Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090612

Approximate Analytic Solutions for the Influence of Mass Flux on Mixed Convection in a Saturated Porous Medium

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Received: 12 Feb. 2015, Revised: 12 Apr. 2015, Accepted: 13 Apr. 2015
Published online: 1 Nov. 2015

Abstract: In this paper, we have considered the nonlinear coupled boundary-layer equations that describe the problem of injection or extraction of fluid along the surface of an inclined wall embedded in a saturated porous medium. We obtain very accurate approximate analytic solutions in closed-form by a direct method for the special cases $\lambda = 0$ and $\lambda = 1$. Furthermore, an accurate analytic series solution is obtained by the modified Adomian decomposition method for the two limiting cases of free and forced convection by setting $\frac{Ra_k}{Pe_x} = 0$ and $\frac{Ra_k}{Pe_x} \neq 0$, respectively.

Keywords: Coupled boundary-layer equations, problem of injection, approximate solution, Adomian decomposition method

1 Introduction

Mixed convection flow in porous media has become one of the most important topics in the theory of heat transfer in the last few decades, and has attracted a great deal of interest from researchers in different areas of science, technology and engineering. The main reason for this interest is the diverse, important applications of the subject. Some of the applications are: nuclear waste disposal, electronic cooling, food processing, thermal insulation industries, oil recovery, and many more industrial applications that can be modeled as transport phenomena in porous media. Wooding (1963) was a pioneer in attempting to solve the governing equations of fluid flow and heat transfer in porous media using certain boundary-layer assumptions. Wooding [1], Prats [2] and Sutton [3] investigated free convection in porous media. Haajizadeh and Tien [4] obtained several analytical and numerical results on mixed convection in a horizontal plate. Kwendakwema and Boehm [5] obtained numerical results for mixed convection about a vertical concentric cylinder. Lai, Prasad and Kulacki [6] investigated the problem for mixed convection in a vertical porous layer, and obtained numerical results. Details about the literature concerning this topic including a summary of the numerical and experimental results can be found in the book by Pop and Ingham [7] and also by Nield and Bejan [8]. The first buoyancy convective problem that had been studied is the free convection boundary-layer flow over a vertical flat plate. Stewartson (1958) and Gill (1965) were the first to provide a description of the boundary-layer flow over a horizontal surface. Jones (1973) was the first to study free convection boundary-layer flow near a flat surface. It is worth noting that the study of the convective problem over vertical surfaces has attracted growing attention in the research literature, while rather less attention has been devoted to convection over inclined surfaces.

Mixed convection flows are characterized by the buoyancy parameter $k = \frac{Ra_k}{Pe_x}$, where Ra_k is the Rayleigh number, while Pe_x is the Peclet number. The parameter k measures the effect of the free convection when compared to the forced convection on the fluid flow. The case $k > 0$ corresponds to a heated plate (assisting the forced flow, while $k < 0$ corresponds to a cooled plate (opposing the flow). When $k \to 0$, it means that the forced convection dominates the transport of heat, and when $k \to \infty$ the free convection dominates the transport. It turns out that the

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mixed convection occurs when the buoyancy parameter is bounded from above and from below by certain quantities, say $k_{\max}$ and $k_{\min}$ such that $k_{\min} \leq k \leq k_{\max}$. When the natural buoyancy forces support the forced flow, it will enhance the surface heat transfer. For details about this subject, we refer the reader to [7] and [8].

The governing nonlinear coupled boundary-layer equations are given in [9]- [10] as a system of two coupled nonlinear differential equations

$$f'' = \pm \frac{Ra_k}{Pe_x}\theta', \tag{1}$$

$$\theta'' + \frac{\lambda+1}{2} f\theta' - \lambda f'\theta = 0, \tag{2}$$

subject to the boundary conditions

$$f(0) = f_w,\ f'(\infty) = 1 \tag{3}$$

and

$$\theta(0) = 1,\ \theta(\infty) = 0, \tag{4}$$

where f_w is the extraction or injection parameter. Positive values ($f_w > 0$) determine flows with extraction, $f_w < 0$ represents flows with injection, and the zero value ($f_w = 0$) corresponds to flow along an impermeable wall with zero mass transfer where λ is a constant.

The derivation of Eq. (1)-Eq. (4) when the permeability $\Omega \to \infty$ and the heat source/sink parameter $\gamma = 0$ can be also found in [11]-[12], and other related problems in [13].

Numerical integration using the fourth-order Runge-Kutta method and the shooting technique have previously been proposed to solve this problem for the three special cases of $\lambda = 0$, which corresponds to a uniform free stream flowing along an isothermal vertical wall with the injection rate varying with $x^{-\frac{1}{2}}$, $\lambda = 1$, which corresponds to the stagnation flow normal to a vertical wall with linear temperature variation and a constant injection / extraction rate and $\lambda = \frac{1}{3}$, which corresponds to a free stream flowing over an inclined wall ($\alpha = 45^0$) having constant heat flux where the injection rate varying with $x^{-\frac{1}{3}}$, where α is the angle of inclination [9]-[10]. The limiting cases of free and forced convection are also presented in [1] and [2].

It may, however, be worthwhile if the physical models can be constructed in such a manner that the coupled system can either be solved analytically or transformed into another system in which the equations are decoupled and solved separately.

The purpose of this work is to present a new technique for solving the above system. We propose an algorithm consisting of two steps that will introduce a preliminary exact closed-form solution to the system, followed by a correction to that solution. Approximate analytic solutions are obtained by a direct method for the special cases of $\lambda = 0$ and $\lambda = 1$. Also, an accurate analytic series solution is obtained by the modified Adomian decomposition method [17]-[38]. Both of these techniques lead to very accurate approximate solutions.

2 Approximate analytic solutions

The system (1)-(4) can be transformed into another equivalent system in which the equations are decoupled and combined into one equation. Indeed, integrating Eq. (1) from 0 to η and taking into account the boundary conditions at infinity $f'(\infty) = 1$ and $\theta(\infty) = 0$, we obtain

$$f' = \pm \frac{Ra_k}{Pe_x}\theta + 1. \tag{5}$$

The substitution of Eq. (5) into Eq. (2) gives

$$f''' + \frac{\lambda+1}{2} ff'' - \lambda (f')^2 + \lambda f' = 0. \tag{6}$$

Listed below are several special cases when the third-order nonlinear equation Eq. (6) subject to the boundary conditions (3) is solvable by the proposed direct method.

2.1 Case 1: $\lambda = 1$

Assuming that the exact solution of Eq. (6), where λ is chosen as 1, is in the form

$$f(\eta) = \alpha + \beta e^{-\eta}, \tag{7}$$

where α and β are two nonzero parameters to be determined. This proposed assumption was built on the fact that the vast majority of the obtained solutions are derived in exponential forms. Because we have assumed that Eq. (7) satisfies Eq. (6), then upon substitution and solving the resulting equation, we obtain $\alpha = 2$.

Using the initial condition $f(0) = f_w$, we obtain $\beta = f_w - 2$.

By substituting the calculated values of α and β into Eq. (7), we immediately find that

$$f(\eta) = 2 + (f_w - 2)e^{-\eta}, \tag{8}$$

which is indeed the preliminary exact closed-form solution of Eq. (6). However, this solution does not satisfy the boundary condition at infinity $f'(\infty) = 1$. In view of this development, we will employ this boundary condition together with Eq. (8) to achieve the second goal of our proposed technique by making a correction to our preliminary exact closed-form solution.

Inserting the obtained solution Eq. (8) and its derivative

$$f'(\eta) = -(f_w - 2)e^{-\eta} \tag{9}$$

into Eq. (6), we obtain

$$f''' + [2 + (f_w - 2)e^{-\eta}]f'' = (f_w - 2)^2 e^{-2\eta} + (f_w - 2)e^{-\eta}. \tag{10}$$

Substituting $f'' = g$ into Eq. (10), we obtain

$$g' + [2 + (f_w - 2)e^{-\eta}]g = (f_w - 2)^2 e^{-2\eta} + (f_w - 2)e^{-\eta}, \quad (11)$$

which is a first-order linear differential equation, where its general solution can readily be found as

$$g = (f_w - 2)e^{-\eta} + c_1 e^{-2\eta + (f_w-2)e^{-\eta}}, \quad (12)$$

where c_1 is a constant of integration.
Upon substitution, we have

$$f'' = (f_w - 2)e^{-\eta} + c_1 e^{-2\eta + (f_w-2)e^{-\eta}}. \quad (13)$$

Integrating Eq. (13) with respect to η, we obtain

$$f'(\eta) = -(f_w - 2)e^{-\eta} + c_1 \left(\frac{1}{(f_w - 2)^2} - \frac{e^{-\eta}}{f_w - 2} \right) e^{(f_w-2)e^{-\eta}} + c_2, \quad (14)$$

where c_2 is another constant of integration. Using the boundary condition $f'(\infty) = 1$, we obtain

$$c_1 \frac{1}{(f_w - 2)^2} + c_2 = 1. \quad (15)$$

Next, integrating Eq. (14) with respect to η, we obtain

$$f(\eta) = (f_w - 2)e^{-\eta} + \frac{c_1}{(f_w - 2)^2} \left(e^{(f_w-2)e^{-\eta}} - Ei((f_w - 2)e^{-\eta}) \right) + c_2\eta + c_3, \quad (16)$$

where Ei denotes the well-known Exponential Integral function and c_3 is yet another constant of integration which can be determined from the boundary condition $f(0) = f_w$ as

$$c_3 = 2 - \frac{c_1}{(f_w - 2)^2} \left(e^{(f_w-2)} - Ei(f_w - 2) \right). \quad (17)$$

Consequently, we have

$$\theta(\eta) = \frac{1}{\pm \frac{Ra_k}{Pe_x}} \left[-(f_w - 2)e^{-\eta} + c_1 \left(\frac{1}{(f_w - 2)^2} - \frac{e^{-\eta}}{f_w - 2} \right) e^{(f_w-2)e^{-\eta}} \right] + \frac{1}{\pm \frac{Ra_k}{Pe_x}} (c_2 - 1). \quad (18)$$

The initial condition $\theta(0) = 1$ leads to

$$c_1 \left(\frac{1}{(f_w - 2)^2} - \frac{1}{f_w - 2} \right) e^{(f_w-2)} + c_2 = \pm \frac{Ra_k}{Pe_x} + f_w - 1. \quad (19)$$

Eq. (15) and Eq. (19) compose a system of two linear equations in two unknown, i.e. c_1 and c_2. This system is solvable when the determinant of the coefficient matrix is nonzero, i.e.

$$\triangle = \frac{1}{(f_w - 2)^2} - \left(\frac{1}{(f_w - 2)^2} - \frac{1}{(f_w - 2)} \right) e^{(f_w-2)} \neq 0. \quad (20)$$

Therefore the coefficient values are

$$c_1 = \frac{2 \mp \frac{Ra_k}{Pe_x} - f_w}{\frac{1}{(f_w-2)^2} - \left(\frac{1}{(f_w-2)^2} - \frac{1}{(f_w-2)} \right) e^{(f_w-2)}} \quad (21)$$

and

$$c_2 = \frac{\frac{1}{(f_w-2)^2} \left(\pm \frac{Ra_k}{Pe_x} + f_w - 1 \right) - \left(\frac{1}{(f_w-2)^2} - \frac{1}{(f_w-2)} \right) e^{(f_w-2)}}{\frac{1}{(f_w-2)^2} - \left(\frac{1}{(f_w-2)^2} - \frac{1}{(f_w-2)} \right) e^{(f_w-2)}}. \quad (22)$$

Thus the pair of approximate analytic closed-form solutions $(f(\eta), \theta(\eta))$ of system (1)-(2) subject to the boundary conditions (3)-(4) is finally obtained.

2.2 *Case 2:* $\lambda = 0$

For $\lambda = 0$, Eq. (2) becomes

$$\theta'' + \frac{1}{2} f\theta' = 0. \quad (23)$$

As stated before, we begin first by assuming that the preliminary exact closed-form solutions of the coupled nonlinear system Eq.(1) and Eq.(23) are in the forms

$$f(\eta) = \frac{1}{\alpha\eta + \beta} \quad (24)$$

and

$$\theta(\eta) = \frac{A}{(\alpha\eta + \beta)^2}, \quad (25)$$

where A, α and β are three parameters to be determined. Because we assume that (24) and (25) satisfy Eq. (1) and Eq. (23), then by substituting them into these equations and solving the resulting equations, we obtain

$$\alpha = \frac{1}{6} \quad (26)$$

and

$$A = -\frac{1}{6K}, \text{ where } K = \pm \frac{Ra_k}{Pe_x}. \quad (27)$$

By substituting the calculated values of α and A into Eq. (24) and Eq. (25), we find that

$$f(\eta) = \frac{1}{\frac{1}{6}\eta + \beta} \quad (28)$$

and

$$\theta(\eta) = -\frac{\frac{1}{6K}}{(\frac{1}{6}\eta + \beta)^2}. \quad (29)$$

Using the given boundary conditions $f(0) = f_w$ and $\theta(0) = 1$, we obtain $\beta = \frac{1}{f_w}$ under the suitable condition

$$\frac{1}{f_w^2} + \frac{1}{6K} = 0, \quad (30)$$

which means that $K=-\frac{1}{6f_w^2}<0$. Thus

$$f(\eta)=\frac{1}{\frac{1}{6}\eta+\frac{1}{f_w}} \tag{31}$$

and

$$\theta(\eta)=-\frac{\frac{1}{6K}}{(\frac{1}{6}\eta+\frac{1}{f_w})^2}. \tag{32}$$

However, the obtained closed-form approximate solution Eq. (31) does not satisfy the boundary condition at infinity $f'(\infty)=1$.
Proceeding as before, we insert the obtained solution for θ from Eq. (32) into the RHS of Eq. (1) to obtain

$$f'(\eta)=1-\frac{1}{6}\frac{1}{(\frac{1}{6}\eta+\frac{1}{f_w})^2}. \tag{33}$$

Integrating Eq. (33) from 0 to η and taking into account that $f(0)=f_w$, we obtain

$$f(\eta)=f_w+\eta+\left(\frac{1}{\frac{1}{6}\eta+\frac{1}{f_w}}-f_w\right), \tag{34}$$

so that

$$f(\eta)=\eta+\frac{1}{\frac{1}{6}\eta+\frac{1}{f_w}}. \tag{35}$$

Thus the approximate closed-form solutions of system (1) and (23), when $\lambda=0$ and under the condition (30), is given as

$$(f(\eta),\ \theta(\eta))=(\eta+\frac{1}{\frac{1}{6}\eta+\frac{1}{f_w}},\ -\frac{\frac{1}{6K}}{(\frac{1}{6}\eta+\frac{1}{f_w})^2}). \tag{36}$$

An error analysis using the error remainder function $ER(\eta)$ for these new approximate analytic solutions $(f(\eta),\theta(\eta))$ is subsequently presented.
The substitution of Eq. (1) into Eq. (23) gives

$$\theta''(\eta)+\frac{1}{2K}f(\eta)f''(\eta)=0, \tag{37}$$

exactly; in other words, the functions representing the exact solutions for $\theta(\eta)$ and $f(\eta)$ upon substitution will make Eq. (37) an identity. However, we have not calculated the exact functions but instead the functions in Eq. (36) are actually very accurate approximations to the respective solutions as we shall subsequently reveal; let us now designate them as $\widetilde{\theta}(\eta)$ and $\widetilde{f}(\eta)$. Thus, instead of Eq. (37), upon substitution of the approximate functions into the left hand side of Eq. (37), we define the nonzero error remainder function $ER(\eta)$ as

$$\widetilde{\theta}''(\eta)+\frac{1}{2K}\widetilde{f}(\eta)\widetilde{f}''(\eta)=\frac{1}{36K}\frac{\eta}{(\frac{1}{6}\eta+\frac{1}{f_w})^3}=ER(\eta). \tag{38}$$

In Tables 1-2, we display the computed values of $ER(\eta)$ for various small and large values of η and different values of f_w and K such that $\frac{1}{f_w^2}+\frac{1}{6K}=0$.

Table 1: The values of the error remainder function $ER(\eta)$ for small values of η when $\frac{1}{f_w^2}+\frac{1}{6K}=0$.

η	$f_w=10$ $\mid ER(\eta)\mid$	$f_w=20$ $\mid ER(\eta)\mid$	$f_w=50$ $\mid ER(\eta)\mid$	$f_w=100$ $\mid ER(\eta)\mid$
0	0.000000	0.000000	0.000000	0.000000
1	0.087890	0.040964	0.010249	0.003022
2	0.040964	0.014794	0.003022	0.000823
3	0.023148	0.007513	0.001422	0.000376
4	0.014794	0.004527	0.000823	0.000215
5	0.010249	0.003022	0.000536	0.000138
6	0.007513	0.002159	0.000376	0.000097
7	0.005740	0.001619	0.000279	0.000071
8	0.004527	0.001259	0.000215	0.000055
9	0.003662	0.001007	0.000170	0.000043
10	0.003022	0.000823	0.000138	0.000035

Table 2: The values of the error remainder $ER(\eta)$ for large values of η when $\frac{1}{f_w^2}+\frac{1}{6K}=0$.

η	$f_w=-20$ $\mid ER(\eta)\mid$	$f_w=-10$ $\mid ER(\eta)\mid$	$f_w=10$ $\mid ER(\eta)\mid$	$f_w=20$ $\mid ER(\eta)\mid$
100	0.00000908	0.00000366	0.00000353	0.00000891
200	0.00000226	0.00000090	0.00000089	0.00000223
300	0.00000100	0.00000040	0.00000039	0.00000099
400	0.00000056	0.00000022	0.00000022	0.00000056
500	0.00000036	0.00000014	0.00000014	0.00000035
600	0.00000025	0.00000010	0.00000009	0.00000024
700	0.00000018	0.00000007	0.00000007	0.00000018
800	0.00000014	0.00000005	0.00000005	0.00000014
900	0.00000011	0.00000004	0.00000004	0.00000011
1000	0.00000009	0.00000003	0.00000003	0.00000008
2000	0.00000002	0.00000000	0.00000000	0.00000002

3 Approximate analytic series solution by the modified Adomian decomposition method

We next propose to solve this boundary value problem for the nonlinear coupled system by the modified Adomian decomposition method (ADM) [17]-[38].

3.1 Case: $K = \pm\frac{Ra_k}{Pe_x} = 0$

For the limiting case of forced convection, the governing equations can be written by setting $\frac{Ra_k}{Pe_x} = 0$ as

$$f = \eta + f_w \tag{39}$$

and

$$\theta'' + \frac{\lambda+1}{2}(\eta + f_w)\theta' - \lambda\theta = 0, \tag{40}$$

subject to the boundary conditions (4).
We rewrite Eq. (40) in Adomian's operator-theoretic notation as

$$L_1\theta(\eta) = R\theta(\eta), \tag{41}$$

where

$$L_1\theta = \frac{d^2\theta}{d\eta^2}(\eta) \text{ and } R\theta = -\frac{\lambda+1}{2}(\eta+f_w)\theta' + \lambda\theta. \tag{42}$$

Next by applying the inverse linear operator

$$L_1^{-1}(.) = \int_0^\eta \int_0^\eta (.)d\eta d\eta \tag{43}$$

to both sides of Eq. (41), we obtain

$$\theta(\eta) = 1 + \gamma\eta - \frac{\lambda+1}{2}\int_0^\eta \int_0^\eta (\eta+f_w)\theta'(\eta)d\eta d\eta + \lambda\int_0^\eta \int_0^\eta \theta(\eta)d\eta d\eta, \tag{44}$$

where $\gamma = \theta'(0)$.
The Adomian decomposition method defines the unknown function $\theta(\eta)$ by the series

$$\theta(\eta) = \sum_{n=0}^{\infty} \theta_n(\eta), \tag{45}$$

which we substitute into Eq. (44) to obtain

$$\sum_{n=0}^{\infty}\theta_n = 1 + \gamma\eta - \frac{\lambda+1}{2}\int_0^\eta \int_0^\eta (\eta+f_w)\sum_{n=0}^{\infty}\theta_n'(\eta)d\eta d\eta + \lambda\int_0^\eta \int_0^\eta \sum_{n=0}^{\infty}\theta_n(\eta)d\eta d\eta. \tag{46}$$

We set the recursive relation as

$$\begin{cases} \theta_0(\eta) = 1 + \gamma\eta, \\ \theta_{n+1}(\eta) = -\frac{\lambda+1}{2}\int_0^\eta \int_0^\eta (\eta+f_w)\theta_n'(\eta)d\eta d\eta \\ \qquad + \lambda\int_0^\eta \int_0^\eta \theta_n(\eta)d\eta d\eta,\ n \geq 0, \end{cases} \tag{47}$$

which in turn gives

$$\begin{cases} \theta_0(\eta) = 1 + \gamma\eta, \\ \theta_1(\eta) = \left(-\frac{\lambda+1}{2}\gamma f_w + \lambda\right)\frac{\eta^2}{2!} \\ \qquad + \left(-\frac{\lambda+1}{2}\gamma\lambda\gamma\right)\frac{\eta^3}{3!}, \\ \qquad .... \end{cases} \tag{48}$$

To determine the constant γ, we use the remaining boundary condition

$$\theta(\eta) = 0,\ \eta \to \infty. \tag{49}$$

It is clear that this condition cannot be applied directly to the series of

$$\theta(\eta) = \theta_0 + \theta_1 + ... \tag{50}$$

We can readily evaluate the constant γ by calculating the Padé approximants of this series [28], which possess the advantage of transforming the polynomial approximation into a rational function of polynomials.
However the use of the Padé approximants requires considerable calculations. Here we will show how the constant γ can be easily determined by employing less computational work. To provide a clear overview of this technique, two examples will be subsequently discussed.

3.2 Examples

Example 1.Consider the case: $\lambda = 0$ *with* $K = 0$, *then Eq. (36)* becomes

$$\theta''(\eta) + \frac{1}{2}(\eta+f_w)\theta'(\eta) = 0, \tag{51}$$

which is a second-order linear differential equation that can be integrated to give

$$\theta'(\eta) = c_1 e^{-\frac{1}{2}(f_w\eta + \frac{\eta^2}{2})}, \tag{52}$$

or

$$\theta(\eta) = c_1\sqrt{\pi}e^{\frac{f_w^2}{2}} erf\left(\frac{f_w+\eta}{2}\right) + c_2, \tag{53}$$

where the c_i, for $i = 1, 2$, are constants of integration and erf denotes the Error Function which is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + ...\right). \tag{54}$$

Using the given boundary conditions $\theta(0) = 1$ and $\theta(\infty) = 0$ and taking into account that $erf(0) = 0$ and $erf(\infty) = 1$, we obtain

$$\sqrt{\pi}c_1 e^{\frac{1}{2}f_w^2} erf\left(\frac{f_w}{2}\right) + c_2 = 1 \tag{55}$$

and

$$\sqrt{\pi}c_1 e^{\frac{1}{2}f_w^2} + c_2 = 0. \tag{56}$$

We recognize that these two equations compose a system of two linear equations in two unknowns, i.e., c_1 and c_2. This system is solvable when the determinant of the coefficient matrix is nonzero, i.e.

$$\triangle = \sqrt{\pi}e^{\frac{f_w^2}{2}}\left(erf\left(\frac{f_w}{2}\right) - 1\right) \neq 0. \tag{57}$$

Therefore the coefficient values are obtained as

$$c_1 = \frac{1}{\sqrt{\pi} e^{\frac{f_w^2}{2}} \left(erf\left(\frac{f_w}{2}\right) - 1 \right)} \quad (58)$$

and

$$c_2 = -\frac{1}{erf\left(\frac{f_w}{2}\right) - 1}. \quad (59)$$

If we choose $f_w = 0$, then $c_1 = -\frac{1}{\sqrt{\pi}}$ and $c_2 = 1$, from which it follows that the exact closed-form solution is

$$\theta(\eta) = 1 - (\frac{\eta}{2}). \quad (60)$$

Following Adomian's decomposition method, we calculate

$$\begin{cases} \theta_0(\eta) = 1 + \gamma\eta, \\ \theta_1(\eta) = -\frac{\gamma}{12}\eta^3, \\ \theta_2(\eta) = \frac{\gamma}{160}\eta^5, \\ \theta_3(\eta) = -\frac{\gamma}{2688}\eta^7, \end{cases} \quad (61)$$

and so on. Thus the solution in a series form is given as

$$\theta(\eta) = 1 + \gamma\eta - \frac{\gamma}{12}\eta^3 + \frac{\gamma}{160}\eta^5 - \frac{\gamma}{2688}\eta^7 + \ldots \quad (62)$$

or

$$\theta(\eta) = 1 + 2\gamma \left(\frac{\eta}{2} - \frac{1}{3}(\frac{\eta}{2})^3 + \frac{1}{10}(\frac{\eta}{2})^5 - \frac{1}{42}(\frac{\eta}{2})^7 + \ldots \right) \quad (63)$$

and in a closed form as

$$\theta(\eta) = 1 + 2\gamma \left(\frac{\sqrt{\pi}}{2}(\frac{\eta}{2}) \right). \quad (64)$$

Then we evaluate this solution at $\eta = \infty$ and use the remaining boundary condition to obtain

$$1 + 2\gamma \left(\frac{\sqrt{\pi}}{2}(\infty) \right) = 0. \quad (65)$$

Hence, we have

$$\gamma = -\frac{1}{\sqrt{\pi}}, \quad (66)$$

which gives the exact solution as

$$\theta(\eta) = 1 - (\frac{\eta}{2}). \quad (67)$$

Example 2.Consider the case when $\lambda = \frac{1}{3}$ and $f_w = 0$ with $K = 0$.

Following Adomian's decomposition method, we calculate

$$\begin{cases} \theta_0(\eta) = 1 + \gamma\eta, \\ \theta_1(\eta) = \frac{1}{3}\frac{\eta^2}{2!} - \gamma\frac{1}{3}\frac{\eta^3}{3!}, \\ \theta_2(\eta) = -\frac{1}{3}\frac{\eta^4}{4!} + \gamma\frac{5}{9}\frac{\eta^5}{5!}, \\ \quad \ldots. \end{cases} \quad (68)$$

Thus the solution in a series form is given as

$$\theta(\eta) = \left(1 + \frac{\frac{1}{3}}{2!}\eta^2 - \frac{\frac{1}{3}}{4!}\eta^4 + \ldots \right) + \gamma \left(\eta - \frac{\frac{1}{3}}{3!}\eta^3 + \frac{\frac{5}{9}}{5!}\eta^5 + \ldots \right). \quad (69)$$

We observe that the first and second terms of this series can be written as

$$1 + \frac{\frac{1}{3}}{2!}\eta^2 - \frac{\frac{1}{3}}{4!}\eta^4 + \ldots = \left(1 - \frac{\frac{\eta^2}{3}}{1!} + \frac{(-\frac{\eta^2}{3})^2}{2!} + \ldots \right) \left(1 + \frac{1}{2!}\eta^2 + \frac{\frac{7}{3}}{4!}\eta^4 + \ldots \right) \quad (70)$$

and

$$\eta - \frac{\frac{1}{3}}{3!}\eta^3 + \frac{\frac{5}{9}}{5!}\eta^5 + \ldots = \left(1 - \frac{\frac{\eta^2}{3}}{1!} + \frac{(-\frac{\eta^2}{3})^2}{2!} + \ldots \right) \left(\eta + \frac{\frac{5}{3}}{3!}\eta^3 + \frac{5}{5!}\eta^5 + \ldots \right) \quad (71)$$

These can be written in closed form as

$$1 + \frac{\frac{1}{3}}{2!}\eta^2 - \frac{\frac{1}{3}}{4!}\eta^4 + \ldots = e^{-\frac{1}{3}\eta^2} F_1(\frac{3}{4};\ \frac{1}{2};\ \frac{\eta^2}{3}) \quad (72)$$

and

$$\eta - \frac{\frac{1}{3}}{3!}\eta^3 + \frac{\frac{5}{9}}{5!}\eta^5 + \ldots = e^{-\frac{1}{3}\eta^2} H_{-\frac{3}{2}}(\frac{x}{\sqrt{3}}), \quad (73)$$

where $F_1(a;\ b;\ \eta)$ is a confluent hypergeometric function of the first kind and H_n is a Hermite polynomial.
Thus the particular solution of this problem is given as

$$\theta(\eta;\gamma) = e^{-\frac{1}{3}\eta^2} F_1(\frac{3}{4};\ \frac{1}{2};\ \frac{\eta^2}{3}) + \gamma e^{-\frac{1}{3}\eta^2} H_{-\frac{3}{2}}(\frac{x}{\sqrt{3}}), \quad (74)$$

which is the indeed the exact solution of this problem, and from which the remaining boundary condition $\theta(\eta) = 0$, $\eta \to \infty$ is satisfied for all values of γ. Consequently, the solution $\theta(\eta;\gamma)$ depends on the parameter γ.

3.3 Case: $K = \pm\frac{Ra_k}{Pe_x} \neq 0$

First, for the purpose of comparison, we consider $K = 1$, where the boundary condition $f'(\infty) = 0$ is chosen instead

of the above nonhomogeneous boundary condition $f'(\infty) = 1$. This problem was originally proposed by Cheng [14,15].
Thus the substitution of Eq. (1) into Eq. (2) gives

$$f''' + \frac{\lambda+1}{2} f f'' - \lambda (f')^2 = 0. \quad (75)$$

Eq. (75) with the boundary conditions $f(0) = f_w$ and $f'(\infty) = 0$ can be transformed into an equivalent problem. Let us consider the following transformation [16]:

$$z(\xi) = f'(\eta),\ \xi = \alpha - f(\eta),\ \alpha = f(\infty). \quad (76)$$

We have

$$z'(\xi) = \frac{dz}{d\xi} = -\frac{f''(\eta)}{f'(\eta)} \quad (77)$$

and

$$z''(\xi) = \frac{d^2z}{d\xi^2} = \frac{f'''(\eta)}{(f')^2(\eta)} - \frac{(f'')^2(\eta)}{(f')^3(\eta)}. \quad (78)$$

Substituting these into Eq. (75), we obtain

$$\frac{d}{d\xi}(\xi \frac{dz}{d\xi}) + \frac{1+\lambda}{2}(\xi - \alpha)\frac{dz(\xi)}{d\xi} - \lambda z(\xi) = 0. \quad (79)$$

Since $f'(\infty) = 0$, $f'(0) = 1$ and $f(0) = f_w$, we obtain $z(0) = 0$ and $z(\alpha - f_w) = 1$. Hence, the boundary value problem Eq. (75) with $f'(\infty) = 0$, $f'(0) = 1$ and $f(0) = f_w$ can be transformed into the new boundary value problem

$$\begin{cases} \frac{d}{d\xi}(z\frac{dz}{d\xi}) + \frac{\lambda+1}{2}(\xi-\alpha)\frac{dz(\xi)}{d\xi} - \lambda z(\xi) = 0, \\ z(0) = 0,\ z(\alpha - f_w) = 1. \end{cases} \quad (80)$$

Our aim is to find the solution $z(\xi)$ and then compute α by means of the modified Adomian decomposition method to obtain accurate quantitative solutions.
By integrating the first equation of Eq. (80) from 0 to ξ and taking into account that the initial condition $z(0) = 0$, we obtain

$$z\frac{dz}{d\xi} + \frac{\lambda+1}{2}(\xi - \alpha)z(\xi) - \frac{3\lambda+1}{2}\int_0^\xi z(\xi)d\xi = 0. \quad (81)$$

Consequently, we have

$$z^2(\xi) + (\lambda+1)\int_0^\xi (\xi-\alpha)z(\xi)d\xi - (3\lambda+1)\int_0^\xi\int_0^\xi z(\xi)d\xi d\xi = 0. \quad (82)$$

Now we seek for a convergent series solution as

$$z(\xi) = \sum_{n=0}^{\infty} C_n \xi^n. \quad (83)$$

Thus assuming the convergence of the Cauchy product, we obtain

$$z^2(\xi) = \sum_{n=0}^{\infty} \xi^n \sum_{k=0}^{n} C_k C_{n-k}. \quad (84)$$

The substitution of (83)-(84) into Eq. (82) yields

$$\sum_{n=0}^{\infty} \xi^n \sum_{k=0}^{n} C_k C_{n-k} + (\lambda+1)\left[\int_0^\xi \sum_{n=0}^{\infty} C_n \xi^{n+1} d\xi - \alpha \int_0^\xi \sum_{n=0}^{\infty} C_n \xi^n d\xi\right] - (3\lambda+1)\int_0^\eta \int_0^\eta \sum_{n=0}^{\infty} C_n \eta^n d\eta d\eta = 0. \quad (85)$$

Carrying out these integrations, we obtain

$$\sum_{n=0}^{\infty} \xi^n \sum_{k=0}^{n} C_k C_{n-k} + (\lambda+1)\left[\sum_{n=0}^{\infty} \frac{C_n}{n+2}\xi^{n+2} - \alpha \sum_{n=0}^{\infty}\frac{C_n}{n+1}\xi^{n+1}\right] - (3\lambda+1)\sum_{n=0}^{\infty}\frac{C_n}{(n+1)(n+2)}\xi^{n+2} = 0. \quad (86)$$

In the second, third and fourth summations on the LHS, n can be replaced by $n-2$, $n-1$ and $n-2$, respectively, to write

$$\sum_{n=0}^{\infty} \xi^n \sum_{k=0}^{n} C_k C_{n-k} + (\lambda+1)\left[\sum_{n=2}^{\infty} \frac{C_{n-2}}{n}\xi^{n} - \alpha \sum_{n=1}^{\infty}\frac{C_{n-1}}{n}\xi^{n}\right] - (3\lambda+1)\sum_{n=2}^{\infty}\frac{C_{n-2}}{n(n-1)}\xi^{n} = 0. \quad (87)$$

Finally, we can equate coefficients of the like powers of ξ on the LHS with those on the RHS to arrive at recurrence relations for the coefficients. Thus

$$\begin{cases} C_0 = 0, \\ C_1 = \alpha\frac{\lambda+1}{2}, \\ C_2 = \frac{\lambda-1}{8}, \\ \quad \dots \end{cases} \quad (88)$$

The general recurrence relations for the coefficients is

$$\sum_{k=0}^{n} C_k C_{n-k} + (\lambda+1)\left[\frac{C_{n-2}}{n} - \alpha\frac{C_{n-1}}{n}\right] - (3\lambda+1)\frac{C_{n-2}}{n(n-1)} = 0,\ \ n \ge 2, \quad (89)$$

where $C_0 = 0$ and $C_1 = \alpha\frac{\lambda+1}{2}$.
The solution can then be written as

$$z(\xi) = C_0 + C_1\xi + C_2\xi^2 + C_3\xi^3 + \dots, \quad (90)$$

where the only unknown constant is α.
In principle, α can be determined by imposing the remaining boundary condition at the second point, that is $z(\alpha - f_w) = 1$.
Thus

$$\sum_{n=0}^{\infty} C_n (\alpha - f_w)^n = 1. \quad (91)$$

This matching equation is a nonlinear algebraic equation in the undetermined coefficient α. Thus, we solve for the first three terms

$$C_0+C_1(\alpha-f_w)+C_2(\alpha-f_w)^2=1 \quad (92)$$

to obtain

$$(5\lambda+3)\alpha^2-2f_w(3\lambda+1)\alpha+f_w^2(\lambda-1)-8=0, \quad (93)$$

which is the same equation that was obtained by Govindarajulu and Malarvizhi [15].
Cheng [14] showed that the range of λ for which the problem is physically realistic is $0\le\lambda\le 1$.
For $\lambda=1$, the coefficients $C_0=0$, $C_1=\alpha\frac{1+\lambda}{2}$ and $C_i=0$, $i\ge 2$. Thus, an analytical solution for $z(\xi)$ can be determined by returning the value of α to the truncated solution of the original equation Eq. (90).

$$z(\xi)=\alpha\frac{1+\lambda}{2}\xi=\alpha\xi, \quad (94)$$

from which it follows that

$$f'(\eta)=\alpha(\alpha-f(\eta)). \quad (95)$$

Consequently,

$$f(\eta)=\alpha+(f_w-\alpha)e^{-\alpha\eta}, \quad (96)$$

where $\alpha^2-2f_w\alpha-1=0$, that is

$$\alpha_{1,2}=f_w\pm\sqrt{f_w^2+1}. \quad (97)$$

Using Eq. (1), we obtain

$$\theta(\eta)=e^{-\alpha\eta}. \quad (98)$$

Thus

$$(f(\eta),\ \theta(\eta))=(\alpha+(f_w-\alpha)e^{-\alpha\eta},\ e^{-\alpha\eta}), \quad (99)$$

which is indeed the exact solution of this problem when $\lambda=1$.
This is once again the same result that was obtained by Govindarajulu and Malarvizhi [15].
Taking the first three components of the solution $z(\xi)$, we obtain

$$z(\xi)=C_0+C_1\xi+C_2\xi^2=\alpha\frac{\lambda+1}{2}\xi+\frac{\lambda-1}{8}\xi^2. \quad (100)$$

We can now return to the original dependent variable $f(\eta)$ to obtain

$$f(\eta)=C_1(\alpha-f(\eta))+C_2(\alpha-f(\eta))^2, \quad (101)$$

which can be written as

$$-(\alpha-f(\eta))'=C_1(\alpha-f(\eta))+C_2(\alpha-f(\eta))^2. \quad (102)$$

Accordingly, the exact solution is therefore explicitly given as

$$f(\eta)=\alpha-\frac{C_1(\alpha-f_w)}{C_2(\alpha-f_w)(e^{C_1\eta}-1)+C_1e^{C_1\eta}}. \quad (103)$$

Hence,

$$\theta(\eta)=\frac{C_1(\alpha-f_w)\left[C_1C_2(\alpha-f_w)e^{C_1\eta}+C_1^2e^{C_1\eta}\right]}{\left[C_2(\alpha-f_w)(e^{C_1\eta}-1)+C_1e^{C_1\eta}\right]^2}. \quad (104)$$

Figures 1 and 2 has been drawn to show the $3rd-$stage approximation of $f(\eta)$ for various values of $f_w=-1$, 0 and 1 when $\lambda=0$ and 1.

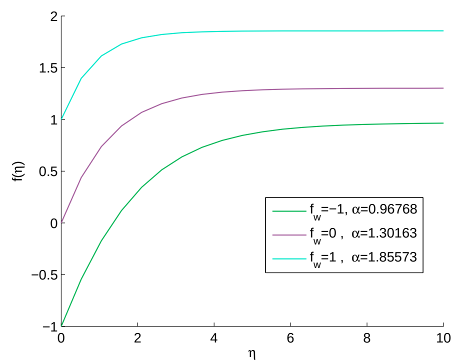

Fig. 1: Plot of the $3rd$-stage approximation obtained for $f(\eta)$ by the ADM when $\lambda=0$.

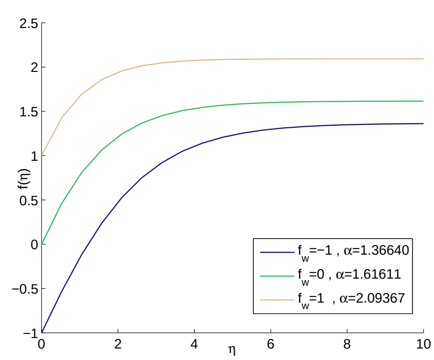

Fig. 2: Plot of the $3rd$-stage approximation obtained of $f(\eta)$ by the ADM when $\lambda=\frac{1}{3}$.

4 Conclusion

In this work, we have considered the nonlinear coupled boundary-layer equations that describe the problem of injection or extraction of fluid along the surface of an inclined wall embedded in a saturated porous medium. We have demonstrated that closed-form analytic approximate solutions can be obtained in a straightforward manner by using a direct method. Also, very good approximate analytic series solutions were obtained by the modified Adomian decomposition method for the two limiting cases of free and forced convection by setting $\frac{Ra_k}{Pe_x} = 0$ and $\frac{Ra_k}{Pe_x} \neq 0$, respectively.

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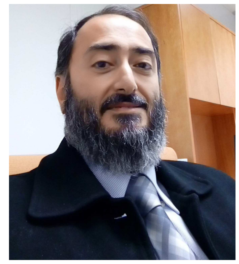

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