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Sequence Spaces Defined By Fractional Difference Operator And Sequence Of Modulus Function In N-Normed Spaces

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Abstract: In this paper we introduce the sequence spaces defined by fractional difference operator and a sequence of modulus function $F = (f_k)$ in n-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

Keywords: Paranorm space, fractional difference operator, modulus function, n-normed space.

1 Introduction and Preliminaries

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_o be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$, respectively.
A sequence $x \in l_\infty$ is said to be almost convergent if all Banach limits of x coincide. Lorentz [1] proved that

$$\hat{c} = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \Big\}.$$

Maddox ([2,3]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [4] has defined the following sequence spaces :

$$[\hat{c}, p] = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^{p_k} = 0, \text{ uniformly in } s \Big\},$$

$$[\hat{c}, p]_0 = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \text{ uniformly in } s \Big\}$$

and

$$[\hat{c}, p]_\infty = \Big\{ x = (x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty \Big\}.$$

The notion of difference sequence spaces was introduced by Kızmaz [5], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [6] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, r be non-negative integers, then for $Z = l_\infty$, c, c_0 we have sequence spaces

$$Z(\Delta_r^m) = \{x = (x_k) \in w : (\Delta_r^m x_k) \in Z\},$$

where $\Delta_r^m x = (\Delta_r^m x_k) = (\Delta_r^{m-1} x_k - \Delta_r^{m-1} x_{k+r})$ and $\Delta_r^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_r^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+rv}.$$

Taking $r = 1$, we get the spaces which were studied by Et and Çolak [6]. Taking $m = r = 1$, we get the spaces which

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were introduced and studied by Kızmaz [5].

In [7] Baliarsingh defined the fractional difference operator as follows:
Let $x=(x_k)\in w$ and α be a real number, then the fractional difference operator $\Delta^{(\alpha)}$ is defined by

$$\Delta^{(\alpha)}x_k=\sum_{i=0}^{k}\frac{(-\alpha)_i}{i!}x_{k-i},$$

where $(-\alpha)_i$ denotes the Pochhammer symbol defined as:

$$(-\alpha)_i=\begin{cases}1, & \text{if } \alpha=0 \text{ or } i=0,\\ \alpha(\alpha+1)(\alpha+2)...(\alpha+i-1), & \text{otherwise.}\end{cases}$$

For more details about difference sequence spaces we may refer to ([8,9,10]) and references therein.
Let X be a linear metric space. A function $p:X\to\mathbb{R}$ is called paranorm, if

1. $p(x)\geq 0$, for all $x\in X$,
2. $p(-x)=p(x)$, for all $x\in X$,
3. $p(x+y)\leq p(x)+p(y)$, for all $x,y\in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ *as* $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n-x)\to 0$ *as* $n\to\infty$, then $p(\lambda_n x_n-\lambda x)\to 0$ *as* $n\to\infty$.

A paranorm p for which $p(x)=0$ implies $x=0$ is called total paranorm and the pair (X,p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [11], Theorem 10.4.2, P-183).
A modulus function is a function $f:[0,\infty)\to[0,\infty)$ such that

1. $f(x)=0$ if and only if $x=0$,
2. $f(x+y)\leq f(x)+f(y)$ for all $x\geq 0$, $y\geq 0$,
3. f is increasing
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x)=\frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x)=x^p$, $0<p<1$, then the modulus $f(x)$ is unbounded. Subseqentially, modulus function has been discussed in ([12,13,14,15,16,17,18,19]) and many others.
The concept of 2-normed spaces was initially developed by Gähler [20] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [21]. Since then, many others have studied this concept and obtained various results, see Gunawan ([22,23]) and Gunawan and Mashadi [24]. For more details about the sequence spaces over n-normed spaces see ([25,26,27,28]). Let $n\in\mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d\geq n\geq 2$. A real valued function $||\cdot,\cdots,\cdot||$ on X^n satisfying the following four conditions:

1. $||x_1,x_2,\cdots,x_n||=0$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent in X;
2. $||x_1,x_2,\cdots,x_n||$ is invariant under permutation;
3. $||\alpha x_1,x_2,\cdots,x_n||=|\alpha|\ ||x_1,x_2,\cdots,x_n||$ for any $\alpha\in\mathbb{K}$, and
4. $||x+x',x_2,\cdots,x_n||\leq||x,x_2,\cdots,x_n||+||x',x_2,\cdots,x_n||$

is called an n-norm on X, and the pair $(X,||\cdot,\cdots,\cdot||)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X=\mathbb{R}^n$ being equipped with the n-norm $||x_1,x_2,\cdots,x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1,x_2,\cdots,x_n$ which may be given explicitly by the formula

$$||x_1,x_2,\cdots,x_n||_E=|\det(x_{ij})|,$$

where $x_i=(x_{i1},x_{i2},\cdots,x_{in})\in\mathbb{R}^n$ for each $i=1,2,\cdots,n$.
Let $(X,||\cdot,\cdots,\cdot||)$ be an n-normed space of dimension $d\geq n\geq 2$ and $\{a_1,a_2,\cdots,a_n\}$ be linearly independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1,x_2,\cdots,x_{n-1}||_\infty=\max\{||x_1,x_2,\cdots,x_{n-1},a_i||: i=1,2,\cdots,n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\cdots,a_n\}$.
A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to converge to some $L\in X$ if

$$\lim_{k\to\infty}||x_k-L,z_1,\cdots,z_{n-1}||=0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty}||x_k-x_p,z_1,\cdots,z_{n-1}||=0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

If every cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $F=(f_k)$ be a sequence of modulus function and $(X,||.,\cdots,.||)$ be a n-normed space. Let $p=(p_k)$ be bounded sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X,||.,\cdots,.||)$. In the present paper we define the following sequence spaces:

$$\Big[\hat{c},F,p,||.,\cdots,.||\Big](\Delta^{(\alpha)})=$$
$$\Big\{x=(x_k)\in S(n-X):\lim_n\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}x_{k+s}-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=0,$$
$$\text{uniformly in } s, \text{ for some } \rho>0 \text{ and } L>0\Big\},$$

$$\left[\hat{c}, F, p, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in s, for some } \rho > 0\Big\},$$

and

$$\left[\hat{c}, F, p, ||., \cdots, .||\right]_\infty (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $F(x) = x$, we have

$$\left[\hat{c}, p, ||., \cdots, .||\right] (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{\Delta^{(\alpha)} x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$\left[\hat{c}, p, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} = 0, \text{ uniformly in s, for some } \rho > 0\Big\},$$

and

$$\left[\hat{c}, p, ||., \cdots, .||\right]_\infty (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, $\forall k \in \mathbb{N}$, we have

$$\left[\hat{c}, F, ||., \cdots, .||\right] (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$\left[\hat{c}, F, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big) = 0, \text{ uniformly in s, for some } \rho > 0\Big\},$$

and

$$\left[\hat{c}, F, ||., \cdots, .||\right]_\infty (\Delta^{(\alpha)}) =$$

$$\Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \le K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and inclusion relation between the sequence spaces $\left[\hat{c}, F, p, ||., \cdots, .||\right] (\Delta^{(\alpha)})$, $\left[\hat{c}, F, p, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)})$ and $\left[\hat{c}, F, p, ||., \cdots, .||\right]_\infty (\Delta^{(\alpha)})$.

2 Main Results

Theorem 2.1 *Let $F = (f_k)$ be a sequence of modulus function and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then the classes of sequence $\left[\hat{c}, F, p, ||., \cdots, .||\right] (\Delta^{(\alpha)})$, $\left[\hat{c}, F, p, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)})$ and $\left[\hat{c}, F, p, ||., \cdots, .||\right]_\infty (\Delta^{(\alpha)})$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in \left[\hat{c}, F, p, ||., \cdots, .||\right]_0 (\Delta^{(\alpha)})$ and β, γ be any scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{\Delta^{(\alpha)} x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0$$

and

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots z_{n-1}||\right)\right]^{p_k}=0.$$

Let $\rho_3 = \max(2|\beta|\rho_1, 2|\gamma|\rho_2)$. Since $F = (f_k)$ is non-decreasing function, by using inequality (1.1), we have

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}(\beta x_{k+s}+\gamma y_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$
$$\leq \lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}(\beta x_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}|| + ||\frac{\Delta^{(\alpha)}(\gamma y_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$
$$\leq \lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}|| + ||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$
$$\leq K\lim_n \frac{1}{n}\sum_{k=1}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\right)\right]^{p_k} + K\lim_n \frac{1}{n}\sum_{k=1}\left[f_k\left(||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$
$$\to 0 \quad \text{as } n\to\infty, \text{ uniformly in } s.$$

So that $(\beta x+\gamma y) \in \left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. This proves that $\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$ is a linear space. Similarly, we can prove that $\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})$ and $\left[\hat{c},F,p,||.,\cdots,.||\right](\Delta^{(\alpha)})$ are linear spaces.

Theorem 2.2 *Let $F = (f_k)$ be a sequence of modulus function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$ is a paranormed space with respect to the paranorm defined by*

$$g(x)=\inf\left\{\rho^{\frac{p_n}{H}} : \left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\},$$

where $H=\max(1,\sup_k p_k<\infty)$.

Proof. Clearly $g(x)\geq 0$ for $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. Since $F(0)=0$, we get $g(0)=0$.
Conversely, suppose that $g(x)=0$, then

$$\inf\left\{\rho^{\frac{p_n}{H}} : \left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}=0.$$

This implies that for a given $\varepsilon>0$, there exists some $\rho_\varepsilon(0<\rho_\varepsilon<\varepsilon)$ such that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_\varepsilon},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Thus

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\varepsilon},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_\varepsilon},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq 1,$$

for each n. Suppose that $x_k\neq 0$ for each $k\in\mathbb{N}$. This implies that $\Delta^{(\alpha)}x_{k+s}\neq 0$, for each $k,s\in\mathbb{N}$. Let $\varepsilon\to 0$, then $||\frac{\Delta^{(\alpha)}x_{k+s}}{\varepsilon},z_1,\cdots,z_{n-1}||\to\infty$. It follows that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\varepsilon},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \to\infty,$$

which is a contradiction. Therefore, $\Delta^{(\alpha)}x_{k+s}=0$ for each k and thus $x_k=0$ for each $k\in\mathbb{N}$. Let $\rho_1>0$ and $\rho_2>0$ be such that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

for each n. Let $\rho=\rho_1+\rho_2$. Then by using Minkowski's inequality, we have

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}(x+y)_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}+\Delta^{(\alpha)}y_{k+s}}{\rho_1+\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}\left[\left(\frac{\rho_1}{\rho_1+\rho_2}\right)f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)f_k\left(||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\right)^{\frac{1}{H}}$$
$$\leq 1.$$

Since ρ's are non-negative, so we have

$$g(x+y)=\inf\left\{\rho^{\frac{p_n}{H}} : \left(\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}(x_{k+s}+y_{k+s})}{\rho},\right.\right.\right.\right.$$

$$z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\},$$

$$\le \inf\Big\{\rho_1^{\frac{p_n}{H}}:\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\}$$

$$+\inf\Big\{\rho_2^{\frac{p_n}{H}}:\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}y_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\}.$$

Therefore,

$$g(x+y)\le g(x)+g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}\lambda x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\}.$$

Then

$$g(\lambda x)=\inf\Big\{(|\lambda|t)^{\frac{p_n}{H}}:\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}x_{k+s}}{t},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\}.$$

where $t=\frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_n}\le\max(1,|\lambda|^{\sup p_n})$, we have

$$g(\lambda x)\le\max(1,|\lambda|^{\sup p_n})\inf\Big\{t^{\frac{p_n}{H}}:\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}x_{k+s}}{t},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality.
This completes the proof of the theorem.

Theorem 2.3 *Let $F=(f_k)$ be a sequence of modulus function. Then the following statements are equivalent:*
(i) $\Big[\hat{c},p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})\subseteq\Big[\hat{c},F,p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$,
(ii) $\Big[\hat{c},p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})\subseteq\Big[\hat{c},F,p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$,
(iii) $\sup_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}<\infty$, *where* $t=||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||>0$.

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Suppose $\Big[\hat{c},p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})\subseteq\Big[\hat{c},F,p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$ and let (iii) does not hold. Then for some $t>0$

$$\sup_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=\infty,$$

and therefore there is a sequence (n_i) of positive integers such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(i^{-1})]^{p_k}>i,\quad i=1,2,... \tag{2}$$

Define $x=(x_k)$ by

$$x_k=\begin{cases}i^{-1}, & 1\le k\le n_i,\quad i=1,2,...\\ 0, & k\ge n_i.\end{cases}$$

Then $x=(x_k)\in\Big[\hat{c},p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})$ but $x=(x_k)\notin\Big[\hat{c},F,p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Suppose $x=(x_k)\in\Big[\hat{c},p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$ and $x=(x_k)\notin\Big[\hat{c},F,p,||.,\cdots,.||\Big]_\infty(\Delta^{(\alpha)})$. Then

$$\sup_{s,n}\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=\infty. \tag{3}$$

Let $t=||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||$ for each k and fixed s, then by eqn.(2.2)

$$\sup_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=\infty,$$

which contradicts (iii). Hence (i) must hold.

Theorem 2.4 *Let $1\le p_k\le\sup_k p_k<\infty$. Then the following statements are equivalent :*
(i) $\Big[\hat{c},F,p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})\subseteq\Big[\hat{c},p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})$,
(ii) $\Big[\hat{c},F,p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})\subseteq\Big[\hat{c},p,||.,\cdots.||\Big]_\infty(\Delta^{(\alpha)})$,
(iii) $\inf_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}>0,\quad t>0.$

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Suppose $\Big[\hat{c},F,p,||.,\cdots,.||\Big]_0(\Delta^{(\alpha)})\subseteq\Big[\hat{c},p,||.,\cdots.||\Big]_\infty(\Delta^{(\alpha)})$ and let (iii) does not hold. Then

$$\inf_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=0,\quad t>0. \tag{4}$$

We can choose an index sequence (n_i) such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(i)]^{p_k}<i^{-1},\quad i=1,2,...$$

Define the sequence $x=(x_k)$ by

$$x_k = \begin{cases} i, & 1\le k\le n_i, \quad i=1,2,...\\ 0, & k\ge n_i. \end{cases}$$

Thus by eqn.(2.3), $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$ but $x=(x_k)\notin\left[\hat{c},p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Let $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. That is,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}=0,$$

$$\text{uniformly in } s. \tag{5}$$

Suppose (iii) hold and $x=(x_k)\notin\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. Then for some number $\varepsilon_0>0$ and index n_0, we have $||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\ge\varepsilon_0$, for some $s>s'$ and $1\le k\le n_0$. Therefore

$$[f_k(\varepsilon_0)]^{p_k}\le\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$

and consequently by eqn.(2.4)

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(\varepsilon_0)]^{p_k}=0,$$

which contradicts (iii). Hence $\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})\subseteq\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$.

Theorem 2.5 *Let $F=(f_k)$ be a sequence of modulus function. Let $1\le p_k\le\sup_k p_k<\infty$. Then*

$$\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})\subseteq\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)}) \text{ hold if}$$

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=\infty,\quad t>0. \tag{6}$$

Proof. Suppose $\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})\subseteq\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$ and let eqn.(2.5) does not hold. Therefore there is a number $t_0>0$ and an index sequence (n_i) such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(t_0)]^{p_k}\le N<\infty,\quad i=1,2,... \tag{7}$$

Define the sequence $x=(x_k)$ by

$$x_k = \begin{cases} t_0, & 1\le k\le n_i, \quad i=1,2,...\\ 0, & k\ge n_i. \end{cases}$$

Clearly, $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})$ but $x=(x_k)\notin\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. Hence eqn.(2.5) must hold.

Conversely, if $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})$, then for each s and n

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z||\right)\right]^{p_k}\le N<\infty. \tag{8}$$

Suppose that $x=(x_k)\notin\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. Then for some number $\varepsilon_0>0$ there is a number s_0

$$||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\ge\varepsilon_0,\ \text{ for } s\ge s_o.$$

Therefore

$$[f_k(\varepsilon_o)]^{p_k}\le\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k},$$

and hence for each k and s we get

$$\frac{1}{n}\sum_{k=1}^{n}[f_k(\varepsilon_o)]^{p_k}\le N<\infty,$$

for some $N>0$, which contradicts eqn. (2.5). Hence

$$\left[\hat{c},F,p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})\subseteq\left[\hat{c},p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)}).$$

This completes the proof.

Theorem 2.6 *Let $F=(f_k)$ be a sequence of modulus function and let $1\le p_k\le\sup_k p_k<\infty$. Then*

$$\left[\hat{c},p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})\subseteq\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)}) \text{ hold if}$$

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=0,\quad t>0 \tag{9}$$

Proof. Let $\left[\hat{c},p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})\subseteq\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. Suppose that eqn.(2.8) does not hold. Then for some $t_0>0$,

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k}=L\neq0. \tag{10}$$

Define $x=(x_k)$ by

$$x_k=t\sum_{v=0}^{k-m}(-1)^m\binom{m+k-v-1}{k-v}$$

for $k=1,2,....$ Then $x=(x_k)\notin\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$ but $x=(x_k)\in\left[\hat{c},p,||.,\cdots,.||\right]_\infty(\Delta^{(\alpha)})$. Hence eqn. (2.8) must hold.

Conversely, let $x=(x_k)\in\left[\hat{c},p,||.,\cdots,.||\right]_{\infty}(\Delta^{(\alpha)})$. Then for every k and s, we have

$$||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\leq N<\infty.$$

Therefore

$$\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\leq[f_k(N)]^{p_k}$$

and

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(||\frac{\Delta^{(\alpha)}x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}$$

$$\leq\lim_n\frac{1}{n}\sum_{k=1}^{n}[f_k(N)]^{p_k}=0.$$

Hence $x=(x_k)\in\left[\hat{c},F,p,||.,\cdots,.||\right]_0(\Delta^{(\alpha)})$. This completes the proof.

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