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Convergence Orders in Length Estimation with Exponential Parameterization and ε-Uniformly Sampled Reduced Data

Ryszard Kozera[1,∗], *Lyle Noakes*[2] *and Piotr Szmielew*[1]

[1] Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences - SGGW, Nowoursynowska str. 159, 02-776 Warsaw, Poland
[2] School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley W.A. 6009, Perth, Australia

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Abstract: We investigate the length approximation of the unknown regular curve in arbitrary Euclidean space upon applying a piecewise-quadratic interpolation based on ε-uniformly sampled reduced data in combination with the exponential parameterization. As proved in this paper, similarly to the trajectory estimation, there is a discontinuity in the quality of length estimation with exponential parameterization performing no better than a blind uniform guess for the unknown knots, except for the case of cumulative chords. The theoretical asymptotic estimates established here for length approximation are also experimentally confirmed to be nearly sharp.

Keywords: Curve interpolation, length approximation, numerical analysis, asymptotics, exponential parameterization, ε-uniform samplings

1 Introduction

Reduced data form an ordered collection of $m+1$ points $q_0,q_1,\ldots,q_m$ in Euclidean n-space E^n upon sampling an unknown but sufficiently smooth and regular curve $\gamma:[0,1]\to E^n$ at $0=t_0<t_1<t_2<\ldots<t_m=1$, where the interpolation knots $t_1,t_2,\ldots,t_{m-1}$ are also assumed to be unknown. Here $q_i=\gamma(t_i)$ for $0\le i\le m$ and any interpolation scheme based on reduced data is described as *nonparametric interpolation*. More precisely, the task is to estimate the unknown curve γ (or its length) by a curve $\hat\gamma:[0,1]\to E^m$ such that $\hat\gamma(\hat t_i)=q_i$ for all $i=0,1,\ldots,m$, where $\hat\gamma$ and the $\hat t_i$ are computed from $q_0,q_1,\ldots,q_m$ exclusively. To emphasize that the knots $\{t_i\}_{i=0}^m$ are not given, we also call reduced data $Q_m=\{q_i\}_{i=0}^m$ as *nonparametric data*. Some applications of nonparametric data interpolation in computer vision, computer graphics, engineering or physics are given e.g. in [1], [2], [3], [4] or [5].

In order to approximate the length (see [6])

$$d(\gamma)=\int_0^1\|\dot\gamma(t)\|dt$$

of the interpolated curve γ it is necessary to assume that our samplings $\{t_i\}_{i=0}^m$ satisfy the so-called *admissibility condition*:

$$\lim_{m\to\infty}\delta_m=0,\quad\text{where}\quad\delta_m=\max_{0\le i\le m-1}(t_{i+1}-t_i).\qquad(1)$$

Remark 1. Recall that, a family $\{f_{\delta_m},\delta_m>0\}$ of functions $f_{\delta_m}:[0,T]\to E$ is said to be of *order* $O(\delta_m^p)$ when there is a constant $K>0$ such that, for some $\bar\delta$, $|f_{\delta_m}(t)|<K\delta_m^p$, for all $\delta_m\in(0,\bar\delta)$ and all $t\in[0,T]$. For the family of vector-valued functions $F_{\delta_m}:[0,T]\to E^n$ (e.g. for $T=1$ and $F_{\delta_m}=\hat\gamma\circ\psi-\gamma$; here $\hat\gamma$ and a special $\psi:[0,1]\to[0,1]$ depend on δ_m) we write that $F_{\delta_m}=O(\delta_m^\alpha)$ when $\|F_{\delta_m}\|_\infty=O(\delta_m^\alpha)$, where $\|F_{\delta_m}\|_\infty=\sup_{t\in[0,T]}\|F_{\delta_m}(t)\|$ and $\|\cdot\|$ denotes the Euclidean norm. The latter holds if there exists constant $\hat K>0$ such that for some $\hat\delta>0$ we have $\|F_{\delta_m}\|\le\hat K\delta_m^\alpha$, for all $\delta_m\in(0,\hat\delta)$ and all $t\in[0,T]$. Here K and $\hat K$ depend on curve γ and possibly on samplings $\{t_i\}_{i=0}^m$. Note that for F_{δ_m} continuous over compact $[0,T]$ we have $\|F_{\delta_m}\|_\infty=\max_{t\in[0,T]}\|F_{\delta_m}(t)\|$. In case of length

∗ Corresponding author e-mail: ryszard_kozera@sggw.pl ryszard.kozera@gmail.com

approximation we set $f_{\delta_m} = d(\hat{\gamma}) - d(\gamma)$, where f_{δ_m} depends only on δ_m but not on $t \in [0,T]$. $\square$

We introduce now two special subfamilies of admissible samplings (1).

Definition 1. *The sampling $\{t_i\}_{i=0}^m$ is* more-or-less uniform *(see e.g. [3], [7] or [8]) when, for some $\beta \in (0,1]$, and all sufficiently large m and all $i = 1,2,\ldots,m$, we have:*

$$\beta\delta_m \leq t_i - t_{i-1} \leq \delta_m. \tag{2}$$

Alternatively, for sampling (2) we have

$$\frac{\beta_0}{m} \leq t_i - t_{i-1} \leq \frac{\beta_1}{m}, \tag{3}$$

for some $0 < \beta_0 \leq \beta_1$, sufficiently large m and all $i = 1,2,\ldots,m$. Necessarily $\beta_1 \geq 1$, by summing the inequalities (3). $\square$

Note that both inequalities (2) and (3) may hold with different constants β, β_0 and β_1 for various more-or-less uniform samplings.

Definition 2. *For each $\varepsilon > 0$, the sampling $\{t_i\}_{i=0}^m$ is coined as* ε-uniform *(see e.g. [9]) when, for some C^∞ diffeomorphism $\phi : [0,1] \to [0,1]$, sufficiently large m and all $0 \leq i \leq m$, we have:*

$$t_i = \phi\left(\frac{i}{m}\right) + O\left(\frac{1}{m^{1+\varepsilon}}\right). \tag{4}$$

The ε-uniformity (4) is more restrictive than more-or-less uniformity (2) (see [3]). Since by (1), $m\delta_m \geq 1$ the second term in (4) reads also as $O(\delta_m^{1+\varepsilon})$. $\square$

Again both ϕ and the term $O(\delta_m^{1+\varepsilon})$ determine each ε-uniform sampling. One of the most frequently used method to approximate the unknown knots $\{t_i\}_{i=0}^m$ is to invoke the so-called *exponential parametrization* (see e.g. [4]), defined as follows:

Definition 3. *Choose $\lambda \in [0,1]$ and set $\tilde{t}_0 = 0$. Then, inductively, for $1 \leq i \leq m$, we define:*

$$\tilde{t}_i = \tilde{t}_{i-1} + \|q_i - q_{i-1}\|^\lambda. \tag{5}$$

Finally, upon normalization set $\hat{t}_i = \tilde{t}_i/\tilde{t}_m$, for $0 \leq i \leq m$. It is implicitly assumed $q_i \neq q_{i+1}$ so that $\tilde{t}_i < \tilde{t}_{i+1}$. $\square$

Varying the parameter $\lambda \in [0,1]$ affects the shape of curve $\hat{\gamma}$ (see [10]). The choice of $\lambda = 0$ yields uniform $\hat{t}_i = i$, corresponding to *a blind guess* of $\{t_i\}_{i=0}^m$, which does not incorporate the geometrical distribution of the interpolation points $\{q_i\}_{i=0}^m$. The latter is manifested by the following (see [9]):

Theorem 1. Let γ be C^4 and let the unknown $\{t_i\}_{i=0}^m$ be sampled ε-uniformly, where $\varepsilon > 0$. If $\hat{\gamma} = \hat{\gamma}_2$ is constructed using piecewise-quadratic Lagrange interpolation based on $\lambda = 0$ (a blind uniform guess) and $m\delta_m = O(1)$ then, for piecewise-C^∞ re-parameterization $\psi : [0,1] \to [0,1]$ (computed only from data Q_m), we have the asymptotic estimate*:*

$$d(\gamma) = d(\hat{\gamma}_2) + O(\delta_m^{\min\{4,4\varepsilon\}}).$$

$\square$

On the other hand, still in the context of length approximation, a much better estimate of $\{t_i\}_{i=0}^m$ follows from the so-called *cumulative chord parameterization* forming a special case of exponential parameterization (5) with $\lambda = 1$. Indeed by [11] we have:

Theorem 2. Let γ be C^4 and let the unknown $\{t_i\}_{i=0}^m$ be sampled ε-uniformly, where $\varepsilon > 0$. If $\hat{\gamma} = \hat{\gamma}_2$ is constructed using piecewise-quadratic Lagrange interpolation based on $\lambda = 1$ (scaled cumulative chord parameterization) and $m\delta_m = O(1)$ then, for piecewise-C^∞ re-parameterization $\psi : [0,1] \to [0,1]$ (computed only from data Q_m), the sharp asymptotic estimate reads as:

$$d(\gamma) = d(\hat{\gamma}_2) + O(\delta_m^{\min\{4,3+\varepsilon\}}). \tag{6}$$

In fact formula (6), for wider class of arbitrary admissible samplings (1), holds with $O(\delta_m^3)$ error. $\square$

So scaled cumulative chord parametrization performs (with $\lambda = 1$) at least at cubic order of length approximation getting accelerated by $\min\{1,\varepsilon\}$ for ε-uniform samplings. At the other extreme (i.e. for $\lambda = 0$)), the asymptotics for the blind uniform guess of Th. 1 are much worse for small values of ε. Upon collating the above opposite cases, one might expect a steady increase in the exponent of δ_m (or of $1/m$) as λ varies from 0 to 1. Unexpectedly the latter does not occur, as proved in Th. 3 constituting *our main result*. Indeed the following generalization of Th. 1 and Th. 2 holds:

Theorem 3. Let γ be C^4 and the unknown knots $\{t_i\}_{i=0}^m$ be sampled ε-uniformly with $\varepsilon > 0$ and $m\delta_m = O(1)$. For $\hat{\gamma} = \hat{\gamma}_2$ defined as previously as a piecewise-quadratic Lagrange interpolant combined with the exponential parameterization (5) with parameter $\lambda \in [0,1]$ and for some piecewise-quadratic-C^∞ re-parameterization $\psi : [0,1] \to [0,1]$ (computed only from reduced data Q_m) the following holds:

$$d(\gamma) = d(\hat{\gamma}_2) + \begin{cases} O(\delta_m^{\min\{4,4\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{4,3+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^4), & t_i = \frac{i}{m}. \end{cases} \tag{7}$$

$\square$

The last section of this paper supplements Th. 3 with the numerical tests indicating a near sharp character of the asymptotics established in (7). In particular a discontinuity in asymptotic orders (7) at $\lambda = 1$ and their independence on $\lambda \in [0,1)$ and dependence on $\varepsilon \in (0,1]$ is also confirmed.

A similar phenomenon of non-steady jump in asymptotics for trajectory estimation based on exponential parameterization (5), ε-uniform (or more-or-less uniform) samplings (4) (or (2)) and piecewise-quadratic Lagrange interpolation has been recently established in [12] and [13].

2 A proof - asymptotics in length approximation for exponential parameterization and ε-uniform samplings

We pass now to the proof of Th. 3, which claim (7) also entails already established two special cases of $\lambda = 0$ (see Th. 1) and of $\lambda = 1$ (see Th. 2).

Proof. As shown in [13] it is sufficient to prove the asymptotics (7) for $\{\hat{t}_i\}_{i=0}^m$ (instead of using unnormalized knots $\{\tilde{t}_i\}_{i=0}^m$ - see (5)). In addition, one can consider shifted knots according to $\hat{t}-\hat{t}_i$ (over each segment $I_i = [t_i, t_{i+2}]$). Therefore we use the same notation for (5) and (8). Let $\psi_i : [t_i, t_{i+2}] \to [\hat{t}_i, \hat{t}_{i+2}] = \hat{I}_i$ be the quadratic polynomial satisfying interpolation conditions $\psi_i(t_{i+j}) = \hat{t}_{i+j}$, with $j = 0, 1, 2$, where

$$\hat{t}_i = 0, \quad \hat{t}_{i+1} = \|q_{i+1} - q_i\|^\lambda, \qquad \hat{t}_{i+2} = \hat{t}_{i+1} + \|q_{i+2} - q_{i+1}\|^\lambda, \tag{8}$$

for $\lambda \in [0, 1]$. The track-sum of $\{\psi_i\}_{i=0}^{m-2}$ (for $i = 0, 2, 4 \ldots m-2$) defines a continuous piecewise-C^∞ mapping $\psi : [0, 1] \to [0, \hat{T}]$, where $\hat{T} = \hat{t}_m$. Note that by [13] the function ψ_i is asymptotically a re-parameterization for each $\varepsilon > 0$ and $\lambda \in [0, 1]$. Noticeably, the latter may not hold for $\varepsilon = 0$ and $\lambda \in [0, 1)$ (see [12]). The proof of Th. 3 is divided into three steps:

2.1 Step 1: difference between interpolant $\hat{\gamma}_2$ and curve γ

Let the interpolant $\hat{\gamma}_2(\hat{t}_i) = q_i$ be defined as a track-sum of quadratics $\hat{\gamma}_{2,i} : [\hat{t}_i, \hat{t}_{i+2}] \to E^n$ satisfying $\hat{\gamma}_{2,i}(\hat{t}_{i+j}) = q_{i+j}$, for $j = 0, 1, 2$ and $i = 2k$, where $k = 0, 1, \ldots, m/2$ (m is assumed here to be even). By [13], the difference between the interpolant $\hat{\gamma} = \hat{\gamma}_2$ and the unknown curve γ over each I_i (and thus over $[0, 1]$ since ψ_i is a re-parameterization) reads as:

$$f_i(t) = (\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t). \tag{9}$$

Thus as $\hat{\gamma}_{2,i}(\hat{t}_{i+j}) = (\hat{\gamma}_{2,i} \circ \psi_i)(t_{i+j})$ (for $j = 0, 1, 2$) we arrive at

$$f_i(t_{i+j}) = \mathbf{0}. \tag{10}$$

An inspection of the proof of Hadamard's Lemma (see [14]; Part 1, Lemma 2.1) combined with (10) leads to:

$$f_i(t) = (t - t_i)(t - t_{i+1})(t - t_{i+2}) g_i(t), \tag{11}$$

where $g_i(t) = O(f_i^{(3)}(t))$, uniformly over compact I_i and $g_i \in C^1$. Consequently, since both ψ_i and $\hat{\gamma}_{2,i}$ are quadratics, the chain rule applied to $f_i^{(3)}$ (see also (9)) yields[1]:

$$g_i(t) = O(\hat{\gamma}''_{2,i}(\hat{t})) \cdot O(\psi_i^{(1)}(t)) \cdot O(\psi_i^{(2)}(t)) + O(1). \tag{12}$$

By [13] each contributing term $O(\hat{\gamma}''_{2,i}(\hat{t}))$, $O(\psi_i^{(1)}(t))$ and $O(\psi_i^{(2)}(t))$ occurring in (12) reads as:

$$\hat{\gamma}''_{2,i}(\hat{t}) = \begin{cases} O(\delta_m^{\min\{2-2\lambda, 1+\varepsilon-2\lambda\}}), & \lambda \in [0,1); \\ O(\delta_m^{2-2\lambda}), & \lambda = 1 \text{ or } t_i = \frac{i}{m}, \end{cases} \tag{13}$$

for $\hat{t} \in \hat{I}_i$ and

$$\psi_i^{(1)}(t) = \begin{cases} O(\delta_m^{-1+\lambda}), & \lambda \in [0,1); \\ \delta_m^{-1+\lambda} + O(\delta_m^{1+\lambda}), & t_i = \frac{i}{m}, \end{cases} \tag{14}$$

together with

$$\psi_i^{(2)}(t) = \begin{cases} O(\delta_m^{\min\{-1+\lambda, -2+\lambda+\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{2, 1+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^{1+\lambda}), & t_i = \frac{i}{m}, \end{cases} \tag{15}$$

for $t \in I_i$. Combining (12) with (13), (14) and (15) yields:

$$g_i(t) = \begin{cases} O(\delta_m^{\min\{0, -2+2\varepsilon\}}), & \lambda \in [0,1); \\ O(1), & \lambda = 1 \text{ or } t_i = \frac{i}{m}. \end{cases} \tag{16}$$

Finally, upon coupling (11) with (16) we arrive at the sharp asymptotics for the trajectory estimation (see [13]):

$$f_i(t) = \begin{cases} O(\delta_m^{\min\{3, 1+2\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^3), & \lambda = 1 \text{ or } t_i = \frac{i}{m}, \end{cases} \tag{17}$$

for $t \in I_i$ and $\hat{t} \in \hat{I}_i$.

2.2 Step 2: difference between $\hat{\gamma}'_2$ and $\gamma^{(1)}$

Furthermore, upon differentiating (11) we obtain (for any $t \in I_i$):

$$f_i^{(1)}(t) = O(\delta_m^2) \cdot g_i(t) + O(\delta_m^3) \cdot g_i^{(1)}(t). \tag{18}$$

Hadamard's Lemma together with the chain rule yield:

$$\begin{aligned} g_i^{(1)}(t) &= O((\hat{\gamma}_{2,i} \circ \psi_i - \gamma)^{(4)}(t)) \\ &= O(\gamma''_{2,i}(\hat{t})(\psi_i^{(2)}(t))^2 - \gamma^{(4)}(t)) \end{aligned} \tag{19}$$

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[1] Derivatives over $\hat{t}$ are denoted by apostrophes, whereas calculated over t use superscript notation.

as both $\gamma_{2,i}$ and ψ_i are quadratics. Thus since $\gamma\in C^4$ over compact $[0,1]$, once we combine (13), (15) and (19) we obtain (over I_i) the following:

$$g_i^{(1)}(t)=\begin{cases} O(\delta_m^{\min\{2-2\lambda,1+\varepsilon-2\lambda\}}) \\ \cdot O(\delta_m^{\min\{-2+2\lambda,-4+2(\varepsilon+\lambda)\}})+O(1), & \lambda\in[0,1);\\ O(1)\cdot O(\delta_m^{\min\{4,2+2\varepsilon\}})+O(1), & \lambda=1;\\ O(\delta_m^{2-2\lambda})\cdot O(\delta_m^{2+2\lambda})+O(1), & t_i=\frac{i}{m}.\end{cases} \tag{20}$$

Taking into account that

$$\min\{2-2\lambda,1+\varepsilon-2\lambda\}+\min\{-2+2\lambda,-4+2\varepsilon+2\lambda\}$$
$$=\begin{cases}-3+3\varepsilon, & 0<\varepsilon<1;\\ 0, & \varepsilon\geq 1,\end{cases}$$

formula (20), for $t\in I_i$, reads as:

$$g_i^{(1)}(t)=\begin{cases} O(\delta_m^{-3+3\varepsilon}), & 0<\varepsilon<1,\ \lambda\in[0,1);\\ O(1), & \varepsilon\geq 1,\ \lambda\in[0,1);\\ O(1), & \lambda=1 \text{ or } t_i=\frac{i}{m};\end{cases}$$
$$=\begin{cases} O(\delta_m^{\min\{0,-3+3\varepsilon\}}), & \lambda\in[0,1);\\ O(1), & \lambda=1 \text{ or } t_i=\frac{i}{m}.\end{cases} \tag{21}$$

The latter combined with (16) and (18) renders (over I_i):

$$f_i^{(1)}(t) = O(\delta_m^2)\cdot\begin{cases} O(\delta_m^{\min\{0,-2+2\varepsilon\}}), & \lambda\in[0,1);\\ O(1), & \lambda=1 \text{ or } t_i=\frac{i}{m};\end{cases}$$
$$+\ O(\delta_m^3)\cdot\begin{cases} O(\delta_m^{\min\{0,-3+3\varepsilon\}}), & \lambda\in[0,1);\\ O(1), & \lambda=1 \text{ or } t_i=\frac{i}{m};\end{cases}$$
$$=\begin{cases} O(\delta_m^{\min\{2,2\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^2), & \lambda=1 \text{ or } t_i=\frac{i}{m}.\end{cases} \tag{22}$$

2.3 Step 3: asymptotics in length approximation

Let $V^{\perp}_{\gamma^{(1)}(t)}$ denote the orthogonal complement of the space spanned by $\gamma^{(1)}(t)$. As γ is regular and can be parameterized by arc-length we have $\|\gamma^{(1)}(t)\|=1$ and therefore over I_i:

$$(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)=\langle(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t),\gamma^{(1)}(t)\rangle\gamma^{(1)}(t)+v(t), \tag{23}$$

where $v(t)$ is the orthogonal projection of $(\gamma_{2,i}\circ\psi_i)^{(1)}(t)$ onto $V^{\perp}_{\gamma^{(1)}(t)}$. Here $\langle\cdot,\cdot\rangle$ denotes a standard Euclidean dot product in E^n. By (9) $(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)=f_i^{(1)}(t)+\gamma^{(1)}(t)$ which combined with (23) and $\|\gamma^{(1)}(t)\|=1$ results in

$$(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)=(1+\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle\gamma^{(1)}(t)+v(t). \tag{24}$$

Hence the latter coupled with (22) leads to (for $t\in I_i$):

$$v(t)=f_i^{(1)}(t)-\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle\gamma^{(1)}(t)$$
$$=\begin{cases} O(\delta_m^{\min\{2,2\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^2), & \lambda=1,\text{ or } t_i=\frac{i}{m}.\end{cases} \tag{25}$$

Subsequently, again by (24) we have:

$$\|(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)\|=\sqrt{A}=\sqrt{B}, \tag{26}$$

where

$$A=(1+\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle)^2+\|v(t)\|^2$$
$$+2(1+\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle)\langle\gamma^{(1)}(t),v(t)\rangle,$$
$$B=1+2\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle+(\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle)^2+\|v(t)\|^2$$

as $\langle\gamma^{(1)}(t),v(t)\rangle=0$. By Taylor expansion we have $h(x)=(1+x)^{1/2}=1+(1/2)x+O(x^2)$ for any x separated from -1. Combining (22) with (25) leads to:

$$x=2\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle+(\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle)^2+\|v(t)\|^2$$
$$=\begin{cases} O(\delta_m^{\min\{2,2\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^2), & \lambda=1 \text{ or } t_i=\frac{i}{m},\end{cases} \tag{27}$$

which is asymptotically separated from -1. Hence by (26), (27) and Taylor expansion of $h(x)$ we arrive at:

$$\|(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)\|=1+\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle$$
$$+\begin{cases} O(\delta_m^{\min\{4,4\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^4), & \lambda=1 \text{ or } t_i=\frac{i}{m}.\end{cases}$$

Thus over each sub-interval I_i (as $\|\gamma^{(1)}(t)\|=1$) we obtain:

$$\|(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)\|-\|\gamma^{(1)}(t)\|=\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle$$
$$+\begin{cases} O(\delta_m^{\min\{4,4\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^4), & \lambda=1 \text{ or } t_i=\frac{i}{m}.\end{cases}$$

Subsequently, over each I_i we have:

$$J_i=\int_{t_i}^{t_{i+2}}(\|(\hat{\gamma}_{2,i}\circ\psi_i)^{(1)}(t)\|-\|\gamma^{(1)}(t)\|)dt$$
$$=\int_{t_i}^{t_{i+2}}\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle dt$$
$$+\begin{cases} \int_{t_i}^{t_{i+2}}O(\delta_m^{\min\{4,4\varepsilon\}})dt, & \lambda\in[0,1);\\ \int_{t_i}^{t_{i+2}}O(\delta_m^4)dt, & \lambda=1 \text{ or } t_i=\frac{i}{m}.\end{cases} \tag{28}$$

The first integral in (28) (denoted as $J_{1,i}$) upon integration by parts and by (11) reads as:

$$J_{1,i}=\int_{t_i}^{t_{i+2}}\langle f_i^{(1)}(t),\gamma^{(1)}(t)\rangle dt$$
$$=\langle f_i(t),\gamma^{(1)}(t)\rangle|_{t_i}^{t_{i+2}}-\int_{t_i}^{t_{i+2}}\langle f_i(t),\gamma^{(2)}(t)\rangle dt$$
$$=-\int_{t_i}^{t_{i+2}}\langle f_i(t),\gamma^{(2)}(t)\rangle dt$$

$$= -\int_{t_i}^{t_{i+2}} (t-t_i)(t-t_{i+1})(t-t_{i+2})\langle g_i(t),\gamma^{(2)}(t)\rangle dt, \tag{29}$$

since at interpolation points q_i and q_{i+2} we have $f_i(t_i) = f_i(t_{i+2}) = \mathbf{0}$. In addition as $g_i^{(1)} \in C^1$ (see (11)) the function $r_i(t) = \langle g_i(t),\gamma^{(2)}(t)\rangle$ satisfying $\langle f_i(t),\gamma(t)\rangle = (t-t_i)(t-t_{i+1})(t-t_{i+2})r_i(t)$ is of class C^1 and thus $r_i(t) = r_i(t_i) + (t-t_i)r_i^{(1)}(\xi_i)$ (for $t \in I_i$ and some $\xi_i \in I_i$). Therefore (29) reads as:

$$J_{1,i} = -\langle g_i(t_i),\gamma^{(2)}(t_i)\rangle \cdot \int_{t_i}^{t_{i+2}} (t-t_i)(t-t_{i+1})(t-t_{i+2})dt$$
$$- \int_{t_i}^{t_{i+2}} (t-t_i)^2(t-t_{i+1})(t-t_{i+2})\langle g_i^{(1)}(\xi_i),\gamma^{(2)}(\xi_i)\rangle dt$$
$$- \int_{t_i}^{t_{i+2}} (t-t_i)^2(t-t_{i+1})(t-t_{i+2})\langle g_i(\xi_i),\gamma^{(3)}(\xi_i)\rangle dt. \tag{30}$$

Again, invoking integration by parts results in $\int_{t_i}^{t_{i+2}}(t-t_i)(t-t_{i+1})(t-t_{i+2})dt$

$$= \int_{t_i}^{t_{i+2}} \frac{(t-t_i)^2(t_{i+1}-t)}{2}dt + \int_{t_i}^{t_{i+2}} \frac{(t-t_i)^2(t_{i+2}-t)}{2}dt$$
$$= \frac{(t_i-t_{i+2})^3(t_{i+2}-t_{i+1})}{6} + \frac{1}{3}\int_{t_i}^{t_{i+2}}(t-t_i)^3dt$$
$$= \frac{(t_i-t_{i+2})^3}{12}(t_{i+2}-2t_{i+1}+t_i)$$
$$= O(\delta_m^3)O(\delta_m^{\min\{2,1+\varepsilon\}})$$
$$= O(\delta_m^{\min\{5,4+\varepsilon\}}), \tag{31}$$

where in the latter ε-uniformity (4) together with Taylor expansion yield $t_{i+2} - 2t_{i+1} + t_i = O(\delta_m^{\min\{2,1+\varepsilon\}})$. For $t_i = i/m$ uniform the last integral vanishes as $t_{i+2} - 2t_{i+1} + t_i = 0$. Note that $\min\{0,-3+3\varepsilon\} \le \min\{0,-2+2\varepsilon\}$ for $\varepsilon > 0$. Consequently, by (16), (21) and (31) we obtain in (30):

$$J_{1,i} = \begin{cases} O(\delta_m^{\min\{5,4+\varepsilon\}}), & t_i \neq i/m; \\ 0, & t_i = \frac{i}{m}; \end{cases}$$
$$\cdot \begin{cases} O(\delta_m^{\min\{0,-2+2\varepsilon\}}), & \lambda \in [0,1); \\ O(1), & \lambda = 1 \text{ or } t_i = \frac{i}{m}; \end{cases}$$
$$+O(\delta_m^5) \cdot \begin{cases} O(\delta_m^{\min\{0,-3+3\varepsilon\}}), & \lambda \in [0,1); \\ O(1), & \lambda = 1 \text{ or } t_i = \frac{i}{m}; \end{cases}$$
$$= \begin{cases} O(\delta_m^{\min\{5,2+3\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^5), & t_i = \frac{i}{m}. \end{cases} \tag{32}$$

The asymptotics of the second integral in (28) (denoted as $J_{2,i}$) amounts to:

$$J_{2,i} = \begin{cases} O(\delta_m^{\min\{5,1+4\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^5), & \lambda = 1 \text{ or } t_i = \frac{i}{m}. \end{cases} \tag{33}$$

Upon recalling that $J_i = J_{1,i} + J_{2,i}$, the combination of (29), (32), (33) leads to (over each I_i):

$$\int_{t_i}^{t_{i+2}} (\|(\hat{\gamma}_2 \circ \psi_i)^{(1)}(t)\| - \|\gamma^{(1)}(t)\|)dt$$
$$= \begin{cases} O(\delta_m^{\min\{5,2+3\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^5), & t_i = \frac{i}{m}; \end{cases}$$
$$+ \begin{cases} O(\delta_m^{\min\{5,1+4\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^5), & \lambda = 1; \\ O(\delta_m^5), & t_i = \frac{i}{m}; \end{cases}$$
$$= \begin{cases} O(\delta_m^{\min\{5,1+4\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^5), & t_i = \frac{i}{m}, \end{cases} \tag{34}$$

as $1 + 4\varepsilon \le 2 + 3\varepsilon$, for $0 < \varepsilon \le 1$ and $5 \le \min\{1+4\varepsilon, 2+3\varepsilon\}$, for $\varepsilon \ge 1$. Finally, by (34) the estimate for error in length approximation stands as:

$$d(\hat{\gamma}_2) - d(\gamma) = \sum_{i=0}^{m=2k} \int_{t_i}^{t_{i+2}} (\|(\hat{\gamma}_2 \circ \psi_i)^{(1)}(t)\| - \|\gamma^{(1)}(t)\|)dt$$
$$= \frac{m}{2} \cdot \begin{cases} O(\delta_m^{\min\{5,1+4\varepsilon\}}), & \lambda \in [0,1); \\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \lambda = 1; \\ O(\delta_m^5), & t_i = \frac{i}{m}. \end{cases} \tag{35}$$

Formula (7) follows from (35) upon resorting to the drawn assumption $m\delta_m = O(1)$. The proof of Th. 3 is henceforth completed. □

3 Experiments

In the closing section of this paper we verify numerically the asymptotics established in Th. 3. More specifically, we test the sharpness of (7), its independence from $\lambda \in [0,1)$ and a discontinuity in asymptotical convergence orders from Th. 3 for length estimation at $\lambda = 1$.

The tests are performed in *Mathematica 9.0* (see [15]) and are carried out on a 2.4GHZ Intel Core 2 Duo computer with 8GB RAM. Observe that since $T = \sum_{i=1}^{m}(t_{i+1} - t_i) \le m\delta$ we have $m^{-\beta} = O(\delta_m^{\beta})$, for $\beta > 0$. Hence the investigated asymptotics in terms of $O(\delta_m^{\beta})$ can be examined in terms of $O(1/m^{\beta})$ asymptotics. Note that for a parametric regular curve $\gamma : [0,1] \to E^n$, $\lambda \in [0,1]$ with m varying between $m_{min} \le m \le m_{max}$ the i-th component of the error for $d(\gamma)$ estimation reads as:

$$E_m^i = \int_{\hat{t}_i}^{\hat{t}_{i+2}} \|\hat{\gamma}_2^{i'}(\hat{s})\| d\hat{s} - \int_{t_i}^{t_{i+2}} \|\gamma^{(1)}(s)\| ds$$
$$= d(\hat{\gamma}_2^i) - d(\gamma|_{[t_i,t_{i+2}]}),$$

where $\hat{\gamma}_2^i : [t_i,t_{i+2}] \to E^n$ is a Lagrange quadratic satisfying $\hat{\gamma}_2|_{[t_i,t_{i+2}]} = \hat{\gamma}_2^i$. Recall that by [13], the function $\psi_i : [t_i,t_{i+2}] \to [\hat{t}_i,\hat{t}_{i+2}]$ (see also the proof of Th. 3) forms a re-parameterization for each $\varepsilon > 0$ and for $\lambda \in [0,1]$. Even more by [11], the quadratic ψ_i defines also a re-parameterization for all admissible samplings (1) in case when $\lambda = 1$. Obviously, the quantity E_m defined here as a sum of E_m^i represents the searched error $d(\hat{\gamma}_2) - d(\gamma) = O(\delta_m^{\beta(\lambda)})$ in length approximation of curve γ. From the set of *absolute errors* $\{E_m\}_{m=m_{min}}^{m_{max}}$ the numerical estimate $\bar{\beta}(\lambda)$ of genuine order $\beta(\lambda)$ is next computed by using *a linear regression* applied to the pair of points $\mathscr{A} = \{(\log(m), -\log(E_m))\}_{m=m_{min}}^{m_{max}}$ (see also [3]). Since piece-wisely $deg(\hat{\gamma}_2) = 2$ the number of interpolation points $\{q_i\}_{i=0}^m$ is assumed to be odd i.e. $m = 2k$ is even as indexing runs over $0 \le i \le m$. The *Mathematica* built-in functions *LinearModelFit* yields the coefficient $\hat{\beta}(\lambda)$ from the computed regression line $y(x) = \hat{\beta}(\lambda)x + b$ based on $\mathscr{A}$. Our experiments use the following two testing families of ε-uniform samplings:

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{m^{1+\varepsilon}}, \tag{36}$$

and

$$t_i = \begin{cases} \frac{i}{m}, & \text{if } i \text{ even;} \\ \frac{i}{m} + \frac{1}{2m^{1+\varepsilon}}, & \text{if } i = 4k+1; \\ \frac{i}{m} - \frac{1}{2m^{1+\varepsilon}}, & \text{if } i = 4k+3. \end{cases} \tag{37}$$

For both samplings (36) and (37) we set $t_0 = 0$ and $t_m = 1$, and hence $t_i \in [0,1]$. Some examples of plotting the distribution of $\{\gamma(t_i)\}_{i=0}^m$ for the above ε-uniform samplings (36) and (37) are presented e.g. in [3]. We pass now to the experiments designed to test numerically the convergence orders established in (7).

3.1 Length estimation for reduced data from planar curves

The first test is performed for length estimation of the cubic curve in E^2.

Example 1. Consider now the following regular *cubic curve* $\gamma_c : [0,1] \to E^2$: $\gamma_c(t) = (\pi t, (\pi t+1)^3(\pi+1)^{-3})$, sampled according to either (36) or (37). For the first sampling we set $m_{min} = 40$ and $m_{max} = 200$, whereas for the second one $m_{min} = 100$ and $m_{max} = 120$. The corresponding length of γ_c reads as $d(\gamma_c) = 3.452$. The linear regression applied to $m_{min} \le m \le m_{max}$ renders computed $\hat{\beta}_\varepsilon(\lambda)$ approximating $\beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ (for $\varepsilon > 0$), which are listed in Table 1 and Table 2.

The sharpness of Th. 3 for either $\lambda = 1$ with $\varepsilon > 0$ or for $\lambda \in [0,1)$ with $\varepsilon = 1$ is confirmed in Table 1 (see the last row or the last column, respectively). On the other hand, Table 2 shows more clearly the discontinuity in convergence orders $\beta_\varepsilon(\lambda)$ at $\lambda = 1$ for $\varepsilon \in (0,1)$, predicted by Th. 3. Table 2 also underlines (see each column) the expected independence of $\beta_\varepsilon(\lambda)$ on λ once ε is fixed. Both Tables 1 and 2 also indicate that for λ fixed, increasing ε from 0 to 1, makes $\hat{\beta}_\varepsilon(\lambda)$ bigger and closer to 4. Upon satisfying inequality $\varepsilon \ge 1$ the quartic orders in convergence are reached. The latter coincides with the asymptotics held by $\{t_i\}_{i=0}^m$ uniform. However, the results obtained in both Tables 1 and 2 for $\lambda \in [0,1)$ suggest faster convergence rates $\beta_\varepsilon(\lambda)$ as compared to (7). In case of $\{t_i\}_{i=0}^m$ uniform the computed convergence orders $\hat{\beta} \approx 4$ (see (7)) as for $\lambda \in \{0, 0.1, 0.33, 0.5, 0.7, 0.9, 1\}$ they are $\{4.036, 4.044, 4.036, 4.037, 4.037, 4.039, 4.074\}$, respectively. Evidently, the sharpness of (7) for uniform sampling is experimentally confirmed. □

Table 1: Estimated $\hat{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_c and sampling (36) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in (0,2]$.

λ	$\varepsilon=0.1$	$\varepsilon=0.33$	$\varepsilon=0.5$	$\varepsilon=0.7$	$\varepsilon=0.9$	$\varepsilon=1.0$	$\varepsilon=2.0$
$\beta_\varepsilon(\lambda)$	**0.400**	**1.320**	**2.000**	**2.800**	**3.600**	**4.000**	**4.000**
0.00	2.597	2.800	3.064	3.456	3.857	4.043	4.095
0.10	2.601	4.977	3.686	4.033	3.981	4.058	4.001
0.33	2.183	2.640	2.664	3.333	3.702	3.963	3.890
0.50	2.196	2.646	2.971	3.346	3.730	3.992	3.909
0.70	2.196	2.644	2.969	3.340	3.718	3.982	3.902
0.90	2.194	2.629	2.936	3.265	3.363	4.139	3.814
$\beta_\varepsilon(1)$	**3.100**	**3.330**	**3.500**	**3.700**	**3.900**	**4.000**	**4.000**
1.00	3.111	3.364	3.541	3.749	3.954	4.056	7.070

Table 2: Estimated $\hat{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_c and sampling (37) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in (0,2]$.

λ	$\varepsilon=0.1$	$\varepsilon=0.33$	$\varepsilon=0.5$	$\varepsilon=0.7$	$\varepsilon=0.9$	$\varepsilon=1.0$	$\varepsilon=2.0$
$\beta_\varepsilon(\lambda)$	**0.400**	**1.320**	**2.000**	**2.800**	**3.600**	**4.000**	**4.000**
0.00	2.379	2.743	3.051	3.449	3.865	4.067	4.044
0.10	2.555	1.196	2.862	3.287	3.480	3.374	4.001
0.33	2.216	2.283	3.026	3.431	3.835	4.025	4.037
0.50	2.221	2.684	3.027	3.432	3.837	4.029	4.037
0.70	2.221	2.684	3.028	3.431	3.836	4.027	4.036
0.90	2.223	2.684	2.025	3.426	3.823	4.008	4.032
$\beta_\varepsilon(1)$	**3.100**	**3.330**	**3.500**	**3.700**	**3.900**	**4.000**	**4.000**
1.00	4.157	4.404	4.578	4.773	4.971	5.078	3.974

We pass now to the second example which tests the asymptotics in (7) on a planar spiral.

Example 2. Consider now *a planar regular spiral* $\gamma_{sp1} : [0,1] \to E^2$ defined here as $\gamma_{sp1}(t) = ((t+0.2)\cos(\pi(1-t)), (t+0.2)\sin(\pi(1-t)))$. To estimate $\beta_\varepsilon(\lambda)$ a linear regression is applied again to $100 = m_{min} \le m \le m_{max} = 120$ and to ε-uniform samplings (37). The pertinent numerical results for $\hat{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda)$ are listed in Table 3. Similarly as in Example 1, Table 3 shows that the computed orders $\hat{\beta}_\varepsilon(\lambda)$ exceed (for $\lambda \in [0,1)$ and $\varepsilon \in (0,1)$) the convergence rates established in Th. 3. However, still the inequality $\hat{\beta}_\varepsilon(\lambda) \ge \beta_\varepsilon(\lambda)$ resulting from Table 3 is

Table 3: Estimated $\hat{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_{sp1} and sampling (37) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in (0,2]$.

λ	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	**0.400**	**1.320**	**2.000**	**2.800**	**3.600**	**4.000**	**4.000**
0.00	2.368	2.704	2.914	3.054	4.110	4.044	4.035
0.10	2.522	2.827	3.618	3.987	4.039	4.041	4.035
0.33	2.220	2.706	3.103	3.639	3.992	4.037	4.035
0.50	2.224	2.701	3.089	3.611	3.983	4.036	4.035
0.70	2.224	2.704	3.095	3.623	3.987	4.037	4.035
0.90	2.233	2.728	3.160	3.729	4.011	4.038	4.035
$\beta_\varepsilon(1)$	**3.100**	**3.330**	**3.500**	**3.700**	**3.900**	**4.000**	**4000**
1.00	4.102	4.100	4.071	4.052	4.043	4.040	4.035

Table 4: Estimated $\hat{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \{4, 4\varepsilon\}$ for γ_h and sampling (36) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in (0,2]$.

λ	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	**0.400**	**1.320**	**2.000**	**2.800**	**3.600**	**4.000**	**4.000**
0.00	2.516	2.685	2.755	5.980	4.066	4.037	4.034
0.10	2.527	3.021	3.749	4.002	4.033	4.034	4.034
0.33	2.191	2.694	3.128	3.706	4.001	4.031	4.033
0.50	2.197	2.686	3.104	3.672	3.994	4.031	4.033
0.70	2.197	2.689	3.114	3.687	3.997	4.031	4.033
0.90	2.212	2.731	3.217	3.803	4.015	4.034	4.033
$\beta_\varepsilon(1)$	**3.100**	**3.330**	**3.500**	**3.700**	**3.900**	**4.000**	**4.000**
1.00	4.056	4.079	4.050	4.037	4.034	4.034	4.033

consistent with (7). On the other hand, the sharpness of (7) is confirmed for $\lambda = 1$ with $\varepsilon > 0$ or for $\lambda \in [0,1)$ with $\varepsilon \geq 1$. Visibly, each column (i.e. with fixed ε) and $\lambda \in (0,1)$ shows almost equal $\hat{\beta}_\varepsilon(\lambda)$. Similarly, each row of Table 3 indicates the increasing tendency in values of $\hat{\beta}_\varepsilon(\lambda)$ while varying ε from 0 to 1. Once $\varepsilon = 1$ is reached, the orders $\hat{\beta}_{\varepsilon=1,2}(\lambda) \approx 4$ are attained. In addition, the expected discontinuity of $\beta_\varepsilon(\lambda)$ at $\lambda = 1$ is also manifested upon inspecting the last three rows of Table 3. Finally, the case when $\{t_i\}_{i=0}^m$ is uniform renders $\hat{\beta} \approx 4$ (see (7)) for $\lambda \in \{0, 0.1, 0.33, 0.5, 0.7, 0.9, 1\}$ equal to $\{4.0352, 4.0351, 4.0350, 4.0349, 4.0348, 4.0348, 4.0348\}$, respectively. The sharpness of (7) for uniform sampling is thus also experimentally confirmed. □

3.2 Length estimation for reduced data from spatial curves

The last experiment verifies the asymptotics in (7) for curves in E^3. However, it should be emphasized here, that the Th. 3 applies to an arbitrary multidimensional reduced data Q_m by sampling regular curve in E^n according to ε-uniform fashion. More examples testing (7) for $n = 2,3$ (including segmentation of medical images) as well as some applications of fitting reduced data for $n > 3$ are discussed e.g. in [2] or [10].

Example 3. Consider *an elliptical helix* $\gamma_h : [0,1] \to E^3$, where $\gamma_h(t) = ((3/2)\cos(2\pi t), \sin(2\pi t), (2\pi t)/4)$, with $t \in [0,1]$ and sampled ε-uniformly in accordance with (36). The respective length of γ_h is $d(\gamma_h) = 8.090$. To approximate $\beta_\varepsilon(\lambda)$, a linear regression is used with $m_{min} = 100 \leq m \leq m_{max} = 120$. The computed estimates $\hat{\beta}_\varepsilon(\lambda)$ approximating $\beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ are presented in Table 4. Visibly the results are consistent with the asymptotics from Th. 3, though the examined sharpness of (7) is again not experimentally confirmed (for $\lambda \in [0,1)$). However, the expected discontinuity in $\beta(\lambda)$ at $\lambda = 1$ together with its independence on $\lambda \in [0,1)$ are again demonstrated in Table 4. □

4 Conclusion

In this work we generalize the existing results for length estimation of the regular curve in E^n via *piecewise-quadratic interpolation based on ε-uniformly sampled reduced data Q_m*. We analyze the dynamics of the above asymptotics once *the exponential parameterization* (5) depending on parameter $\lambda \in [0,1]$ is invoked. Such parameterization of the reduced data Q_m is commonly used in computer graphics for curve modeling - see e.g. [4]. The case when $\lambda = 0$ is studied in [9] yielding the upper bounds on convergence rates equal to $\beta_\varepsilon(0) = \min\{4, 4\varepsilon\}$. At the other extreme when $\lambda = 1$ the so-called cumulative chords are used. The latter is already analyzed in [3] or [11] yielding sharp convergence orders $\beta_\varepsilon(1) = \{4, 3+\varepsilon\}$ in length approximation. In this paper we extend these two results to all parameters $\lambda \in [0,1]$ (determining the full class of exponential parameterizations (5)) given reduced data Q_m are ε-uniformly sampled. As established and numerically verified in this paper the upper bounds on convergence rates $\beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ are independent on $\lambda \in [0,1)$ but are sensitive to the variation of $0 < \varepsilon < 1$. In addition, the discontinuity in $\beta(\lambda)$ occurs at $\lambda = 1$ and $0 < \varepsilon < 1$ with the expected jumps from at least 4ε to 4.

A similar phenomenon occurs in asymptotic behaviour of $\alpha(\lambda)$ measuring the convergence orders in trajectory approximation - i.e. determining the following difference $(\hat{\gamma}_2 \circ \psi)(t) - \gamma(t) = O(\delta_m^{\alpha(\lambda)})$ over $[0,T]$. Indeed, as shown in [12], for samplings (2) the sharp estimates $\alpha(\lambda) = 1$ (for all $\lambda \in [0,1)$) and $\alpha(1) = 3$ hold. In addition (see [9]), for samplings (4) the following sharp orders $\alpha_{\varepsilon>0}(\lambda) = \{3, 1+2\varepsilon\}$ (for all $\lambda \in [0,1)$) and $\alpha_{\varepsilon>0}(1) = 3$ are established.

An obvious open problem stemming out of Th. 3 concerns the analysis of $\varepsilon = 0$ case and the issue of sharpness in asymptotics (7). Another possible extension of this work is to invoke smooth interpolation schemes (see [16]) combined with reduced data exponential parameterization (see [4]). Certain clues may be given in [17], where complete C^2 splines are dealt with for $\lambda = 1$, to obtain the fourth orders of convergence in length estimation. The analysis of C^1 interpolation for reduced data with cumulative chords (i.e. again with $\lambda = 1$) can additionally be found in [3] or [18].

Different classes of parameterizations applied mainly on sparse reduced data Q_m (though also equally applicable to the dense Q_m) include *blending parameterization* [19], *monotonicity or convexity preserving ones* [4] or an alternative approach discussed in [20].

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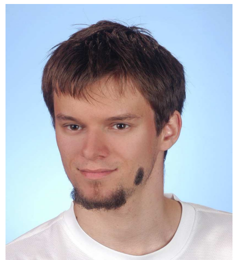
Ryszard Kozera is a Professor at the Faculty of Applied Informatics and Mathematics at Warsaw University of Life Sciences - SGGW. He is also the Adjunct Associate Professor at the School of Information Science and Software Engineering at The University of Western Australia in Perth. He received his M.Sc. degree in Pure Mathematics at Warsaw University, Poland (1985), a Ph.D. degree in Computer Science at Flinders University of South Australia, Adelaide, Australia (1991) and his D.Sc. degree in Computer Science (a Habilitation) at The Silesian University of Technology, Gliwice, Poland (2006). Professor Kozera was awarded three times Alexander von Humboldt Fellowships between 1995-2004. He delivered about 60 seminars at various prestigious international research centers, published 90 refereed journal and conference papers, organized international conferences and participated in the collaborative international grants. He has 21 postgraduate supervisions including Ph.D. and M.Sc. students. Professor Kozera's research interests include computer vision, numerical analysis, optimization, interpolation and approximation, partial differential equations and artificial intelligence. More information can be found under Professor Kozera's homepage (see URL: http://www.wzim.sggw.pl/ryszard_kozera or URL: http://www.csse.uwa.edu.au/˜ryszard).

Lyle Noakes is a Professor and Head of the School of Mathematics and Statistics at The University of Western Australia. He received his BScHons at Otago University, New Zealand (1969), an M.Sc. at Oxford University, UK (1970), and a DPhil. at Oxford University (1974). His doctoral thesis, in algebraic topology, was awarded the top Oxford Senior Mathematics Prize for 1974 and the Johnson Prize. Professor Noakes has 115 refereed publications and 18 postgraduate supervisions, many on applications of differential geometry and topology to problems in approximation theory, mathematical physics, control theory, and computer vision, including work in shape-from-shading, interpolation of scattered data, and the motion of rigid bodies. More information can be found under Professor Noakes' homepage (see URL: http://staffhome.ecm.uwa.edu.au/˜lyle).



Piotr Szmielew received his M.Sc. degree in Computer Science from Warsaw University of Life Sciences - SGGW (2013). He is currently working on completion of his Ph.D. degree in Computer Science. His research interests include various aspects of interpolation, approximation and numerical analysis as well as their applications e.g. in medical image processing.