On an Inequality Related to Capacity of $MA(1)$ Gaussian Channel with Feedback

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There are several inequalities related to capacity of Gaussian channel with feedback. We give an answer for unsolved problem under some condition. And also we give a new inequality in the case of $MA(1)$ Gaussian noise.

Keywords: Gaussian channel, capacity, feedback.

1 Gaussian Channels

The following model for a discrete time Gaussian channel with feedback is considered:

$$Y_n = S_n + Z_n, \ n = 1, 2, \ldots,$$

where $Z = \{Z_n; n = 1, 2, \ldots\}$ is a non-degenerate, zero mean Gaussian process representing the noise and $S = \{S_n; n = 1, 2, \ldots\}$ and $Y = \{Y_n; n = 1, 2, \ldots\}$ are stochastic processes representing input signals and output signals, respectively. The channel is with noiseless feedback, so $S_n$ is a function of a message to be transmitted and the output signals $Y_1, \ldots, Y_{n-1}$. For a code of rate $R$ and length $n$, with code words $x^n(W, Y^{n-1})$, $W \in \{1, \ldots, 2^{nR}\}$, and a decoding function $g_n: \mathbb{R}^n \rightarrow \{1, \ldots, 2^{nR}\}$, the probability of error is

$$P_{e(n)} = \text{Pr}\{g_n(Y^n) \neq W; Y^n = x^n(W, Y^{n-1}) + Z^n\},$$

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where $W$ is uniformly distributed over $\{1, \ldots, 2^R\}$ and independent of $Z^n$. The signal is subject to an expected power constraint

$$\frac{1}{n} \sum_{i=1}^{n} E[S_i^2] \leq P,$$

and the feedback is causal, i.e., $S_i$ is dependent of $Z_1, \ldots, Z_{i-1}$ for $i = 1, 2, \ldots, n$. Similarly, when there is no feedback, $S_i$ is independent of $Z^n$. We denote by $R_X^{(n)}$ and $R_Z^{(n)}$ the covariance matrices of $X$ and $Z$, respectively. It is well known that a finite block length capacity is given by

$$C_{n,FB,Z}(P) = \max \frac{1}{2n} \log \frac{|R_X^{(n)} + R_Z^{(n)}|}{|R_Z^{(n)}|},$$

where the maximum is taken over all symmetric, nonnegative definite matrix $R_X^{(n)}$ and strictly lower triangular matrix $B$, such that

$$\text{Tr}[(I + B)R_X^{(n)}(I + B^t) + BR_Z^{(n)}B^t] \leq nP.$$

Similarly, let $C_{n,Z}(P)$ be the maximal value when $B = 0$, i.e. when there is no feedback. Under these conditions, Cover and Pombra proved the following.

**Proposition 1.1** (Cover and Pombra [6]). *For every $\epsilon > 0$ there exist codes, with block length $n$ and $2^{n(C_{n,FB,Z}(P)-\epsilon)}$ codewords, $n = 1, 2, \ldots$, such that $P_e^{(n)} \rightarrow 0$, as $n \rightarrow \infty$. Conversely, for every $\epsilon > 0$ and any sequence of codes with $2^{n(C_{n,FB,Z}(P)+\epsilon)}$ codewords and block length $n$, $P_e^{(n)}$ is bounded away from zero for all $n$. The same theorem holds in the special case without feedback upon replacing $C_{n,FB,Z}(P)$ by $C_{n,Z}(P)$.\*

When block length $n$ is fixed, $C_{n,Z}(P)$ is given exactly.

**Proposition 1.2** (Gallager [10]).

$$C_{n,Z}(P) = \frac{1}{2n} \sum_{i=1}^{k} \log \frac{nP + r_1 + \cdots + r_k}{kr_i},$$

where $0 < r_1 \leq r_2 \leq \cdots \leq r_n$ are eigenvalues of $R_Z^{(n)}$, and $k (\leq n)$ is the largest integer satisfying $nP + r_1 + r_2 + \cdots + r_k > kr_k$.

## 2 Mixed Gaussian Channels with Feedback

Let $Z_1, Z_2$ be Gaussian processes with mean 0 and covariance matrices $R_{Z_1}^{(n)}, R_{Z_2}^{(n)}$, respectively. A mixed Gaussian channel is defined by an additive Gaussian channel with noise $\tilde{Z}$ whose mean is 0 and whose covariance matrix is

$$R_{\tilde{Z}}^{(n)} = \alpha R_{Z_1}^{(n)} + \beta R_{Z_2}^{(n)},$$
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where \( \alpha, \beta \geq 0 \) (\( \alpha + \beta = 1 \)). Let \( C_{n,Z}(P) \) be the capacity of mixed Gaussian channel and \( C_{n,FB,Z}(P) \) the capacity of mixed Gaussian channel with feedback.

**Theorem 2.1** (Y-C-Y [20], Y-Y-C [21], C-Y [4]). For any \( P > 0 \),

\[
C_{n,Z}(P) \leq \alpha C_{n,Z_1}(P) + \beta C_{n,Z_2}(P).
\]

**Theorem 2.2** (Y-C-Y [20], Y-Y-C [21], C-Y [4]). For any \( P > 0 \), there exist \( P_1, P_2 \geq 0 \) (\( P = \alpha P_1 + \beta P_2 \)) such that

\[
C_{n,FB,Z}(P) \leq \alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2).
\]

These theorems are proved by the property that \( \log(1 + t^{-1}) \) is an operator convex function. But we have the following conjecture.

**Conjecture 2.1.** For any \( P > 0 \),

\[
C_{n,FB,Z}(P) \leq \alpha C_{n,FB,Z_1}(P) + \beta C_{n,FB,Z_2}(P).
\]

We solved the above conjecture partially.

**Theorem 2.3** (Yanagi, Yu, and Chao [21]). If one of the following conditions is satisfied, then the conjecture holds.

1. \( R_{Z_1}^{(n-1)} = R_{Z_2}^{(n-1)}. \)
2. \( R_Z \) is white.

### 3 Kim’s Result

Let \( Z = \{Z_i; i = 1, 2, \ldots\} \) be a discrete time first order moving average Gaussian process that we denote by MA(1). MA(1) can be characterized in the following three properties.

1. \( Z_i = \alpha U_{i-1} + U_i, \; i = 1, 2, \ldots, \) where \( U_i \sim N(0, 1) \) are i.i.d.
2. Spectral density function (SDF) is given by

\[
f(\lambda) = \frac{1}{2\pi} |1 + \alpha e^{-i\lambda}|^2 = \frac{1}{2\pi}(1 + \alpha^2 + 2\alpha \cos \lambda).
\]
3. \( Z_n = (Z_i, \ldots, Z_n) \sim N_n(0, K_Z) \) for each \( n \), where covariance matrix \( K_Z \) is given by the following:

\[
K_Z = \begin{pmatrix}
1 + \alpha^2 & \alpha & 0 & \cdots & 0 \\
\alpha & 1 + \alpha^2 & \alpha & \cdots & 0 \\
0 & \alpha & 1 + \alpha^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \alpha \\
0 & 0 & 0 & \cdots & 1 + \alpha^2
\end{pmatrix}.
\]
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We define the capacity of Gaussian channel with the MA(1) Gaussian noise by the following:

\[ C_{FB,Z}(P) = \lim_{n \to \infty} C_{n,FB,Z}(P) \]

Recently Kim obtained \( C_{FB,Z}(P) \) in above conditions, which is the first result of feedback capacity.

**Theorem 3.1 (Kim [13]).**

\[ C_{FB,Z}(P) = -\log x_0, \]

where \( x_0 \) is a unique positive root of

\[ Px^2 = (1 - x^2)(1 - |\alpha|^2). \]  \hspace{1cm} (3.1)

![Graph](image)

Figure 3.1: Graph of \( Px^2 = (1 - x^2)(1 - |\alpha|^2) \), where \( P = 1, \alpha = 0.5 \)

4 An Inequality Related to Conjecture 2.1

The following inequality holds:

\[ R_{\alpha Z + \beta W} \leq \alpha R_Z + \beta R_W \leq R_{\sqrt{\alpha}Z + \sqrt{\beta}W}, \]

where

\[ Z \sim MA(1,p), \quad Z_i = U_i + pU_{i-1}, \quad 0 < p \leq 1, \]

\[ W \sim MA(1,q), \quad W_i = U_i + qU_{i-1}, \quad 0 < q \leq 1. \]

Since

\[ \alpha R_Z + \beta R_W = R_{\alpha Z + \beta W} + \alpha \beta R_{Z-W}, \]

we have

\[ R_{\alpha Z + \beta W} \leq \alpha R_Z + \beta R_W. \]
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On the other hand we have

$$\alpha R_Z + \beta R_W + \sqrt{\alpha \beta} (R_{ZW} + R_{WZ}) = R_{\sqrt{\alpha Z} + \sqrt{\beta W}},$$

where

$$R_{ZW} + R_{WZ} = \begin{pmatrix} 2 + 2pq & p + q & 0 & \ldots & 0 \\ p + q & 2 + 2pq & p + q & \ldots & 0 \\ 0 & p + q & 2 + 2pq & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & p + q \\ 0 & 0 & 0 & \ldots & 2 + 2pq \end{pmatrix}.$$ 

The eigenvalues $r_i$ of this covariance matrix are represented as follows.

$$r_i = 2 + 2pq - 2(p + q) \cos \frac{i\pi}{n + 1} \quad (i = 1, 2, \ldots, n)$$

$$\geq 2 + 2pq - 2(p + q) = 2(1 - p)(1 - q) \geq 0.$$

Since $R_{ZW} + R_{WZ} \geq 0$, we have $\alpha R_Z + \beta R_W \leq R_{\sqrt{\alpha Z} + \sqrt{\beta W}}$.

**Proposition 4.1.** The following inequality holds.

$$C_{FB,\sqrt{\alpha Z}+\sqrt{\beta W}}(P) \leq C_{FB,\hat{Z}}(P) \leq C_{FB,\alpha Z+\beta W}(P),$$

where $R_{\hat{Z}} = \alpha R_Z + \beta R_W$.

We put $V = \sqrt{\alpha Z} + \sqrt{\beta W}$. Then

$$V_i = (\sqrt{\alpha} + \sqrt{\beta})U_i + (\sqrt{\alpha}p + \sqrt{\beta}q)U_{i-1}.$$

And we also put

$$Y_i = U_i + \frac{\sqrt{\alpha}p + \sqrt{\beta}q}{\sqrt{\alpha} + \sqrt{\beta}} U_{i-1}.$$

Then

$$Y = \frac{\sqrt{\alpha Z} + \sqrt{\beta W}}{\sqrt{\alpha} + \sqrt{\beta}} \sim MA\left(1, \frac{\sqrt{\alpha}p + \sqrt{\beta}q}{\sqrt{\alpha} + \sqrt{\beta}}\right).$$

$$C_{n,FB,V}(P) = \max \left\{ \frac{1}{2n} \log \frac{|R_{S+V}|}{|R_V|}; Tr[R_S] \leq nP \right\}$$

$$= \max \left\{ \frac{1}{2n} \log \frac{|R_{S+(\sqrt{\alpha}+\sqrt{\beta})Y}|}{|R_{(\sqrt{\alpha}+\sqrt{\beta})Y}|}; Tr[R_S] \leq nP \right\}$$

$$= \max \left\{ \frac{1}{2n} \log \frac{|R_{S/(\sqrt{\alpha}+\sqrt{\beta})Y}|}{|R_Y|}; Tr[R_{S/(\sqrt{\alpha}+\sqrt{\beta})}] \leq \frac{nP}{(\sqrt{\alpha} + \sqrt{\beta})^2} \right\}$$

$$= C_{n,FB,Y}\left(\frac{P}{(\sqrt{\alpha} + \sqrt{\beta})^2}\right).$$

We propose Conjecture 4.1 which is weaker than Conjecture 2.1.
Conjecture 4.1. For any $P > 0$,

$$C_{FB, \sqrt{\pi Z} + \sqrt{\pi W}}(P) \leq \alpha C_{FB, Z}(P) + \beta C_{FB, W}(P).$$

In particular we prove the Conjecture in the case of $\alpha = \beta = 1/2$.
Since we can represent (3.1) as

$$|\alpha| = \frac{1}{x} - \frac{\sqrt{P}}{\sqrt{1 - x^2}},$$

we put the function

$$f(t, P) = \frac{1}{t} - \frac{\sqrt{P}}{\sqrt{1 - t^2}}$$

in order to prove the Conjecture. Then there uniquely exist $0 < a < b < 1$ such that
$f(a, P) = 1$, $f(b, P) = 0$. That is

$$1 = \frac{1}{a} - \frac{\sqrt{P}}{\sqrt{1 - a^2}}, \quad 0 = \frac{1}{b} - \frac{\sqrt{P}}{\sqrt{1 - b^2}}.$$ (4.1)

![Figure 4.1: Graph of $f(t, P) = 1/t - \sqrt{P}/\sqrt{1-t^2}$, where $P = 1$.](image)

However, since $f(t, P)$ is not convex function of $t(a \leq t \leq b)$, we put the following concave function

$$g(t, P) = t \left(1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}}\right), \quad \frac{1}{b} \leq t \leq \frac{1}{a}.$$ (4.2)

Now we put $L = \sqrt{(1-a)^2(1-a^2) + a^2}$. Then $b$ and $P$ can be represented as the following functions of $a$:

$$b = \frac{a}{L}, \quad P = \frac{L^2}{a^2} - 1.$$ (4.3)

Lemma 4.1. For any $P > 0$,

$$\frac{\sqrt{P}}{\sqrt{1 - a^2}} \geq \frac{1}{2} \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}.$$
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![Graph of $g(t, P) = t \left(1 - \sqrt{P}/\sqrt{t^2 - 1}\right)$, where $P = 1$.](image)

**Lemma 4.2.** For any $t, s$ ($1/b \leq t \leq s \leq 1/a$),

$$\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \geq \frac{\sqrt{s} - \sqrt{t}}{\sqrt{s} + \sqrt{t}}$$

**Proof.** Since

$$2(2 - \sqrt{2}) > 1$$

and $L > a$,

$$2(2 - \sqrt{2}) \left(\frac{1}{a} - 1\right) > \frac{1}{a} - 1 > \frac{1}{\sqrt{a}} - 1 > \frac{1}{\sqrt{L}} - 1.$$  

And since $L < 1$,

$$\frac{1 - a}{a} > \frac{1}{2(2 - \sqrt{2})} \left(\frac{1}{\sqrt{L}} - 1\right) = \frac{1}{2 - \sqrt{2}} \frac{1 - \sqrt{L}}{2\sqrt{L}}> \frac{1}{2 - \sqrt{2}} \frac{1 - \sqrt{L}}{1 + \sqrt{L}}.$$  

By (4.1),

$$\frac{1 - a}{a} = \frac{\sqrt{P}}{\sqrt{1 - a^2}}.$$  

The inequality is proved by putting $L = a/b$.  

**Lemma 4.2.** For any $t, s$ ($1/b \leq t \leq s \leq 1/a$),

$$\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \geq \frac{\sqrt{s} - \sqrt{t}}{\sqrt{s} + \sqrt{t}}$$

**Proof.** Since

$$\sqrt{b} = \min_{1/b \leq t \leq s \leq 1/a} \sqrt{t/s},$$

the following inequality is obtained.

$$\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} = 2 \left(\frac{\sqrt{b}}{\sqrt{a} + \sqrt{b}} - \frac{1}{2}\right) = 2 \left(\frac{1}{\sqrt{a/b} + 1} - \frac{1}{2}\right)$$

$$\geq 2 \left(\frac{1}{\sqrt{t/s} + 1} - \frac{1}{2}\right) = 2 \left(\frac{\sqrt{s}}{\sqrt{t} + \sqrt{s}} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{s} - \sqrt{t}}{\sqrt{s} + \sqrt{t}}.$$  

□
Lemma 4.3. For any $t, s$ $(1/b \leq t \leq s \leq 1/a)$

$$\frac{1}{2} g(t, P) + \frac{1}{2} g(s, P) \leq g\left(\sqrt{ts}, \frac{P}{2}\right).$$

Proof. Since $g(t, P)$ is concave function of $t$,

$$\frac{\sqrt{s}}{\sqrt{t} + \sqrt{s}} g\left(\frac{t}{2}, P\right) + \frac{\sqrt{t}}{\sqrt{t} + \sqrt{s}} g\left(\frac{s}{2}, P\right) \leq g\left(\sqrt{ts}, \frac{P}{2}\right).$$

Then we have to show the following inequality:

$$\frac{1}{2} g(t, P) + \frac{1}{2} g(s, P) \leq \frac{\sqrt{s}}{\sqrt{t} + \sqrt{s}} g\left(\frac{t}{2}, P\right) + \frac{\sqrt{t}}{\sqrt{t} + \sqrt{s}} g\left(\frac{s}{2}, P\right).$$

By Lemma 4.1 and Lemma 4.2

$$\frac{\sqrt{P}}{\sqrt{1 - a^2}} \geq \frac{1}{2} \frac{\sqrt{s} - \sqrt{t}}{2 - \sqrt{2} \sqrt{s} + \sqrt{t}} = \frac{2}{2 - \sqrt{2}} \left(\frac{\sqrt{s} - \sqrt{t}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2}\right).$$

Since, for any $t, s (1/b \leq t \leq s \leq 1/a)$,

$$0 \leq s \left(1 - \frac{\sqrt{P}}{s^2 - 1}\right) - t \left(1 - \frac{\sqrt{P}}{t^2 - 1}\right) \leq 1,$$

we have the following inequality:

$$\left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sqrt{P}}{\sqrt{1 - a^2}} \geq \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2}$$

$$\geq \left(\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2}\right) \left\{s \left(1 - \frac{\sqrt{P}}{s^2 - 1}\right) - t \left(1 - \frac{\sqrt{P}}{t^2 - 1}\right)\right\}.$$

Since

$$\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} \frac{\sqrt{P}}{\sqrt{1 - 1/t^2}} + \left(1 - \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}}\right) \frac{\sqrt{P}}{\sqrt{1 - 1/s^2}} \geq \frac{\sqrt{P}}{\sqrt{1 - a^2}},$$

we have

$$\left(\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2}\right) \left\{t \left(1 - \frac{\sqrt{P}}{t^2 - 1}\right) - s \left(1 - \frac{\sqrt{P}}{s^2 - 1}\right)\right\}$$

$$+ \left(1 - \frac{1}{\sqrt{2}}\right) \left\{\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} \frac{\sqrt{P}}{\sqrt{1 - 1/t^2}} + \left(1 - \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}}\right) \frac{\sqrt{P}}{\sqrt{1 - 1/s^2}}\right\} \geq 0.$$

Therefore

$$\left(\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2}\right) t \left(1 - \frac{\sqrt{P}}{t^2 - 1}\right) + \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} \frac{\sqrt{P}t}{\sqrt{t^2 - 1}}$$

$$+ \left(\frac{1}{2} - \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}}\right) s \left(1 - \frac{\sqrt{P}}{s^2 - 1}\right) + \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sqrt{t}}{\sqrt{s} + \sqrt{t}} \frac{\sqrt{Ps}}{\sqrt{s^2 - 1}} \geq 0.$$
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Then
\[
\left( \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} - \frac{1}{2} \right) t \left( 1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}} \right) + \left( 1 - \frac{1}{\sqrt{2}} \right) \left( \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} \right) \sqrt{P t} \\
+ \left( \frac{\sqrt{t}}{\sqrt{t + \sqrt{s}} - \frac{1}{2}} \right) s \left( 1 - \frac{\sqrt{P}}{\sqrt{s^2 - 1}} \right) + \left( 1 - \frac{1}{\sqrt{2}} \right) \left( \frac{\sqrt{t}}{\sqrt{t + \sqrt{s}}} \right) \sqrt{P s} \geq 0.
\]

Thus
\[
\frac{\sqrt{s}}{\sqrt{t} + \sqrt{s}} \left\{ t \left( 1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}} \right) + \left( 1 - \frac{1}{\sqrt{2}} \right) \sqrt{P t} \right\} \\
+ \frac{\sqrt{t}}{\sqrt{t} + \sqrt{s}} \left\{ s \left( 1 - \frac{\sqrt{P}}{\sqrt{s^2 - 1}} \right) + \left( 1 - \frac{1}{\sqrt{2}} \right) \sqrt{P s} \right\} \\
\geq \frac{1}{2} t \left( 1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}} \right) + \frac{1}{2} s \left( 1 - \frac{\sqrt{P}}{\sqrt{s^2 - 1}} \right).
\]

Therefore
\[
\frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}} \left( 1 - \frac{\sqrt{P/2}}{\sqrt{t^2 - 1}} \right) + \frac{\sqrt{t}}{\sqrt{t} + \sqrt{s}} s \left( 1 - \frac{\sqrt{P/2}}{\sqrt{s^2 - 1}} \right) \\
\geq \frac{1}{2} t \left( 1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}} \right) + \frac{1}{2} s \left( 1 - \frac{\sqrt{P}}{\sqrt{s^2 - 1}} \right)
\]

Then
\[
\frac{1}{2} g(t, P) + \frac{1}{2} g(s, P) = \frac{1}{2} t \left( 1 - \frac{\sqrt{P}}{\sqrt{t^2 - 1}} \right) + \frac{1}{2} s \left( 1 - \frac{\sqrt{P}}{\sqrt{s^2 - 1}} \right) \\
\leq \frac{\sqrt{s}}{\sqrt{t} + \sqrt{s}} \left( 1 - \frac{\sqrt{P/2}}{\sqrt{t^2 - 1}} \right) + \frac{\sqrt{t}}{\sqrt{t} + \sqrt{s}} s \left( 1 - \frac{\sqrt{P/2}}{\sqrt{s^2 - 1}} \right) \\
= \frac{1}{2} g(t, \frac{P}{2}) + \frac{1}{2} g(s, \frac{P}{2}).
\]

Now we have the following theorem.

**Theorem 4.1.** For any $P > 0$,
\[
C_{FB,(Z+W)/\sqrt{2}}(P) \leq \frac{1}{2} C_{FB,Z}(P) + \frac{1}{2} C_{FB,W}(P).
\]

**Proof.** Let $C_{FB,Z}(P) = -\log x$ and $C_{FB,W}(P) = -\log y$. By putting $s = 1/x$ and $t = 1/y$ in Lemma 4.3, we have
\[
\frac{1}{2} f(x, P) + \frac{1}{2} f(y, P) \leq f \left( \sqrt{xy}, \frac{P}{2} \right). 
\]

Since $Z \sim MA(1,p)$, $0 < p \leq 1$ and $W \sim MA(1,q)$, $0 < q \leq 1$,
\[
p = \frac{1}{x} - \frac{\sqrt{P}}{\sqrt{1-x^2}} = f(x, P),
\]
\[ q = \frac{1}{y} - \frac{\sqrt{P}}{\sqrt{1 - y^2}} = f(y, P). \]

We take \( z \) such that
\[ \frac{p + q}{2} = f\left(\frac{z}{2}, P\right). \]

Then by (4.2)
\[ f\left(\frac{z}{2}, P\right) \leq f\left(\sqrt{\frac{xy}{y}}, \frac{P}{2}\right). \]

Since \( f(t, P/2) \) is a decreasing function of \( t \), we have \( z \geq \sqrt{xy} \). Then we have the following:
\[
C_{FB, (Z+W)/\sqrt{2}}(P) = C_{FB, (Z+W)/2}\left(\frac{P}{2}\right)
= -\log z \\
\leq \frac{1}{2} \left(-\log x\right) + \frac{1}{2} \left(-\log y\right) \\
= \frac{1}{2} C_{FB, Z}(P) + \frac{1}{2} C_{FB, W}(P). \]

References


