

Numerical Solution and Exponential Decay to Von Kármán System with Frictional Damping

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Abstract: In this work we consider the Von Kármán system with frictional damping acting on the displacement and using the Method of Nakao we prove the exponential decay of the solution. The numerical scheme is presented for calculate the solution and to verify the long-time decay energy.

Keywords: Von Kármán system, Method of Nakao, Decay of solutions, Numerical solution, Finite differences method.

1 Introduction

For several years the system of Theodor von Kármán [19] was studied in different situations and methods.

The exponential decay of the energy to the von Kármán equations with memory in noncylindrical domains was studied by Park and Kang [17] in 2009 using the same method as in [18].

To the models of von Kármán taking into account for rotational forces, Bradley and Lásiecka [6] in 1994 showed the uniform decay rates for the solutions in cylindrical domain

The uniform decay of the solution was considered for frictional dissipative at the boundary, for example, in the works of Horn and Lásiecka [8] in 1994, Horn and Lásiecka [9] in 1995, and Horn, Favini, Lásiecka and Tataru [7] in 1996.

Applying multipliers method Avalos and Lásiecka [3] in 1987 showed the uniform decay for two-dimensional linear thermoelastic plates and Avalos and Lásiecka [2] in 1998 the one-dimensional thermoelastic von Kármán model was studied.

For thermal damping Menzala and Zuazua [10] in 1998 proved the exponential decay by the semigroup properties.

For Viscoelastic plates with memory, using energy method, we cite Rivera and Menzala [15] in 1999, and the work of Rivera, Oquendo and Santos [16] in 2005 where was proved that the energy decays uniformly,

exponentially or algebraically with the same rate of decay of the relaxation function.

Based on multipliers method, the exponential decay of solution for the full von Kármán System of Dynamic Thermoelasticity was proved by Benabdallah and Lásiecka [4] in 2000.

For von Kármán System with memory Raposo and Santos [13] in 2011 obtained the General Decay of solution using the idea of Messaoudi [11] in the study of the asymptotic behavior of viscoelastic equations.

For the numerical scheme we mention for example Reinhart [14] in 1982 where was studied the approximation of the von Kármán equations stationary by the mixed finite element. The work of Yosibash, Kirby and Gottlieb [20] in 2004 where was studied the von Kármán system over rectangular domains and numerically solved using both the Chebyshev-collocation and Legendre-collocation methods for the spacial discretization and the implicit Newmark- β scheme combined with a non-linear fixed point algorithm for the temporal discretization, and Bilbao [5] in 2007 used numerical stability for numerical methods for the von Kármán system, through the use of energy-conserving methods.

What distinguishes this paper from other related works is that we apply the Method of Nakao in the von Kármán system to prove the exponential decay of the solution and we present an numerical scheme by finite

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differences method to numerical solution and the long-time decay energy, in this sense, there is few result in the literature.

The remainder of this paper is organized as follows. In section 2 we present the result of existence of weak solution, in the section 3 we prove the exponential decay of the solution, in the section 4 we applied the Finite Difference Method in the von Kármán system and finally in the section 5 we give the conclusion.

2 Existence of solution

We use the standard Lebesgue space and Sobolev Space with their usual properties as in Adams (1975) [1] and in this sense (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ denote the inner product in L^2 and H_0^1 respectively. By $|\cdot|$ we denote the usual norm in L^2 . Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of the plane with regular boundary Γ . For a real number $T > 0$ we denote $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. Here $u = u(x, y, t)$ is the displacement, $v = v(x, y, t)$ the Airy stress function and η is the unit normal external in Ω and $u' = u_t$. With this notation we have the following system

$$u'' - \Delta^2 u - [u, v] + u' = f \quad \text{in } Q, \tag{1}$$

$$\Delta^2 v + [u, u] = 0 \quad \text{in } Q, \tag{2}$$

and

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega, \tag{3}$$

$$u = v = 0, \quad \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \Sigma, \tag{4}$$

where

$$[u, v] = \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2}$$

Now using the same idea as in [10] we have the following result of existence of solution.

Theorem 1. For $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$ there exists $u, v : Q \rightarrow \mathbb{R}$ such that

$$u, v \in L^\infty(0, T; H_0^2(\Omega)), \quad u' \in L^\infty(0, T; L^2(\Omega)),$$

and u, v weak solution (1)-(4).

3 Asymptotic behavior

In this section, we will use the Method of Nakao (1978) [12] to prove the exponential decay of the solution. First we define

$$E(t) = |u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \tag{5}$$

Lemma 1. The functional of energy $E(t)$ is limited.

Proof. Multiplying (1) by u' and integrating in Ω , we have

$$\frac{1}{2} \frac{d}{dt} [|u'(t)|^2 + |\Delta u(t)|^2] - \langle [u(t), v(t)], u'(t) \rangle + |u'(t)|^2 = (f(t), u'(t))$$

Using (2) we obtain

$$\begin{aligned} \langle [u(t), v(t)], u'(t) \rangle &= \langle [u(t), u'(t)], v(t) \rangle \\ &= \frac{1}{2} \left\langle \frac{d}{dt} [u(t), u(t)], v(t) \right\rangle \\ &\quad - \frac{1}{2} \langle \Delta^2 v'(t), v(t) \rangle \\ &= \frac{1}{4} \frac{d}{dt} |\Delta v(t)|^2 \end{aligned}$$

from where follows

$$\frac{d}{dt} \left[|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right] + 2|u'(t)|^2 = 2(f(t), u'(t)) \tag{6}$$

Performing integration from 0 to t follows by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 + 2 \int_0^t |u'(s)|^2 ds \leq \\ \int_0^t |u'(s)|^2 ds + \int_0^t |f(s)|^2 ds + |u_1|^2 + |\Delta u_0|^2 + \frac{1}{2} |\Delta v_0|^2 \end{aligned}$$

then

$$E(t) + \int_0^t |u'(s)|^2 ds \leq E(0) + \int_0^t |f(s)|^2 ds$$

from where follows $E(t) \leq C$ with C constant independently of t .

Now we introduce a new functional.

Lemma 2. The functional

$$F^2(t) = E(t) - E(t+1) + \int_t^{t+1} |f(s)|^2 ds$$

satisfies

$$\int_t^{t+1} |u'(s)|^2 ds \leq F^2(t)$$

Proof. Integrating (6) from τ_1 to τ_2 with $0 < \tau_1 < \tau_2$, we obtain

$$\begin{aligned} E(\tau_2) + 2 \int_{\tau_1}^{\tau_2} |u'(s)|^2 ds = E(\tau_1) \\ + 2 \int_{\tau_1}^{\tau_2} (f(s), u'(s)) ds \tag{7} \end{aligned}$$

and for all $t > 0$

$$\begin{aligned} E(t+1) + 2 \int_t^{t+1} |u'(s)|^2 ds = E(t) + 2 \int_t^{t+1} (f(s), u'(s)) ds \\ \leq E(t) + \int_t^{t+1} |f(s)|^2 ds + \int_t^{t+1} |u'(s)|^2 ds \end{aligned}$$

then

$$\int_t^{t+1} |u'(s)|^2 ds \leq E(t) - E(t+1) + \int_t^{t+1} |f(s)|^2 ds = F^2(t) \quad (8)$$

Lemma 3. *The functional*

$$G^2(t) = 8C \operatorname{ess\,sup}_{s \in [t, t+1]} |\Delta u(s)| F(t) + 2(1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + 2C^2 \int_{t_1}^{t_2} |f(t)|^2 dt$$

satisfies

$$\int_{t_1}^{t_2} \left(|\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq G^2(t)$$

Proof. First we note that

$$\langle [u(t), v(t)], u(t) \rangle = \langle [u(t), u(t)], v(t) \rangle = -\langle \Delta^2 v(t), v(t) \rangle = -|\Delta v(t)|^2 \quad (9)$$

From (8) there exists $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t + 1]$ such that

$$|u'(t_i)| \leq 2F(t), \quad i = 1, 2 \quad (10)$$

Multiplying (1) by u and integrating in Ω , we have

$$\frac{d}{dt} (u'(t), u(t)) - |u'(t)|^2 + |\Delta u(t)|^2 - \langle [u(t), v(t)], u(t) \rangle + (u'(t), u(t)) = (f(t), u(t))$$

Performing integration from t_1 to t_2 and using (9) we have

$$\int_{t_1}^{t_2} (|\Delta u(t)|^2 + |\Delta v(t)|^2) dt = (u'(t_1), u(t_1)) - (u'(t_2), u(t_2)) + \int_{t_1}^{t_2} (|u'(t)|^2 - (u'(t), u(t))) dt + \int_{t_1}^{t_2} (f(t), u(t)) dt$$

Now, choosing C such that $|u| \leq C|\Delta u|$ and applying Cauchy-Schwarz inequality we get

$$\int_{t_1}^{t_2} \left(\frac{1}{2} |\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt \leq C \operatorname{ess\,sup}_{s \in [t, t+1]} (|\Delta u(s)| (|u'(t_1)| + |u'(t_2)|)) + (1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + C^2 \int_{t_1}^{t_2} |f(t)|^2 dt,$$

using (10),

$$\int_{t_1}^{t_2} \left(\frac{1}{2} |\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt \leq 8C \operatorname{ess\,sup}_{s \in [t, t+1]} |\Delta u(s)| F(t) + 2(1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + 2C^2 \int_{t_1}^{t_2} |f(t)|^2 dt,$$

and then

$$\int_{t_1}^{t_2} \left(|\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq G^2(t) \quad (11)$$

Theorem 2. *For $f \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$ with $\int_0^t |f(s)| ds \leq \alpha_1 e^{-\alpha_2 t}$, for all $t \geq 1$ and $\alpha_1, \alpha_2 > 0$, then the solution (u, v) satisfies*

$$|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 + \int_t^{t+1} |u'(s)|^2 ds \leq k_1 e^{-k_2 t}, \quad (12)$$

for almost every $t \geq 1$, with $k_1, k_2 > 0$, constants independently from t .

Proof. From (8) and (11) we concludes

$$\int_{t_1}^{t_2} \left(|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq F^2(t) + G^2(t)$$

There is $t^* \in [t_1, t_2]$ such that

$$E(t^*) = |u'(t^*)|^2 + |\Delta u(t^*)|^2 + \frac{1}{2} |\Delta v(t^*)|^2 \leq 2(F^2(t) + G^2(t)) \quad (13)$$

From (7) we get

$$E(t_1) = E(t^*) + 2 \int_{t_1}^{t^*} |u'(s)|^2 ds - 2 \int_{t_1}^{t^*} (f(s), u'(s)) ds$$

Then

$$E(t) = E(t^*) + 2 \int_t^{t^*} |u'(s)|^2 ds - 2 \int_t^{t^*} (f(s), u'(s)) ds \leq E(t^*) + 3 \int_t^{t+1} |u'(s)|^2 ds + \int_t^{t+1} |f(s)|^2 ds,$$

and

$$\operatorname{ess\,sup}_{s \in [t, t+1]} E(s) \leq E(t^*) + 3 \int_t^{t+1} |u'(s)|^2 ds + \int_t^{t+1} |f(s)|^2 ds$$

Now using (8) and (13) we obtain

$$\begin{aligned} \operatorname{ess\,sup}_{s \in [t, t+1]} E(s) &\leq 2(F^2(t) + G^2(t)) + 3F^2(t) + \int_t^{t+1} |f(s)|^2 ds \\ &\leq 5F^2(t) + 16C \operatorname{ess\,sup}_{s \in [t, t+1]} |\Delta u(s)| F(t) \\ &\quad + 4(1 + C^2) \int_t^{t+1} |u'(s)|^2 ds \\ &\quad + (1 + 4C^2) \int_t^{t+1} |f(s)|^2 ds \\ &\leq (9 + 4C^2) F^2(t) + \frac{1}{2} \operatorname{ess\,sup}_{s \in [t, t+1]} E(s) \\ &\quad + 128C^2 F^2(t) + (1 + 4C^2) \int_t^{t+1} |f(s)|^2 ds, \end{aligned}$$

from where follows

$$\text{ess sup}_{s \in [t, t+1]} E(s) \leq (274 + 8C^2)F^2(t) + (2 + 8C^2) \int_t^{t+1} |f(s)|^2 ds,$$

and then

$$E(t) \leq C_1(E(t) - E(t+1)) + C_2 \int_t^{t+1} |f(s)|^2 ds,$$

where $C_i, i = 1, 2$, constants independently from t .

Without lost of generality, we can suppose $C_1 > 1$ and for $0 < \beta = \frac{1}{C_1} < 1$ we, have

$$E(t+1) \leq (1 - \beta)E(t) + \beta C_2 \int_t^{t+1} |f(s)|^2 ds$$

For $t \geq 1$ and $n \in \mathbb{N}$ such that $n \leq t \leq n+1$

$$\begin{aligned} E(t) &\leq (1 - \beta)E(t-1) + \beta C_2 \int_{t-1}^t |f(s)|^2 ds \\ &\leq (1 - \beta)^n E(t-n) + \beta C_2 \int_{t-n}^t |f(s)|^2 ds \end{aligned}$$

Now

$$(1 - \beta)^{n+1} < (1 - \beta)^t \quad \text{implies} \quad (1 - \beta)^n < (1 - \beta)^{t-1}$$

Then

$$\begin{aligned} E(t) &\leq (1 - \beta)^{t-1} \text{ess sup}_{s \in [0,1]} E(s) + \beta C_2 \int_0^t |f(s)|^2 ds \\ &= \frac{m_0}{1 - \beta} (1 - \beta)^t + \beta C_2 \int_0^t |f(s)|^2 ds, \end{aligned}$$

with $m_0 = \text{ess sup}_{s \in [0,1]} E(s) < \infty$.

Now we have

$$\begin{aligned} E(t) &< \frac{m_0}{1 - \beta} e^{t \ln(1 - \beta)} + \beta C_2 \int_0^t |f(s)|^2 ds \\ &< \frac{m_0}{1 - \beta} e^{-\beta_1 t} + \beta C_2 \alpha_1 e^{-\alpha_2 t}, \end{aligned}$$

for almost every $t \geq 1$, with $\beta_1 = -\ln(1 - \beta) > 0$ and then

$$|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 < \gamma_1 e^{-\gamma_2 t} \quad (14)$$

From (8), we have

$$\int_t^{t+1} |u'(s)|^2 ds \leq E(t) + \int_t^{t+1} |f(s)|^2 < \gamma_3 e^{-\gamma_4 t} \quad (15)$$

with $\gamma_i > 0$ constants. Finally we concludes from (14) and (15) that there is constants $k_1, k_2 > 0$ such that

$$|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 + \int_t^{t+1} |u'(s)|^2 ds < k_1 e^{-k_2 t}$$

This completes the proof.

4 Numerical solution

For a given small constant $\varepsilon > 0$ we define a thin plate by

$$\Omega \times (-\varepsilon, \varepsilon) = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Omega, z \in (-\varepsilon, \varepsilon)\}$$

whose midsurface is identified with Ω .

We resolve the von Kármán system in a square thin elastic plate by Finite Difference Method, subjected to a perpendicular load f and boundary condition of clamped type.

4.1 Discrete formulation

Consider the discrete domain the midsurface of the square plate, $\Omega_h = (0, \pi)^2$ with uniform grid $x_i = ih, y_j = jh, i, j = 0, \dots, N + 1, h = \pi/(N + 1)$. The internal point are $x_i = ih, y_j = jh, 1 \leq i, j \leq N$. The boundary of Ω_h is denoted Γ_h . The temporal discretization of interval $I_k = (0, T)$ is given by $t_n = nk, n = 0, \dots, M + 1, k = C_0 h/2$ where C_0 is a positive constant and $T = k(M + 1)$. Denote by $u_{i,j}^n$ and $v_{i,j}^n$ the functions u and v evaluate in the point (x_i, y_j) and at the instant t_n , respectively. It also, denoted by $Q_h^k = \Omega_h \times I_k$ and $\Sigma_h^k = \Gamma_h \times I_k$.

We show in Figure 1 the pattern mesh of Ω with its points: internal (circles), boundary (squares) and ghost (diamonds). We define the following discrete differential

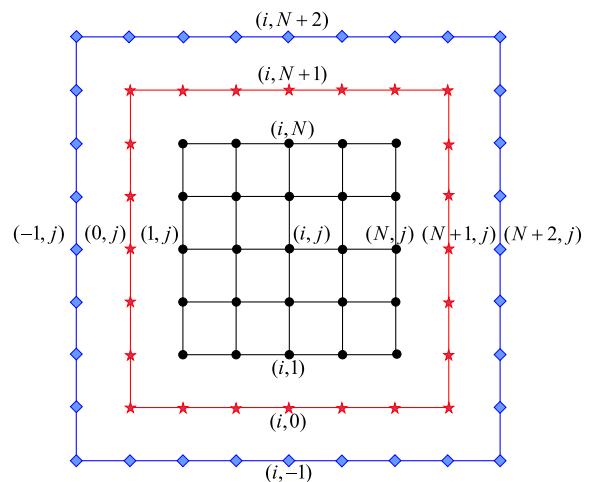


Fig. 1: The pattern mesh of Ω with internal, boundary and ghost points.

operators:

$$\begin{aligned} \delta_t u_{i,j}^n &= \frac{1}{k} (u_{i,j}^n - u_{i,j}^{n-1}), \\ \delta_t^2 u_{i,j}^n &= \frac{1}{k^2} (u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}) \end{aligned} \quad (16)$$

$$\begin{aligned} \delta_x^2 u_{i,j}^n &= \frac{1}{h^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n), \\ \delta_y^2 u_{i,j}^n &= \frac{1}{h^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) \end{aligned} \quad (17)$$

$$\begin{aligned} \delta_{xy} u_{i,j}^n &= \delta_x (\delta_y u_{i,j}^n) = \\ &= \frac{1}{4h^2} (u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n) \end{aligned} \quad (18)$$

$$\begin{aligned} \delta_x^4 u_{i,j}^n &= \delta_x^2 (\delta_x^2 u_{i,j}^n) = \\ &= \frac{1}{h^4} (u_{i+2,j}^n - 4u_{i+1,j}^n + 6u_{i,j}^n - 4u_{i-1,j}^n + u_{i-2,j}^n) \end{aligned} \quad (19)$$

$$\begin{aligned} \delta_y^4 u_{i,j}^n &= \delta_y^2 (\delta_y^2 u_{i,j}^n) = \\ &= \frac{1}{h^4} (u_{i,j+2}^n - 4u_{i,j+1}^n + 6u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n) \end{aligned} \quad (20)$$

$$\begin{aligned} \delta_{xy}^2 u_{i,j}^n &= \delta_x^2 (\delta_y^2 u_{i,j}^n) = \\ &= \frac{1}{h^4} (u_{i+1,j+1}^n - 2u_{i+1,j}^n + u_{i+1,j-1}^n - 2u_{i,j+1}^n + 4u_{i,j}^n \\ &\quad - 2u_{i,j-1}^n + u_{i-1,j+1}^n - 2u_{i-1,j}^n + u_{i-1,j-1}^n) \end{aligned} \quad (21)$$

The discrete biharmonic operator is given by

$$\Delta_h^2 u_{i,j}^n = \delta_x^4 u_{i,j}^n + \delta_y^4 u_{i,j}^n + 2\delta_{xy}^2 u_{i,j}^n \quad (22)$$

The discrete bracket operator is given by

$$\begin{aligned} [u_{i,j}^n, v_{i,j}^n]_h &= \delta_x^2 u_{i,j}^n \delta_y^2 v_{i,j}^n + \delta_y^2 u_{i,j}^n \delta_x^2 v_{i,j}^n - \\ &= 2\delta_{xy} u_{i,j}^n \delta_{xy} v_{i,j}^n \end{aligned} \quad (23)$$

Using the equations (16) to (23) we obtain the discrete model of von Kármán system,

$$\begin{aligned} \delta_t^2 u_{i,j}^n - a_1^2 \Delta_h^2 u_{i,j}^n - a_2^2 [u_{i,j}^n, v_{i,j}^n]_h + a_3^2 \delta_t u_{i,j}^n \\ = f_{i,j}^n \quad \text{in } \mathcal{Q}_h^k, \end{aligned} \quad (24)$$

$$\Delta_h^2 v_{i,j}^n + [u_{i,j}^n, u_{i,j}^n]_h = g_{i,j}^n \quad \text{in } \mathcal{Q}_h^k, \quad (25)$$

$$u_{i,j}^0 = (u_0)_{i,j}, \quad \delta_t u_{i,j}^0 = (u_1)_{i,j} \quad \text{in } \Omega_h \quad (26)$$

$$u_{i,j}^n = v_{i,j}^n = 0 \quad \text{on } \Sigma_h^k, \quad (27)$$

$$\left(\frac{\partial u}{\partial \eta} \right)_{i,j}^n = \left(\frac{\partial v}{\partial \eta} \right)_{i,j}^n = 0 \quad \text{on } \Sigma_h^k, \quad (28)$$

where a_1, a_2 and a_3 are constants. a_3 is the damping parameter, f and g are perpendicular and horizontal loads, respectively. Note that we obtain the discrete model of equations (1) to (4) when $a_1 = a_2 = a_3 = 1$ and $g = 0$.

For the temporal computation of the equations (24) to (28), we have splitted in three levels: 0, 1 and 2.

For calculate u in the level 0, we use (27), i.e.,

$$u_{i,j}^0 = (u_0)_{i,j} \quad (29)$$

For calculate u in the level 1, we use (27) and (29), i.e.,

$$u_{i,j}^1 = u_{i,j}^0 + k(u_1)_{i,j} \quad (30)$$

For calculate u in the level 2, we first calculate

$$\Delta_h^2 v_{i,j}^n = -[u_{i,j}^n, u_{i,j}^n]_h + g_{i,j}^n, \quad n = 1, \dots, M. \quad (31)$$

Using (22) for function v , the equation (31) is given by

$$Av_{i,j}^n = Bv_{i,j}^{*n} + Dv_{i,j}^{\circ n} - [u_{i,j}^n, u_{i,j}^n]_h + g_{i,j}^n, \quad n = 1, \dots, M \quad (32)$$

where $A = (a_{i,j})_{N^2 \times N^2}$ is a symmetric matrix, $B = (b_{i,j})_{N^2 \times N_b}$ and $D = (d_{i,j})_{N^2 \times N_g}$, where N_b is the number the boundary points, N_g is the number the ghost points and, v^* and v° denote the function v evaluate in the boundary and ghost points, respectively. Thus, for all $n = 1, \dots, M$

$$v^{*n} = 0 \quad (33)$$

$$v^{\circ n} = v^{*n} \quad (34)$$

The equations (33) and (34) are due to a clamped boundary condition, given by (27) and (28), and because the exterior normal coincides with canonical vectors. Thus, the linear system (32) is resolved by the SOR method.

Once known v we can calculate u for all $n = 1, \dots, M$

$$\begin{aligned} u_{i,j}^{n+1} &= \mu_1 \tilde{u}_{i,j}^n + \mu_2 (\omega_{i,j}^n - (1/8)\omega_{2,i,j}^n + \omega_{3,i,j}^n) \\ &\quad + \mu_3 u_{i,j}^n + \mu_4 u_{i,j}^{n-1} + \mu_5 f_{i,j}^n \end{aligned} \quad (35)$$

where,

$$\begin{aligned} \mu_1 &= a_1^2 k^2 / h^4, \quad \mu_2 = a_2^2 k^2 / h^4, \quad \mu_3 = 2 - a_3^2 k \\ \mu_4 &= a_3^2 k - 1, \quad \mu_5 = k^2, \end{aligned}$$

$$\begin{aligned} \tilde{u}_{i,j}^n &= u_{i+2,j}^n + 2u_{i+1,j+1}^n - 8u_{i+1,j}^n + 2u_{i+1,j-1}^n + u_{i,j+2}^n \\ &\quad - 8u_{i,j+1}^n + 20u_{i,j}^n - 8u_{i,j-1}^n + u_{i,j-2}^n + 2u_{i-1,j+1}^n \\ &\quad - 8u_{i-1,j}^n + 2u_{i-1,j-1}^n + u_{i-2,j}^n, \end{aligned}$$

$$\omega_{1,i,j}^n = (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n)(v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n),$$

$$\begin{aligned} \omega_{2,i,j}^n &= (u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n) \\ &\quad (v_{i+1,j+1}^n - v_{i+1,j-1}^n - v_{i-1,j+1}^n + v_{i-1,j-1}^n), \end{aligned}$$

$$\omega_{3,i,j}^n = (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)(v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n)$$

4.2 Numerical tests

Consider the following analytical solution of the equations (24) - (28)

$$u_a(x, y, t) = \sin^2 x \sin^2 y e^{-t} \quad (36)$$

$$v_a(x, y, t) = \sin^2 x \sin^2 y \quad (37)$$

with loads given by

$$\begin{aligned} f(x, y, t) &= e^{-t} \left[(1 - a_3^2) \sin^2 x \sin^2 y + 8a_1^2 (\cos 2x \sin^2 y \right. \\ &\quad \left. - \cos 2x \cos 2y + \cos 2y \sin^2 x) \right. \\ &\quad \left. - 2a_2^2 (4 \cos 2x \sin^2 y \cos 2y \sin^2 x - \sin^2 2x \sin^2 2y) \right] \end{aligned} \quad (38)$$

$$\begin{aligned} g(x, y, t) &= -8(\cos 2x \sin^2 y - \cos 2x \cos 2y + \cos 2y \sin^2 x) \\ &\quad + 2e^{-2t} (4 \cos 2x \sin^2 y \cos 2y \sin^2 x - \sin^2 2x \sin^2 2y) \end{aligned} \quad (39)$$

We use for size of mesh $N = 19$. For case $g = 0$ the mechanical energy is given by

$$E(t) = |u'(t)|^2 + a_1^2 |\Delta u(t)|^2 + \frac{1}{2} a_2^2 |\Delta v(t)|^2 \quad (40)$$

Example 1. The first problem we consider the following data: $a_1 = 0.01$, $a_2 = 0.5$ and $a_3 = 1.25$. f and g are given in (38) and (39), respectively. u_0 and u_1 are obtained from (36) and u and v satisfies the clamping conditions. For constant $C_0 = 0.4$ a convergence is attained in $T = k(M + 1) \approx 0.031416(830) \approx 26.075$ s. In Figure 2 we present for all $t \in (0, T]$, the *absolute error* defined by $|u(t) - u_a(t)|$. In Figure 3, we show the long-time behavior of the transversal displacement in the point $(\pi/2, \pi/2)$. In Figure 4, we show in 3D the transversal displacement of plate for different time steps: $t_0 = 0$, $t_{25} \approx 0.785$, $t_{70} \approx 2.199$, $t_{830} \approx 26.075$. We initially observe that the deflections are larger in the corners of the plate and then they reduce smoothly and expand rapidly near the boundary.

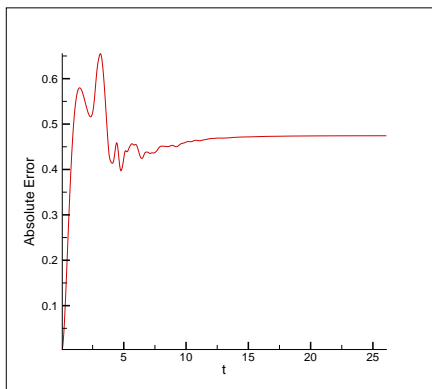


Fig. 2: The absolute error at L^2 norm the long-time.

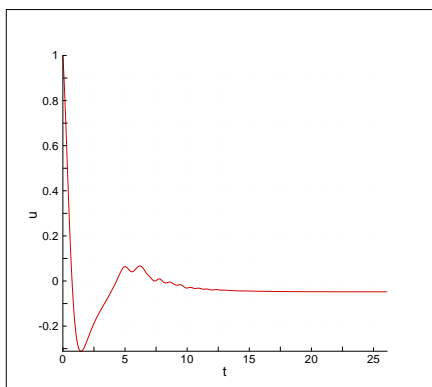


Fig. 3: Transversal displacement in the point $(\pi/2, \pi/2)$.

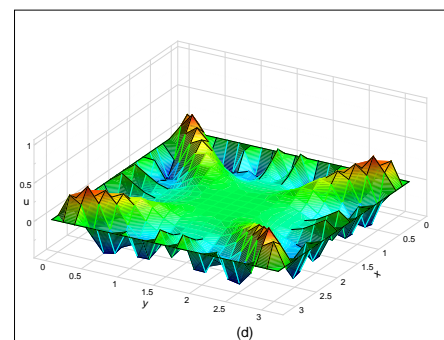
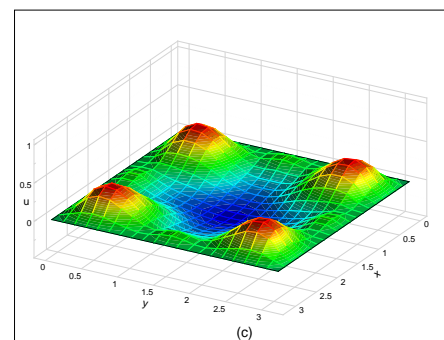
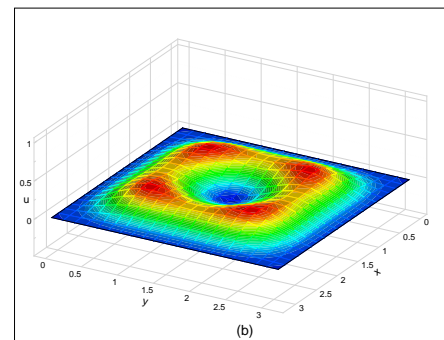
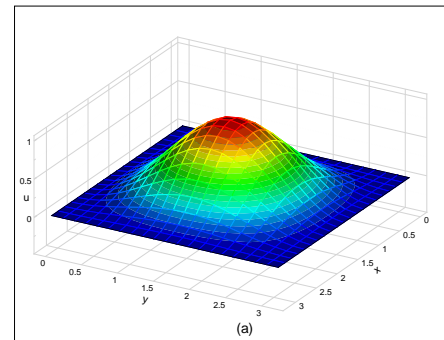


Fig. 4: Transversal displacement of plate. (a) $t = 0$ s, (b) $t_{25} \approx 0.785$ s, (c) $t_{70} \approx 2.199$ s and (d) $t_{830} \approx 26.075$ s.

Example 2. In this example we consider $a_1 = 0.01$, $a_2 = 1$ and $a_3 = 1$. f is given in (38) and $g = 0$.

u_0 and u_1 are obtained from (36) and u and v satisfy the clamping conditions. For constant $C_0 = 0.2$ a convergence is attained in $T = k(M + 1) \approx 0.0314159(1862) \approx 58.496$ s. For calculate of energy of system, given by (40), we have computed in all the plate using the Composite Simpson's rule. In Figure 5 and 6 we show the long-time behavior of the solution in the point $(\pi/2, \pi/2)$ and energy of system, respectively. Note that energy converge to 0. In Figure 7, we show the transversal displacement of the plate, in 3D, for different time steps: $t_0 = 0, t_{45} \approx 1.414, t_{140} \approx 4.398, t_{1843} \approx 58.496$. We initially observe that the deflections are larger near the boundary, but in long-time these deflections disappear.

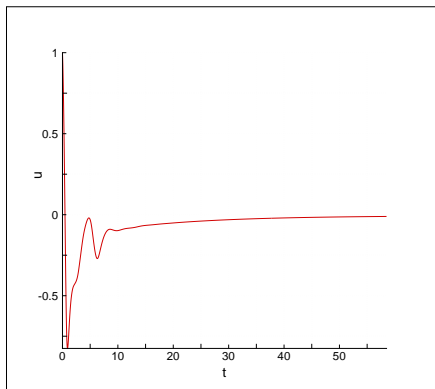


Fig. 5: Transversal displacement in the point $(\pi/2, \pi/2)$.

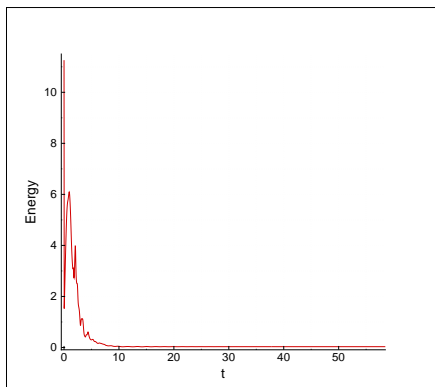


Fig. 6: The energy of system through long-time.

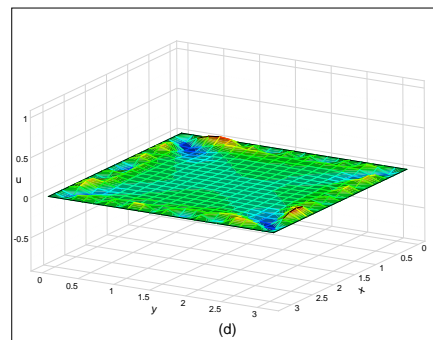
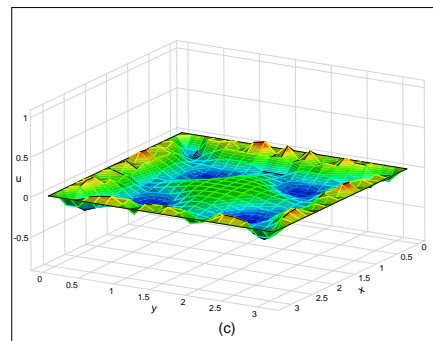
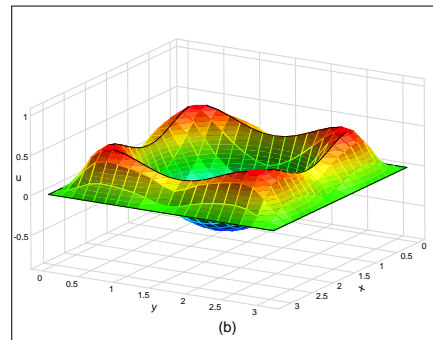
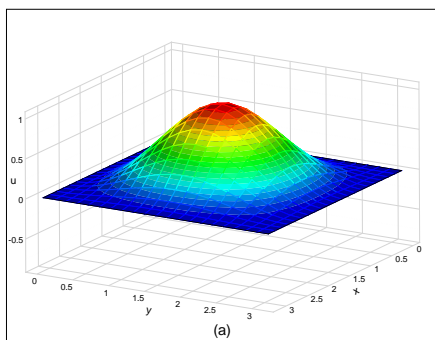


Fig. 7: Transversal displacement of plate. (a) $t = 0$ s, (b) $t_{45} \approx 1.414$ s, (c) $t_{140} \approx 4.398$ and (d) $t_{1843} \approx 58.496$ s.

5 Conclusion

The Nakao's method proved to be an efficient method for the demonstration of the exponential decay of the solution of the system of von Kármán. Numerical tests have shown the decay of the mechanical energy of the system.

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