha_2 t}, \end{aligned}$$

for almost every $t\geq 1$, with $\beta_1=-\ln(1-\beta)>0$ and then

$$|u'(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2<\gamma_1e^{-\gamma_2 t} \tag{14}$$

From (8), we have

$$\int_t^{t+1}|u'(s)|^2ds \leq E(t)+\int_t^{t+1}|f(s)|^2<\gamma_3e^{-\gamma_4 t} \tag{15}$$

with $\gamma_i>0$ constants. Finally we concludes from (14) and (15) that there is constants $k_1,k_2>0$ such that

$$|u'(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2+\int_t^{t+1}|u'(s)|^2ds<k_1e^{-k_2 t}$$

This completes the proof.

4 Numerical solution

For a given small constant $\varepsilon>0$ we define a thin plate by

$$\Omega\times(-\varepsilon,\varepsilon)=\{(x,y,z)\in\mathbb{R}^3:(x,y)\in\Omega,\ z\in(-\varepsilon,\varepsilon)\}$$

whose midsurface is identified with Ω.
We resolve the von Kármán system in a square thin elastic plate by Finite Difference Method, subjected to a perpendicular load f and boundary condition of clamped type.

4.1 Discrete formulation

Consider the discrete domain the midsurface of the square plate, $\Omega_h=(0,\pi)^2$ with uniform grid $x_i=ih, y_j=jh, i,j=0,\dots,N+1, h=\pi/(N+1)$. The internal point are $x_i=ih, y_j=jh, 1\leq i,j\leq N$. The boundary of Ω_h is denoted Γ_h. The temporal discretization of interval $I_k=(0,T)$ is given by $t_n=nk$, $n=0,\dots,M+1$, $k=C_0h/2$ where C_0 is a positive constant and $T=k(M+1)$. Denote by $u^n_{i,j}$ and $v^n_{i,j}$ the functions u and v evaluate in the point (x_i,y_j) and at the instant t_n, respectively. It also, denoted by $Q^k_h=\Omega_h\times I_k$ and $\Sigma^k_h=\Gamma_h\times I_k$.
We show in Figure 1 the pattern mesh of Ω with its points: internal (circles), boundary (squares) and ghost (diamonds). We define the following discrete differential

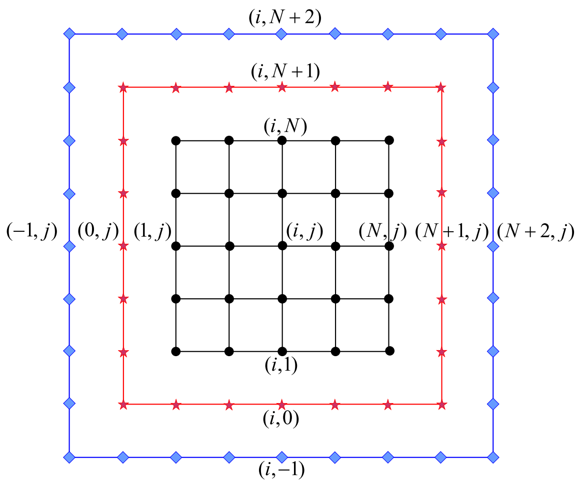


Fig. 1: The pattern mesh of Ω with internal, boundary and ghost points.

operators:

$$\begin{aligned} \delta_t u^n_{i,j} &= \frac{1}{k}(u^n_{i,j}-u^{n-1}_{i,j}), \\ \delta^2_t u^n_{i,j} &= \frac{1}{k^2}(u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}) \end{aligned} \tag{16}$$

$$\delta_x^2 u_{i,j}^n = \frac{1}{h^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n),$$
$$\delta_y^2 u_{i,j}^n = \frac{1}{h^2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) \tag{17}$$

$$\delta_{xy} u_{i,j}^n = \delta_x(\delta_y u_{i,j}^n) = \frac{1}{4h^2}(u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n) \tag{18}$$

$$\delta_x^4 u_{i,j}^n = \delta_x^2(\delta_x^2 u_{i,j}^n) = \frac{1}{h^4}(u_{i+2,j}^n - 4u_{i+1,j}^n + 6u_{i,j}^n - 4u_{i-1,j}^n + u_{i-2,j}^n) \tag{19}$$

$$\delta_y^4 u_{i,j}^n = \delta_y^2(\delta_y^2 u_{i,j}^n) = \frac{1}{h^4}(u_{i,j+2}^n - 4u_{i,j+1}^n + 6u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n) \tag{20}$$

$$\delta_{xy}^2 u_{i,j}^n = \delta_x^2(\delta_y^2 u_{i,j}^n) = \frac{1}{h^4}(u_{i+1,j+1}^n - 2u_{i+1,j}^n + u_{i+1,j-1}^n - 2u_{i,j+1}^n + 4u_{i,j}^n - 2u_{i,j-1}^n + u_{i-1,j+1}^n - 2u_{i-1,j}^n + u_{i-1,j-1}^n) \tag{21}$$

The discrete biharmonic operator is given by

$$\Delta_h^2 u_{i,j}^n = \delta_x^4 u_{i,j}^n + \delta_y^4 u_{i,j}^n + 2\delta_{xy}^2 u_{i,j}^n \tag{22}$$

The discrete bracket operator is given by

$$[u_{i,j}^n, v_{i,j}^n]_h = \delta_x^2 u_{i,j}^n\, \delta_y^2 v_{i,j}^n + \delta_y^2 u_{i,j}^n\, \delta_x^2 v_{i,j}^n - 2\,\delta_{xy} u_{i,j}^n\, \delta_{xy} v_{i,j}^n \tag{23}$$

Using the equations (16) to (23) we obtain the discrete model of von Kármán system,

$$\delta_t^2 u_{i,j}^n - a_1^2 \Delta_h^2 u_{i,j}^n - a_2^2 [u_{i,j}^n, v_{i,j}^n]_h + a_3^2 \delta_t u_{i,j}^n = f_{i,j}^n \quad \text{in} \quad Q_h^k, \tag{24}$$
$$\Delta_h^2 v_{i,j}^n + [u_{i,j}^n, u_{i,j}^n]_h = g_{i,j}^n \quad \text{in} \quad Q_h^k, \tag{25}$$

$$u_{i,j}^0 = (u_0)_{i,j},\ \delta_t u_{i,j}^0 = (u_1)_{i,j} \quad \text{in} \quad \Omega_h \tag{26}$$
$$u_{i,j}^n = v_{i,j}^n = 0 \quad \text{on} \quad \Sigma_h^k, \tag{27}$$
$$\left(\frac{\partial u}{\partial \eta}\right)_{i,j}^n = \left(\frac{\partial v}{\partial \eta}\right)_{i,j}^n = 0 \quad \text{on} \quad \Sigma_h^k, \tag{28}$$

where a_1, a_2 and a_3 are constants. a_3 is the damping parameter, f and g are perpendicular and horizontal loads, respectively. Note that we obtain the discrete model of equations (1) to (4) when $a_1 = a_2 = a_3 = 1$ and $g = 0$.

For the temporal computation of the equations (24) to (28), we have splitted in three levels: 0, 1 and 2.
For calculate u in the level 0, we use (27), i.e.,

$$u_{i,j}^0 = (u_0)_{i,j} \tag{29}$$

For calculate u in the level 1, we use (27) and (29), i.e.,

$$u_{i,j}^1 = u_{i,j}^0 + k\,(u_1)_{i,j} \tag{30}$$

For calculate u in the level 2, we first calculate

$$\Delta_h^2 v_{i,j}^n = -[u_{i,j}^n, u_{i,j}^n]_h + g_{i,j}^n, \quad n = 1, \ldots, M. \tag{31}$$

Using (22) for function v, the equation (31) is given by

$$Av_{i,j}^n = Bv_{i,j}^{\star n} + Dv_{i,j}^{\diamond n} - [u_{i,j}^n, u_{i,j}^n]_h + g_{i,j}^n,\ n = 1, \ldots, M \tag{32}$$

where $A = (a_{i,j})_{N^2 \times N^2}$ is a symmetric matrix, $B = (b_{i,j})_{N^2 \times N_b}$ and $D = (d_{i,j})_{N^2 \times N_g}$, where N_b is the number the boundary points, N_g is the number the ghost points and, $v^\star$ and $v^\diamond$ denote the function v evaluate in the boundary and ghost points, respectively. Thus, for all $n = 1, \ldots, M$

$$v^{\star n} = 0 \tag{33}$$
$$v^{\diamond n} = v^{\star n} \tag{34}$$

The equations (33) and (34) are due to a clamped boundary condition, given by (27) and (28), and because the exterior normal coincides with canonical vectors. Thus, the linear system (32) is resolved by the SOR method.
Once known v we can calculate u for all $n = 1, \ldots, M$

$$u_{i,j}^{n+1} = \mu_1 \tilde{u}_{i,j}^n + \mu_2 \left(\omega_{1\,i,j}^n - (1/8)\omega_{2\,i,j}^n + \omega_{3\,i,j}^n\right) + \mu_3 u_{i,j}^n + \mu_4 u_{i,j}^{n-1} + \mu_5 f_{i,j}^n \tag{35}$$

where,

$$\mu_1 = a_1^2 k^2/h^4,\ \ \mu_2 = a_2^2 k^2/h^4,\ \ \mu_3 = 2 - a_3^2 k$$
$$\mu_4 = a_3^2 k - 1,\ \ \mu_5 = k^2,$$

$$\tilde{u}_{i,j}^n = u_{i+2,j}^n + 2u_{i+1,j+1}^n - 8u_{i+1,j}^n + 2u_{i+1,j-1}^n + u_{i,j+2}^n - 8u_{i,j+1}^n + 20u_{i,j}^n - 8u_{i,j-1}^n + u_{i,j-2}^n + 2u_{i-1,j+1}^n - 8u_{i-1,j}^n + 2u_{i-1,j-1}^n + u_{i-2,j}^n,$$

$$\omega_{1\,i,j}^n = (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n)(v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n),$$
$$\omega_{2\,i,j}^n = (u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n)(v_{i+1,j+1}^n - v_{i+1,j-1}^n - v_{i-1,j+1}^n + v_{i-1,j-1}^n),$$
$$\omega_{3\,i,j}^n = (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)(v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n)$$

4.2 Numerical tests

Consider the following analytical solution of the equations (24) - (28)

$$u_a(x,y,t) = \sin^2 x \sin^2 y\, e^{-t} \tag{36}$$
$$v_a(x,y,t) = \sin^2 x \sin^2 y \tag{37}$$

with loads given by

$$f(x,y,t) = e^{-t}\Big[(1 - a_3^2)\sin^2 x \sin^2 y + 8a_1^2(\cos 2x \sin^2 y - \cos 2x \cos 2y + \cos 2y \sin^2 x) - 2a_2^2(4\cos 2x \sin^2 y \cos 2y \sin^2 x - \sin^2 2x \sin^2 2y)\Big] \tag{38}$$

$$g(x,y,t) = -8(\cos 2x \sin^2 y - \cos 2x \cos 2y + \cos 2y \sin^2 x) + 2e^{-2t}(4\cos 2x \sin^2 y \cos 2y \sin^2 x - \sin^2 2x \sin^2 2y) \tag{39}$$

We use for size of mesh $N = 19$. For case $g = 0$ the mechanical energy is given by

$$E(t) = |u'(t)|^2 + a_1^2|\Delta u(t)|^2 + \frac{1}{2}a_2^2|\Delta v(t)|^2 \quad (40)$$

Example 1. The first problem we consider the following data: $a_1 = 0.01, a_2 = 0.5$ and $a_3 = 1.25$. f and g are given in (38) and (39), respectively. u_0 and u_1 are obtained from (36) and u and v satisfies the clamping conditions. For constant $C_0 = 0.4$ a convergence is attained in $T = k(M+1) \approx 0.031416(830) \approx 26.075$ s. In Figure 2 we present for all $t \in (0,T]$, the *absolute error* defined by $|u(t) - u_a(t)|$. In Figure 3, we show the long-time behavior of the transversal displacement in the point $(\pi/2, \pi/2)$. In Figure 4, we show in 3D the transversal displacement of plate for different time steps: $t_0 = 0$, $t_{25} \approx 0.785$, $t_{70} \approx 2.199$, $t_{830} \approx 26.075$. We initially observe that the deflections are larger in the corners of the plate and then they reduce smoothly and expand rapidly near the boundary.

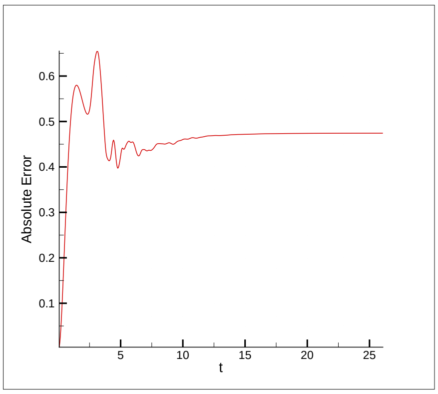


Fig. 2: The absolute error at L^2 norm the long-time.

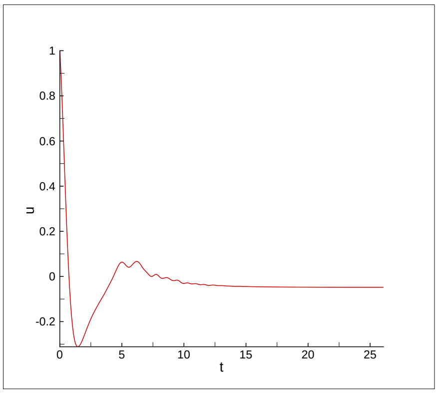


Fig. 3: Transversal displacement in the point $(\pi/2, \pi/2)$.

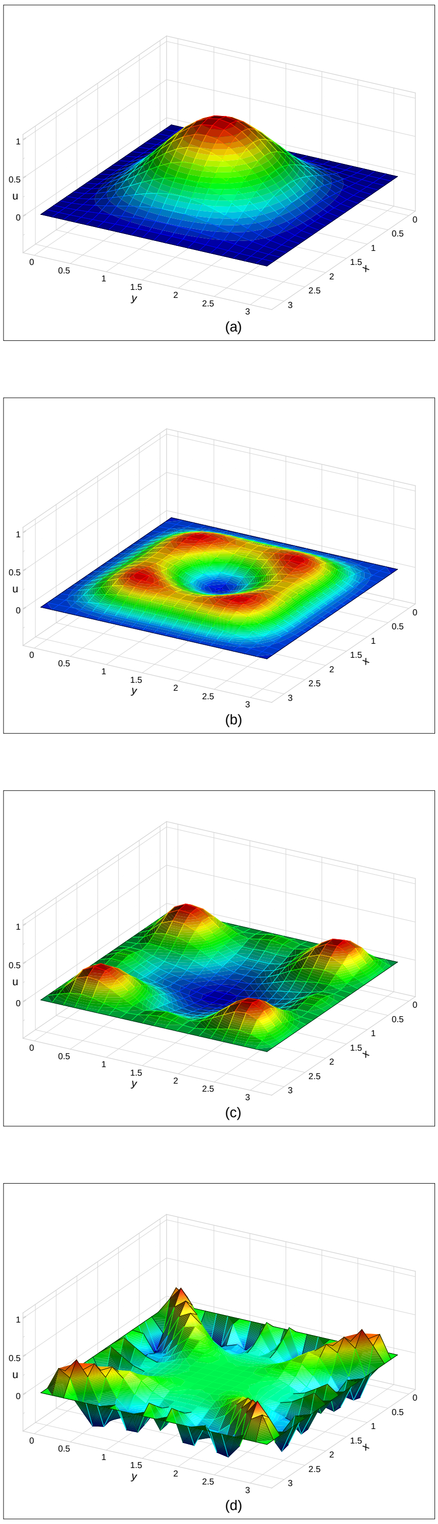


Fig. 4: Transversal displacement of plate. (a) $t = 0$ s, (b) $t_{25} \approx 0.785$ s, (c) $t_{70} \approx 2.199$ s and (d) $t_{830} \approx 26.075$ s.

Example 2. In this example we consider $a_1 = 0.01, a_2 = 1$ and $a_3 = 1$. f is given in (38) and $g = 0$.

u_0 and u_1 are obtained from (36) and u and v satisfy the clamping conditions. For constant $C_0 = 0.2$ a convergence is attained in $T = k(M+1) \approx 0.0314159(1862) \approx 58.496$ s. For calculate of energy of system, given by (40), we have computed in all the plate using the Composite Simpson's rule. In Figure 5 and 6 we show the long-time behavior of the solution in the point $(\pi/2, \pi/2)$ and energy of system, respectively. Note that energy converge to 0. In Figure 7, we show the transversal displacement of the plate, in 3D, for different time steps: $t_0 = 0, t_{45} \approx 1.414, t_{140} \approx 4.398, t_{1843} \approx 58.496$. We initially observe that the deflections are larger near the boundary, but in long-time these deflections disappear.

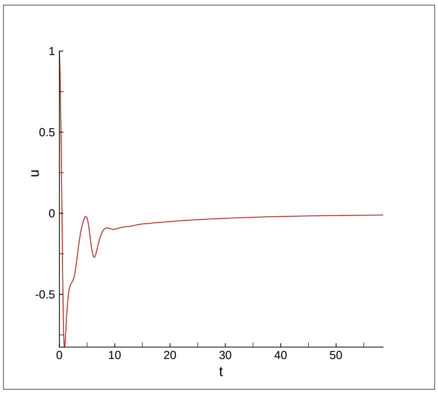


Fig. 5: Transversal displacement in the point $(\pi/2, \pi/2)$.

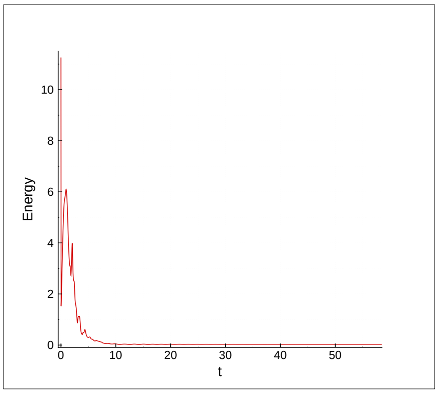
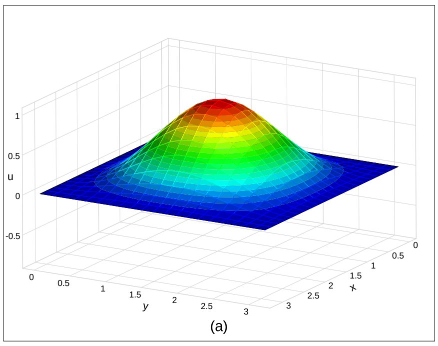


Fig. 6: The energy of system through long-time.



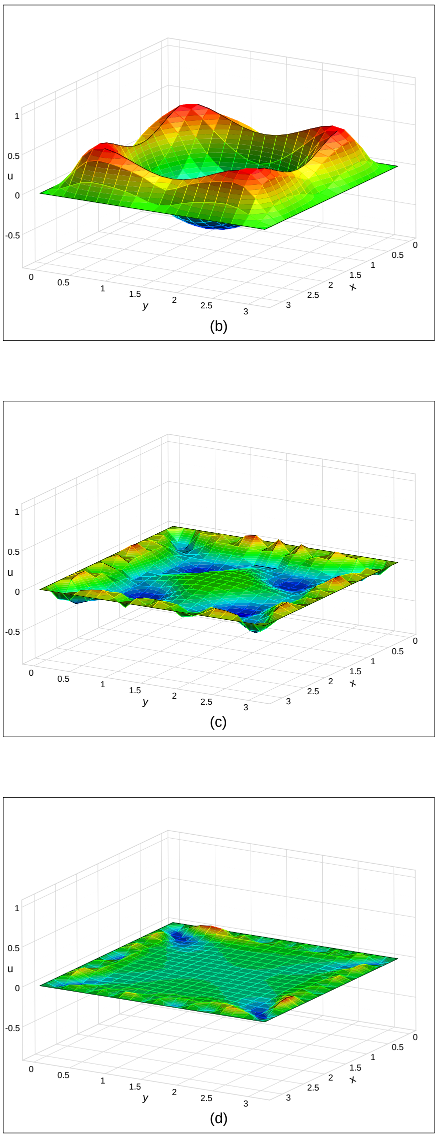


Fig. 7: Transversal displacement of plate. (a) $t = 0$ s, (b) $t_{45} \approx 1.414$ s, (c) $t_{140} \approx 4.398$ and (d) $t_{1843} \approx 58.496$ s.

5 Conclusion

The Nakao's method proved to be an efficient method for the demonstration of the exponential decay of the solution of the system of von Kármán. Numerical tests have shown the decay of the mechanical energy of the system.

References

[1] Adams, R. A.; Academic Press, New York, (1975).
[2] Avalos, G.; Lasiecka, I.; Exponential stability of a thermoelastic plates with free boundary conditions and without mechanical dissipation, SIAM J. Math. Anal., **29**, 155-182 (1998).

[3] Avalos, G.; Lasiecka, I.; Exponential stability of a thermoelastic system without mechani- cal dissipation, Rend. Istit. Mat. Univ. Trieste., **28**, 1-28 (1987).
[4] Benabdallah, A.; Lasiecka, I.; Exponential Decay Rates for a Full von Kármán System of Dynamic Thermoelasticity, Journal of Differential Equations, **160**, 51-93 (2000).
[5] Bilbao, S.; A Family of Conservative Finite Difference Schemes for the Dynamical von Kármán Plate Equations, Numerical Methods for Partial Differential Equations, **24**, 193-216 (2007).
[6] Bradley, M. E.; and Lasiecka, I.; Global decay rates for the solutions to a von Kármán plate without geometric conditions, J. Math. Anal. Appl., 181-254 (1994).
[7] Horn, M.; Favini, A.; Lasiecka, I.; Tataru, D.; Global existence, uniqueness and regularity to a Von Kármán system with nonlinear boundary dissipation, Diff. and Int. Eq., **9**, 267-294 (1996).
[8] Horn, M.; Lasiecka, I.; Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation, Diff. and Int. Equations, **7**, 885-908 (1994).
[9] Horn, M.; Lasiecka, I.; Global stabilization of a dynamical Von Kármán plate with Nonlinear Boundary Feedback, Appl. Math. Optimization, **31**, 57-84 (1995).
[10] Menzala, G. P.; Zuazua, E.; Energy decay rates for the von Kármán system of thermoelastic plates, Differential and Integral Equations, **11**, 755-770 (1998).
[11] Messaoudi, S.A.; General decay of solutions of a viscoelastic equation, Journal of Mathematical Analysis and Applications, **341**, 1457-1467 (2008).
[12] Nakao, M.; A difference inequalit and its application to nonolinear evolution equation, J. Math. Soc. Japan, **30**, 747-762 (1978).
[13] Raposo, C. A.; Santos, M. L.; General decay to a von Kármán System with memory, Nonlinear Analysis, **74**, 937-945 (2011).
[14] Reinhart, L.; On the Numerical Analysis of the Von Kármán Equations: Mixed Finite Element Approximation and Continuation Techniques, Numer. Math., **39**, 371-404 (1982).
[15] Rivera, J. E. M.; Menzala, G.P.; Decay rates of solutions of a von Kármán system for viscoelastic plates with memory, Quarterly of Applied Mathematics. v. LVII, **1**, 181-200 (1999).
[16] Rivera, J. E. M.; Oquendo, H. P.; Santos, M. L.; Asymptotic behavior to a von Kármán plate with boundary memory conditions, Nonlinear Analysis, **62**, 1183-1205 (2005).
[17] Park, J. Y.; Kang, J. R.; Global existence and stability for a von Kármán equations with memory in noncylindrical domains, J. Math. Phys., **50**, 112-701 (2009).
[18] Santos, M. L.; Ferreira, J.; Raposo, C. A.; Existence and uniform decay for a nonlinear beam equation with nonlinearity of Kirchhoff type in domains with moving boundary, Abstr. Appl. Anal., 901-919 (2005).
[19] von Kármán, T.; Festigkeitsprobleme im Maschinenbaum, Encyklopadie der Math. Wiss. V/4C, Leipzig, 311-385 (1910).
[20] Yosibash, Z.; Kirby, R.M.; Gottlieb, D.; Collocation methods for the solution of von-Kármán dynamic non-linear plate systems, Journal of Computational Physics, **200**, 432-461 (2004).

Ducival C. Pereira, PhD in Sciences from Federal University of Rio de Janeiro and post-doctoral in Mathematics by the National Laboratory of Scientific Computing (LNCC). Professor of Pará State University and researcher in Mathematics, with emphasis on analysis, working mainly in the solution problems of nonlinear hyperbolic partial differential equations.

Carlos Alberto Raposo da Cunha, PhD in Mathematics from UFRJ - Federal University of Rio de Janeiro. Full Professor at the Department of Mathematics and Statistics from UFSJ - Federal University of São João del-Rei. Has experience in Mathematics with emphasis in Partial Differential Equations. The interests are mainly in decay of energy for dissipative systems, transmission problems, problems in viscoelasticity with memory, thermoelasticity and asymptotic behavior of solutions of PDEs. Reviewer for the Mathematical Reviews / AMS - American Mathematical Society. Has served as referee for several international mathematical journals.

Jorge Andrés Julca Avila, PhD in Mechanical Engineering from USP - University of São Paulo. Assistant Professor at the Department of Mathematics and Statistics from UFSJ - Federal University of São João del-Rei. Has experience in Applied Mathematics emphasizing Numerical Analysis, Functional Analysis and PDEs. For the moment, the major interests are in the Navier-Stokes Equations, Non Newtonian Fluids and Finite Elements. Is also experienced in Mechanical Engineering with emphasis in Fluid Mechanics and Fluid Dynamics. Is member of the Scientific Committee of UFSJ.