Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/080101

Some Distributional Properties of Sample Coefficient of Determination and its Adjusted Version

Manoj Kumar [1], *Vikas Bist* [2], *Man Inder Kumar* [1,*] *and Jerome P. Keating* [3]

[1]Department of Statistics, Panjab University, Chandigarh, India
[2]Department of Mathematics, Panjab University, Chandigarh, India
[3]Department of Management Science and Statistics, University of Texas at San Antonio, Texas, USA

Received: 25 Apr. 2020, Revised: 2 Jun. 2020, Accepted: 21 Aug. 2020
Published online: 1 Jan. 2021

Abstract: In this paper, the efforts have been made to analyze the skewness & kurtosis of sampling distribution of R^2 and R_a^2 evolving the large sample asymptotic theory. The distributional properties such as mean and variance of the coefficient of determinations and its adjusted version have been re-visited to affirm the efficiency of proposed work. In section 2, we describe the model, population and sample parameters for the goodness of fit along with other related terms. The sampling distributions of statistics along with distributional properties have been discussed in section 3. Lastly, the derivations of results utilized for deriving main results are provided in the appendix.

Keywords: Regression models, Coefficient of determination, Skewness & Kurtosis

1 Introduction

The regression is most widely and commonly used technique for deriving relation between study variable with set of independent variables. The sample coefficient of determination (R^2) and its adjusted version is well-known measure for explaining goodness of fit of model. The fundamental of explaining the goodness of fit of a model depends on the amount of variability being explained by the the set of independent variables. Therefore, they exhibit the adequacy of model along with the capability of predicting the value of dependent or study variables in regression relationship.

The R^2 and its adjusted version have vital role in selection of independent variables during the process of model building. For an illustration, let us suppose a researcher have several competing models and wish to make a choice for best suitable model (with least lack of fit or maximum proportion of variability explained by variables). Then, R^2 and R_a^2 render their assistance in the development of model from a set of competing / similar models based on amount of variability explained. Some of the researchers have pointed out interesting applications of R^2 and R_a^2 and facts pertaining to the hypothesis testing of regression coefficients, see; [12,11,20,19,18,13,10].

There are some limitations and well-known cautions related with the practical use of R^2 and R_a^2 for drawing information from a set of data. In spite of these limitations researchers keep applying R^2 and R_a^2 for citing the quality of the models, see; [6], [17], [14], [15]. Also, Cramer [3] studied the sampling distribution of the R^2 and R_a^2, which are widely used for measuring the goodness of fit of linear regression model. The exact expression derived in [3] turned out to be sufficiently complex and difficult for researchers to deduce any clear information. Therefore, findings pertain to numerical evaluation have a curtailed scope. A comparatively simple and alternative solution of large sample asymptotic approximations has been narrated in literature see; [9]. On parallel lines [7] utilized asymptotic approximations to draw inferences. Also see; [1,2,8,9,4,5].

In this paper, the efforts have been made to analyze the skewness & kurtosis of sampling distribution of R^2 and R_a^2 along with a re-visit to mean, variance. In the next section, we describe the model, population and sample statistics for goodness

* Corresponding author e-mail: maninderbajarh@gmail.com

of fit along with related terms. The sampling distributions of statistics are considered and the four salient features are studied. Lastly, the derivations of results utilized for deriving main results have been provided are given.

2 Model and measures of goodness of fit

Let us postulate the linear regression model as

$$y = \alpha e + X\beta + u \tag{1}$$

where y is a vector of $n \times 1$ observations on the variable to be explained, α is a scalar representing the intercept term, e is a $n \times 1$ vector with all elements of unity, X is a $n \times p$ full column rank matrix of n-observations on p-explanatory variables, β is a $p \times 1$ vector of the coefficients associated with them and u is a $n \times 1$ vector of disturbances.

Further, it is assumed that the elements of disturbance vector (u) are distributed independently and identically following a normal distribution with zero mean and unknown positive variance σ^2. Based on literature, let us allow to denote the matrix A as follows

$$A = \left(I_n - \frac{1}{n}ee'\right) \tag{2}$$

The sample coefficient of determinations is defined as

$$R^2 = \frac{y'AX(X'AX)^{-1}X'Ay}{yA'y} = \frac{SSR}{SSR + SSE} \tag{3}$$

While the adjusted version of sample coefficient of determination, which is popularly known as R_a^2, obtained by applying the correction for degree of freedom as specified here-under

$$R_a^2 = R^2 - \left(\frac{p}{n-p-1}\right)(1-R^2) \leq R^2 \tag{4}$$

Both R^2 and R_a^2 are generally applied as measure for the goodness of the fit in linear regression model.

Further, [16] have provided the population version of the coefficient of determination, θ, as specified below

$$\theta = \frac{\beta'X'AX\beta}{n\sigma^2 + \beta'X'AX\beta} = \frac{\frac{\beta'X'AX\beta}{n}}{\sigma^2 + \frac{\beta'X'AX\beta}{n}} = \frac{\beta'S\beta}{\sigma^2 + \beta'S\beta}; \tag{5}$$

where

$$S = \frac{1}{n}X'AX \tag{6}$$

3 Distributional properties

[3] analyzed the some of the distributional properties of R^2 and R_a^2. However, the exact expressions for the first two moments are found to be such that it is difficult to draw any clear inference due to complexity of mathematical expression. Therefore, [3] has evaluated the results numerically and brought out some interesting findings. With a view to investigate the large sample asymptotic properties of R^2 and R_a^2; Let us first introduce the following set of notations:

$$z = \frac{u'AX\beta}{n^{\frac{1}{2}}\sigma^2} \tag{7}$$

$$w = \left(\frac{u'u}{n^{\frac{1}{2}}\sigma^2} - n^{1/2}\right) \tag{8}$$

$$\theta = \left(1 + \frac{n\sigma^2}{\beta'X'AX\beta}\right)^{-1} \tag{9}$$

$$M = \left(AX(X'AX)^{-1}X'A + \frac{\theta}{n}ee'\right) \tag{10}$$

It is also assumed that the independent variables are asymptotically co-operative such that the limiting form of the matrix $(\frac{X'AX}{n})$, as n grows large enough, is finite and non-singular. By utilizing the results of [9], we can express R^2 and its adjusted version as follows

$$R^2 = \theta + (1-\theta)\left[\frac{f}{n^{\frac{1}{2}}} + \frac{g}{n} + \frac{h}{n^{\frac{3}{2}}}\right] + O(n^{-2}) \tag{11}$$

$$R_a^2 = \theta + (1-\theta)\left[\frac{f}{n^{\frac{1}{2}}} + \frac{g-p}{n} + \frac{h+pf}{n^{\frac{3}{2}}}\right] + O(n^{-2}) \tag{12}$$

where

$$f = 2(1-\theta)z - \theta w \tag{13}$$

$$g = \left(\frac{u'Mu}{\sigma^2}\right) - 4(1-\theta)^2 z^2 - 2(1-\theta)(1-2\theta)zw + \theta(1-\theta)w^2 \tag{14}$$

$$h = (1-\theta)\left[\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 + (1-\theta)(2zw+w)^2\right][2(1-\theta)z - \theta w] - \left(\frac{1-\theta}{\sigma^2}\right)(2z+w)u'Mu \tag{15}$$

It is necessary to observe that the quantities defined above i.e. f, g and h are of order $O(1)$.

3.1 Mean

The sample mean is fundamentally employed to reveal basic properties of an estimator. [3] has also studied this property of sample coefficient of determination. Here, an attempt has been made to first verify the authenticity of our method and assumptions laid down to study properties of sampling distributions of R^2 and its adjusted version. Utilizing (11) and (12) defined in earlier part of this section, the mean of R^2 and R_a^2 can be observe as given by

$$E(R^2) = \theta + (1-\theta)\left[\frac{E(f)}{n^{\frac{1}{2}}} + \frac{E(g)}{n} + \frac{E(h)}{n^{\frac{3}{2}}}\right] \tag{16}$$

$$E(R_a^2) = \theta + (1-\theta)\left[\frac{E(f)}{n^{\frac{1}{2}}} + \frac{E(g)-p}{n} + \frac{E(h)}{n^{\frac{3}{2}}}\right] \tag{17}$$

Utilizing the results of appendix, the expected values of f, g and h are substituted in (16) and (17) to obtain the expression for the mean of sample coefficient of determination and its adjusted version up-to the order $O(n^{-3/2})$ as given by

$$E(R^2) = \theta + \frac{(1-\theta)[p - \theta(1-2\theta)]}{n} \tag{18}$$

$$E(R_a^2) = \theta - \frac{\theta(1-\theta)(1-2\theta)}{n} \tag{19}$$

Looking at aforesaid equations, it can be observed that both R^2 and R_a^2 have same asymptotic mean θ. However, the inference drawn differs when higher order terms are retained. For instance, if we consider mean up-to the order $O(n^{-3/2})$, this is observed that R_a^2 has smaller magnitude of mean in comparison to R^2 when

$$\theta > \frac{(1-\theta)}{n}\left[\theta(1-2\theta) - \frac{p}{2}\right] \tag{20}$$

is satisfied for all values of θ & p and **align with the results available in literature**.

3.2 Variance

The variance of sample coefficient of determination and its adjusted version is re-visited in this section due to underlying facts cited in previous sub-section. The expressions of variance for R^2 along with R_a^2 are derived in this sub-section. Further, by using the results obtained in appendix and (11), the following expression up-to the order $O(n^{-2})$ are derived as under

$$\begin{aligned}
V(R^2) &= E[R^2 - E(R^2)]^2 \\
&=(1-\theta)^2 E\left[\frac{f}{n^{1/2}} + \frac{g-p+\theta(1-2\theta)}{n} + \frac{h}{n^{3/2}})\right]^2 \\
&=(1-\theta)^2 E\Big[\frac{f^2}{n} + \frac{2f(g-p)+2f\theta(1-2\theta)}{n^{3/2}} + \frac{(g-p+\theta(1-2\theta))^2+2fh}{n^2}\Big] \\
&=(1-\theta)^2\Big[\frac{2\theta(2-\theta)}{n^2}\Big] + \frac{(1-\theta)^2}{n^2}\Big[2E(fh)+E(g^2)+p^2-2pE(g)+\theta^2(1-2\theta)^2+(2\theta-4\theta^2)(E(g)-p)+2E(fg)\Big] \\
&=X_1 + Y_1 Z_1 + O_p(n^{-2})
\end{aligned} \tag{21}$$

where

$$X_1 = \frac{2\theta(2-\theta)(1-\theta)^2}{n^2} \tag{22}$$

$$Y_1 = \frac{(1-\theta)^2}{n^2} \tag{23}$$

$$Z_1 = 2E(fh)+E(g^2)+p^2-2pE(g)+\theta^2(1-2\theta)^2+2\theta(1-2\theta)(E(g)-p)+2E(fg) \tag{24}$$

Again, using the results pertaining to expectations of various combination of f, g and h from the section 5 (appendix) and (12), the expression for the variance of R_a^2 up-to the order $O(n^{-2})$ given by

$$\begin{aligned}
V(R_a^2) &= E[R_a^2 - E(R_a^2)]^2 \\
&=(1-\theta)^2 E\left[\frac{f}{n^{1/2}} + \frac{g-p+\theta(1-2\theta)}{n} + \frac{h+pf}{n^{3/2}}\right]^2 \\
&=(1-\theta)^2 E\Big[\frac{f^2}{n} + \frac{2f(g-p)+2f\theta(1-2\theta)}{n^{3/2}} + \frac{(g-p+\theta(1-2\theta))^2+2fh+2pf^2}{n^2}\Big] \\
&=(1-\theta)^2\Big[\frac{2\theta(2-\theta)}{n^2}\Big] + \frac{(1-\theta)^2}{n^2}\Big[2E(fh)+E(g^2)+p^2-2pE(g)+\theta^2(1-2\theta)^2+2\theta(1-2\theta)(E(g)-p)+2E(fg)\Big] \\
&\quad + \frac{(1-\theta)^2}{n^2}2pE(f^2) \\
&=X_1 + Y_1 Z_1 + [4p\theta(2-\theta)]Y_1 + O_p(n^{-2})
\end{aligned} \tag{25}$$

By utilizing (21) and (25) for comparing both versions, it is easy to draw the inference that R^2 has always smaller variance than its adjusted version (R_a^2) as shown here-under

$$\begin{aligned}
V(R_a^2) - V(R^2) &= E[R_a^2 - E(R_a^2)]^2 - E[R^2 - E(R^2)]^2 \\
&= 4p\theta(2-\theta)\left(\frac{1-\theta}{n}\right)^2
\end{aligned} \tag{26}$$

the results demonstrated in (26) above, **coincides with the results reported in Cramer (1987)**.

3.3 Skewness

Generally in Statistics, the measure of asymmetry of any distribution about its mean is termed as skewness. The Pearson's measure of skewness is being utilized to study the amount of asymmetry in sample coefficient of determination and its adjusted version, as given by

$$Sk(R^2) = \frac{E[R^2 - E(R^2)]^3}{\{E[R^2 - E(R^2)]^2\}^{3/2}} \tag{27}$$

$$Sk(R_a^2) = \frac{E[R_a^2 - E(R_a^2)]^3}{\{E[R_a^2 - E(R_a^2)]^2\}^{3/2}} \tag{28}$$

In order to derive expression for skewness, the expression for third central moment of R^2 and R_a^2 is requisite. Therefore, the third central moment of R^2 up-to order $O(n^{-5/2})$ is given by

$$\begin{aligned}
E[R^2 - E(R^2)]^3 &= (1-\theta)^3 E\left[\frac{f}{n^{1/2}} + \frac{g-p+\theta(1-2\theta)}{n} + \frac{h}{n^{3/2}}\right]^3 \\
=&(1-\theta)^3 E\Big[\frac{f^3}{n^{3/2}} + \frac{3f^2(g-p)+3f^2(\theta)(1-2\theta)}{n^2} + \frac{3f^2h+3f(g-p)^2+3f\theta^2(1-2\theta)^2+6f\theta(1-2\theta)(g-p)}{n^{5/2}}\Big] \\
=&\frac{(1-\theta)^3}{n^{3/2}}E(f^3) + \frac{(1-\theta)^3}{n^2}[3E(f^2g) - 3pE(f^2) + 3\theta(1-2\theta)E(f^2)] + \frac{(1-\theta)^3}{n^{5/2}}[3E(f^2h) + 3E(fg^2) - 6pE(fg) \\
&+ 6\theta(1-2\theta)E(fg)] \\
=&X_2 + Y_2 + Z_2
\end{aligned} \tag{29}$$

where

$$X_2 = \frac{(1-\theta)^3 E(f^3)}{n^{3/2}} \tag{30}$$

$$Y_2 = \frac{(1-\theta)^3}{n^2}[3E(f^2g) - 3pE(f^2) + 3\theta(1-2\theta)E(f^2)] \tag{31}$$

$$Z_2 = \frac{(1-\theta)^3}{n^{5/2}}[3E(f^2h) + 3E(fg^2) - 6pE(fg) + 6\theta(1-2\theta)E(fg)] \tag{32}$$

The results of appendix are utilized in aforesaid expression for substituting the expected values of various combinations of f, g and h, as and when required.

In a similar manner, the third central moment of R_a^2 can be obtained up-to the order $O(n^{-5/2})$ as under

$$\begin{aligned}
E[R_a^2 - E(R_a^2)]^3 &= (1-\theta)^3 E\left[\frac{f}{n^{1/2}} + \frac{g-p+\theta(1-2\theta)}{n} + \frac{h+pf}{n^{3/2}}\right]^3 \\
=&(1-\theta)^3 E\Big[\frac{f^3}{n^{3/2}} + \frac{3f^2(g-p)+3f^2\theta(1-2\theta)}{n^2} + \frac{3f^2h+3pf^3+3f(g-p)^2+3f\theta^2(1-2\theta)^2}{n^{5/2}} + \frac{6f(g-p)\theta(1-2\theta)}{n^{5/2}}\Big] \\
=&\frac{(1-\theta)^3}{n^{3/2}}E(f^3) + \frac{(1-\theta)^3}{n^2}\Big[3E(f^2g) - 3pE(f^2) + 3\theta(1-2\theta)E(f^2)\Big] + \frac{(1-\theta)^3}{n^{5/2}}\Big[3E(f^2h) + 3E(fg^2) - 6pE(fg) \\
&+ 6\theta(1-2\theta)E(fg)\Big] + \frac{(1-\theta)^3}{n^{5/2}}3pE(f^3) \\
=&X_2 + Y_2 + Z_2 + \frac{(1-\theta)^3}{n^{5/2}}3pE(f^3)
\end{aligned} \tag{33}$$

The third central moment of all symmetric distributions is always zero. The first inference drawn out by looking at (29) and (33) is that both versions of sample coefficient of determination are non-zero. In other-words, departure of symmetry exists in both versions.

The comparison of (29) and (33) clearly reveals that the third central moment of R^2 is greater than R_a^2 up to order $O(n^{-5/2})$. Also, from (24) and (25) it is clear that second central moment of R_a^2 is greater than that of R^2. Therefore, this implies that the $Sk(R^2) > Sk(R_a^2)$. Further, values of skewness of R^2 and adjusted R^2 are calculated using the expressions

Table 1: Skewness of R^2 and R_a^2 for different values of n, p and θ.

Sk.	$\theta = 0.2$		$\theta = 0.4$		$\theta = 0.6$		$\theta = 0.8$	
	Sk(R^2)	Sk(R_a^2)	Sk(R^2)	Sk(R_a^2)	Sk(R^2)	Sk(R_a^2)	Sk(R^2)	Sk(R_a^2)
$p = 2$								
n=10	-0.59	-0.39	-4.24	-2.65	-11.77	-7.47	-33.30	-21.56
n=20	-0.71	-0.56	-3.29	-2.54	-9.09	-7.05	-26.50	-20.64
n=50	-0.51	-0.46	-2.15	-1.92	-6.03	-5.40	-18.05	-16.16
$p = 5$								
n=10	-1.23	-0.59	-4.90	-2.01	-11.86	-4.91	-29.16	-12.82
n=20	-1.18	-0.73	-3.74	-2.17	-9.25	-5.36	-24.63	-14.57
n=50	-0.79	-0.62	-2.38	-1.85	-6.16	-4.77	-17.53	-13.60
$p = 7$								
n=10	-1.49	-0.65	-5.51	-1.87	-12.33	-4.19	-27.51	-10.18
n=20	-1.42	-0.78	-4.12	-2.05	-9.53	-4.74	-23.79	-12.21
n=50	-0.96	-0.70	-2.57	-1.83	-6.29	-4.47	-17.27	-12.32
$p = 9$								
n=10	-1.69	-0.69	-6.22	-1.81	-13.03	-3.78	-26.47	-8.59
n=20	-1.62	-0.83	-4.55	-1.99	-9.92	-4.32	-23.19	-10.58
n=50	-1.12	-0.76	-2.77	-1.82	-6.46	-4.23	-17.08	-11.29

Remarks: *Sign of third order central moment is indicator of type of skewness.*

derived in this section and appendix results. The numerically calculated values for different values of θ, $nandp$ are shown in **Table 4.1**.

The numerical difference of $Sk(R^2) - Sk(R_a^2)$ for different values of n, p and θ using MATLAB tool is tabulated in **Table 4.2**. The values for different combinations of n, p and θ reveals that R^2 is more negatively skewed than adjusted version of R^2. Further, with increase in the value of θ for same combination of n and p degree of difference in skewness witness an increase trend. It can also be observed from table 4.2 that for given value of θ and n there is clearly an increasing trend as value of p increases. The value of difference between skewness of R^2 and that of R_a^2 is positive (ignoring negative sign) for different values of n, p and θ, which supports our theoretical findings that R^2 is more skewed (negatively).

3.4 Kurtosis

The Pearson measure of Kurtosis is widely being employed to measure the degree of flatness or peakness of any probability distribution. The Pearson's measures of Kurtosis of R^2 and R_a^2 up-to the order $O(n^{-1})$ can be expressed as

$$K(R^2) = \frac{E[R^2 - E(R^2)]^4}{\{E[R^2 - E(R^2)]^2\}^2} - 3 \tag{34}$$

Table 2: Difference of Skewness of R^2 and Adjusted R^2 using different values of n, p and θ.

Diff. Skewness	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$
$n = 10, p = 2$	0.20	1.59	4.30	11.74
$n = 20, p = 2$	0.16	0.75	2.04	5.86
$n = 50, p = 2$	0.05	0.22	0.63	1.88
$n = 10, p = 5$	0.63	2.89	6.94	16.34
$n = 20, p = 5$	0.45	1.58	3.89	10.07
$n = 50, p = 5$	0.17	0.53	1.39	3.92
$n = 10, p = 7$	0.84	3.65	8.14	17.33
$n = 20, p = 7$	0.63	2.07	4.79	11.57
$n = 50, p = 7$	0.26	0.74	1.83	4.95
$n = 10, p = 9$	1.01	4.40	9.26	17.89
$n = 20, p = 9$	0.79	2.56	5.60	12.61
$n = 50, p = 9$	0.35	0.95	2.23	5.79

Remarks: *Ignoring the negative signs of third order central moments as sign is indicator of type of skewness.*

$$K(R_a^2) = \frac{E[R_a^2 - E(R_a^2)]^4}{\{E[R_a^2 - E(R_a^2)]^2\}^2} - 3 \tag{35}$$

In order to find kurtosis of R^2, consider the fourth central moment of R^2 up-to the order $O(n^{-3})$ as given by

$$\begin{aligned}
&E[R^2 - E(R^2)]^4 = (1-\theta)^4 E\left[\frac{f}{n^{1/2}} + \frac{g-p+\theta(1-2\theta)}{n} + \frac{h}{n^{3/2}}\right]^4 \\
&=(1-\theta)^4 E\Big[\frac{f^4}{n^2} + \frac{4f^3(g-p)+4f^3\theta(1-2\theta)}{n^{5/2}} + \frac{4f^3h+6f^2(g-p)^2+6\theta^2(1-2\theta)^2f^2}{n^3} + \frac{12\theta(1-2\theta)f^2(g-p)}{n^3}\Big] \\
&=\frac{(1-\theta)^4}{n^2}E(f^4) + \frac{(1-\theta)^4}{n^{5/2}}\Big[4E(f^3(g-p)) + 4E(f^3)\theta(1-2\theta)\Big] + \frac{(1-\theta)^4}{n^3}\Big[4E(f^3h) + 6E(f^2(g-p)^2) + 6\theta^2(1-2\theta)^2 \\
&\quad E(f^2) + 12\theta(1-2\theta)E(f^2(g-p))\Big] \\
&=X_3 + Y_3 + Z_3
\end{aligned} \tag{36}$$

where

$$X_3 = \frac{(1-\theta)^4}{n^2}E(f^4) \tag{37}$$

$$Y_3 = \frac{(1-\theta)^4}{n^3}[4E(f^3h) + 6E(f^2(g-p)^2) + 6\theta^2(1-\theta)^2E(f^2) + 12\theta(1-2\theta)E(f^2(g-p))] \tag{38}$$

$$Z_3 = \frac{(1-\theta)^4}{n^{5/2}}[4E(f^3(g-p)) + 4\theta(1-2\theta)E(f^3)] \tag{39}$$

Table 3: Kurtosis of R^2 and R_a^2 for different values of n, p and θ.

Kr.	$\theta=0.2$		$\theta=0.4$		$\theta=0.6$		$\theta=0.8$	
	K(R^2)	K(R_a^2)	K(R^2)	K(R_a^2)	K(R^2)	K(R_a^2)	K(R^2)	K(R_a^2)
				$p=2$				
n=10	14.22	7.39	18.31	9.76	31.16	17.17	75.92	42.73
n=20	7.41	5.22	9.47	6.82	17.19	12.44	44.40	32.06
n=50	3.13	2.71	4.06	3.56	7.58	6.65	19.97	17.37
				$p=5$				
n=10	11.04	3.19	26.48	7.34	53.38	15.78	122.02	39.89
n=20	7.96	3.73	15.20	7.13	31.57	15.13	79.69	39.35
n=50	4.17	2.92	6.81	4.86	14.33	10.25	38.43	27.49
				$p=7$				
n=10	9.35	1.87	29.76	5.98	64.07	14.02	140.43	35.86
n=20	7.85	2.92	18.23	6.76	39.71	15.21	97.86	39.66
n=50	4.65	2.87	8.48	5.33	18.53	11.72	49.50	31.54
				$p=9$				
n=10	7.99	1.02	31.93	4.90	72.34	12.40	152.59	32.11
n=20	7.59	2.27	20.75	6.27	46.87	14.78	112.69	38.67
n=50	5.01	2.74	10.04	5.61	22.52	12.71	59.68	34.21

Similarly, to obtain the expression for kurtosis of R_a^2, the fourth central moment of R_a^2 to order $O(n^{-3})$ is derived as follows:

$$
\begin{aligned}
E(R_a^2-E(R_a^2))^4 &= (1-\theta)^4 E\left[\frac{f}{n^{1/2}}+\frac{g-p+\theta(1-2\theta)}{n}+\frac{h+pf}{n^{3/2}}\right]^4 \\
&=(1-\theta)^4 E\left[\frac{f^4}{n^2}+\frac{4f^3(g-p)+4f^3\theta(1-2\theta)}{n^{5/2}}+\frac{4f^3h+4pf^4+6f^2(g-p)^2+6f^2\theta^2(1-2\theta)^2}{n^3}+\frac{12f^2(g-p)\theta(1-2\theta)}{n^3}\right] \\
&=\frac{(1-\theta)^4}{n^2}E(f^4)+\frac{(1-\theta)^4}{n^{5/2}}\left[4E(f^3(g-p))+4(f^3\theta(1-2\theta))\right]+\frac{(1-\theta)^4}{n^3}\Big[4E(f^3h)+6E(f^2(g-p)^2)+6\theta^2(1-2\theta)^2 \\
&\quad E(f^2)+12\theta(1-2\theta)E(f^2(g-p))\Big]+\frac{(1-\theta)^4}{n^3}4pE(f^4) \\
&=X_3+Y_3+Z_3+\frac{(1-\theta)^4}{n^3}4pE(f^4)
\end{aligned}
\tag{40}
$$

The comparison using (36) and (40) reveal that the fourth central order moment of R^2 is greater than that of R_a^2. However, exact value is calculated using equations (34) and (35) by inserting values of (36), (40), (24), (25) and results cited in Appendix.

The numerically calculated values of kurtosis for both version of sample measure of goodness of fit are tabulated for some values of n, p and θ as **Table 4.3**. Also, the difference of $K(R^2)$ and $K(R_a^2)$ is tabulated under **Table 4.4** using different values of n, p and θ. The kurtosis of R^2 is seems to be higher than its adjusted version. Further, with increase in the value of θ for given value of n and p the difference values is showing an increasing trend. However, for given value of θ and p

Table 4: Difference of Kurtosis of R^2 and R_a^2 using different values of θ, n and p.

Diff. Kurtosis	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$
$n = 10, p = 2$	6.83	8.55	13.98	33.19
$n = 20, p = 2$	2.18	2.65	4.75	12.34
$n = 50, p = 2$	0.41	0.49	0.94	2.60
$n = 10, p = 5$	7.86	19.14	37.60	82.13
$n = 20, p = 5$	4.23	8.07	16.45	40.34
$n = 50, p = 5$	1.24	1.94	4.07	10.94
$n = 10, p = 7$	7.48	23.78	50.04	104.56
$n = 20, p = 7$	4.93	11.47	24.51	58.19
$n = 50, p = 7$	1.78	3.15	6.81	17.96
$n = 10, p = 9$	6.97	27.02	59.94	120.48
$n = 20, p = 9$	5.32	14.48	32.08	74.02
$n = 50, p = 9$	2.27	4.44	9.81	25.47

Remarks: The difference between Kurtosis of R^2 and R_a^2 is positive for different values of n, p and θ.

with increase in value of n the difference has observed a decreasing trend.

Acknowledgement

The authors are grateful to the anonymous referees for their constructive suggestions for improving the quality of this paper.

References

[1] A. van der Linde, G. Tutz, On association in regression: the coefficient of determination revisited Statistics, **42 (1)**, 1-24 (2008).

[2] A.K. Srivastava, Shobhit, A family of estimators for the coefficient of determination in linear regression models J. Appl. Statist. Sci., **11 (2)**, 133-144 (2002).

[3] Cramer, J.S., Mean and Variance of R^2 in small and moderate samples. Journal of Econometrics, **35**, 253-266 (1987).

[4] C.-L. Cheng, Shalabh, G. Garg, Coefficient of determination for multiple measurement error models, Journal of Multivariate Analysis,**Volume 126**, 137-152 (2014).

[5] Luke J. Saunders, Richard A. Russell, David P. Crabb, The Coefficient of Determination: What Determines a Useful R2 Statistic?. Invest. Ophthalmol. Vis. Sci. **53(11)**, 6830-6832 (2012).

[6] Srivastava, A.K.; Srivastava V.K.; Ullah, A., The Coefficient of determination and its adjusted version in linear regression models. Econometric Reviews, **14**, 229-240 (1995).

[7] Smith, M.D., Comparing Approximations to the Expectation of a Ratio of Quadratic forms in Normal variables. Econometric Reviews, **15**, 81-95 (1996).

[8] Ullah, A.; Srivastava, V.K., Moments of the ration of quadratic forms in non-normal variables with econometric examples. Journal of Econometrics, **62**, 129-142 (1994).

[9] Srivastava, V. K., R. Tiwari., Evaluation of Expectations of Product of Stochastic Matrices. Scandinavian Journal of Statistics, **3**, 135-138 (1976).

[10] Draper, N.R. and Smith, H., Applied Regression Analysis, New York: John Wiley, 1981.

[11] Healy, M.J.R., The Use of R2 as a Measure of Goodness of Fit. Journal of Royal Statistical Society,**A(147)**, 608 -609 (1984).

[12] Judge, G.G., Griiffithss, W.E., Hill, R.C., Luthkepphl, H. and Lee, T. The Theory and Practice of Econometrics. (John Wiley and Sons, New York) (1985).

[13] Johnston, J. Econometric Methods, McGraw Hill, New York, 1991.

[14] Keating, J.P. and Mason, R.L., Pitman Nearness Comparison of the traditional Estimator of the Coefficient of Determination and Its Adjusted Version in Linear Regression Models. Communication in Statistics- Theory and Methods, **34**, 367-374 (2005).
[15] Knautz, H. and Wilde, J., R Square in Restricted Linear Models. Communications in Statistics-Theory and Methods, **22**, 641-645 (1993).
[16] Koerts, J. and Abrahamse, A. P. J. The Correlation Coefficient in the General Linear Model. European Economic Review, **1(3)**, 401-427 (1970).
[17] Kumar, M. and Srivastava, V.K. Pitman Nearness and Concentration Probability Comparision of the Sample Coefficient of Determination and Its Adjusted Version in Linear Regression Models. Communication in Statistics-Theory and Methods, **Vol. 33 (7)**, 1629-1641 (2004).
[18] Kvalseth, T. O., Cautionary Note About R2. The American Statistician, **39**, 279-285 (1985).
[19] Montgomery, D.B. and Morrison, D. G., A Note On Adjusting R2. The Journal of Finance, **28(4)**, 1009-1013 (1973)
[20] Rao, C.R., Linear Statistical Inference and Its Application, John Wiley& Sons, 2002.

Appendix: Derivation of results

In order to find the expectations of various combinations of f, g and h based on the equations (13), (14) and (15), Let us allow to state following theorem.

Theorem 4: The expression for the expectations of the f, g, h and various combinations of n, p, θ up-to $O(n^{-1/2})$ by virtue of normality of disturbances are given as

$$
\begin{aligned}
&E(f)=0\\
&E(f^2)=2\theta(2-\theta)\\
&E(f^3)=-\frac{8\theta^3}{n^{1/2}}(3-2\theta)\\
&E(f^4)=12\theta^2(2-\theta)^2+\frac{48\theta^3}{n}(4-3\theta)\\
&E(g)=p-\theta(1-2\theta)\\
&E(g^2)=p^2+2p(1-\theta+2\theta^2)-\theta(6-17\theta+20\theta^2-12\theta^3)\\
&E(fg)=-\frac{2\theta}{n^{\frac{1}{2}}}(p+4-11\theta+8\theta^2)\\
&E(f^2g)=2\theta[(2-\theta)p+4-14\theta+17\theta^2-6\theta^3]\\
&E(f^3g)=\frac{4\theta^2}{n^{1/2}}(-12p+228\theta-72+7p\theta-241\theta^2+80\theta^3)\\
&E(fg^2)=\frac{4\theta^2}{n^{1/2}}(-p^2+20p\theta-10p-14p\theta^2-43\theta^2+34\theta^3-20\theta^4+42\theta)\\
&E(f^2g^2)=2\theta(-17\theta^5+62\theta^4-46\theta^3+25\theta^2-36\theta+16)+2p\theta(-p\theta+34\theta^2-12\theta^3-29\theta+2p+12)\\
&E(h)=0\\
&E(fh)=-2\theta(1-\theta)(p+6\theta^2-5\theta)\\
&E(f^2h)=\frac{4\theta(1-\theta)}{n^{1/2}}(9p\theta+4p-9\theta^2+4\theta^3+6\theta+8)\\
&E(f^3h)=\theta^2(670\theta^2-940\theta^3-288\theta+128+430\theta^4)+12\theta^2\sigma^2(-\theta^2-2\theta-p\theta+2p+10)
\end{aligned}
\tag{41}
$$

using (13), (14) and (15).

Proof.: In order to obtain the results of theorem; let us first state, when u_i are independently and identically distributed with $N(0,\sigma^2)$, following results:

$$
\begin{aligned}
&E\left[(u'I_n u)\right] = \sigma^2\left[n\right]\\
&E\left[(u'I_n u)(u'I_n u)\right] = \sigma^4\left[n^2+2n\right]\\
&E\left[(u'I_n u)(u'I_n u)(u'I_n u)\right] = \sigma^6\left[n^3+6n^2+8n\right]\\
&E\left[(u'I_n u)(u'I_n u)(u'I_n u)(u'I_n u)\right] = \sigma^8\left[n^4+12n^3+44n^2+48n\right]
\end{aligned}
\tag{42}
$$

where I is a symmetric matrices (identity matrix) of order $n \times n$.

3.5 Expectations of various orders of $\mathbf{z}$

The expected value of z, z^2, z^3, z^4, z^5 and z^6 are obtained, by using (7) and results of (42), as follows

$$E(z) = \frac{1}{n^{1/2}\sigma^2}E(u'AX\beta) = \frac{1}{n^{1/2}\sigma^2}E(u'A)X\beta = \frac{1}{n^{1/2}\sigma^2}E(u')l \quad Where; \;\; l = AX\beta = 0 \tag{43}$$

$$E(z^2) = \frac{1}{n\sigma^4}E[(u'AX\beta)'(u'AX\beta)] = \frac{1}{n\sigma^4}E[(\beta'X'A'uu'AX\beta)] = \frac{1}{n\sigma^4}\sigma^2\sigma^2 n\left(\frac{\theta}{1-\theta}\right) = \left(\frac{\theta}{1-\theta}\right) \tag{44}$$

$$E(z^3) = \frac{1}{n^{3/2}\sigma^6}E(u'l)^3 = 0$$

$$
\begin{aligned}
\because \quad E[u'l]^3 =& E[u_1 l_1 + u_2 l_2 + \ldots + u_m l_m]^3\\
=& E\left[\sum_i (l_i u_i)^3 + \sum_{i\neq j} l_i^2 l_j u_i^2 u_j + \sum_{i\neq j\neq k} l_i l_j l_k u_i u_j u_k\right]\\
=& E\left[\sum_i l_i^3 E(u_i)^3 + \sum_{i\neq j} l_i^2 l_j E(u_i^2)E(u_j) + \sum_{i\neq j\neq k} l_i l_j l_k E(u_i)E(u_j)E(u_k)\right]\\
=& 0 \qquad Where; \qquad l = AX\beta
\end{aligned}
\tag{45}
$$

$$E(z^4) = \frac{1}{n^2\sigma^8}E[(u'AX\beta)'(u'AX\beta)]^2 = \frac{1}{n^2\sigma^4}(\beta'X'AX\beta)^2 = 3\left(\frac{\theta}{1-\theta}\right)^2 \tag{46}$$

$$E(z^5) = \frac{1}{n^{3/2}\sigma^6}E(u'l)^5 = \frac{1}{n^{3/2}\sigma^6}E(u'll'uu'll'uu'l) = 0;\ \text{ref. eq. (45).} \tag{47}$$

$$
\begin{aligned}
E(z^6) =& E\left[\frac{1}{n\sigma^4}(u'AX\beta)'(u'AX\beta)\right]^3 = \frac{1}{n^3\sigma^{12}}E\left[(u'll'u)(u'll'u)(u'll'u)\right]\\
=& \frac{1}{n^3\sigma^{12}}\left[n^3\sigma^6\left(\frac{\theta}{1-\theta}\right)^3 + 6n^3\sigma^6\left(\frac{\theta}{1-\theta}\right)^3 + 8n^3\sigma^6\left(\frac{\theta}{1-\theta}\right)^3\right]\\
=& 15\left(\frac{\theta}{1-\theta}\right)^3
\end{aligned}
\tag{48}
$$

3.6 Expectations of various orders of $\mathbf{w}$

The expected value of w, w^2, w^3, w^4 and w^5 by using (8) and utilizing results of (42), we get

$$E(w) = E\left[\frac{u'u}{n^{1/2}\sigma^2} - n^{1/2}\right] = \frac{1}{n^{1/2}\sigma^2}E(u'u) - E(n^{1/2}) = n^{1/2} - n^{1/2} = 0 \tag{49}$$

$$E(w^2) = E\left[\frac{u'u}{n^{1/2}\sigma^2} - n^{1/2}\right]^2 = \frac{1}{n\sigma^4}E(u'uu'u) + E(n) - \frac{2}{\sigma^2}E(u'u) = n+2+n-2n = 2 \tag{50}$$

$$
\begin{aligned}
E(w^3) &= E\left[\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right]^3 = E\left[\frac{u'uu'uu'u}{n^{3/2}\sigma^6}-n^{3/2}-\frac{3u'uu'u}{n^{1/2}\sigma^4}+\frac{3n^{1/2}u'u}{\sigma^2}\right] \\
&= \frac{E(u'uu'uu'u)}{n^{3/2}\sigma^6}-n^{3/2}-\frac{3E(u'uu'u)}{n^{1/2}\sigma^4}+\frac{3n^{1/2}E(u'u)}{\sigma^2} = 6n^{1/2}+\frac{8}{n^{1/2}}-3n^{3/2}-6n^{1/2}+3n^{3/2}=\frac{8}{n^{1/2}}
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
E(w^4) &= E\left[\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right]^4 = E\left[\frac{A^4}{n^2\sigma^8}+n^2+\frac{2A^2}{\sigma^4}+\frac{4A^2}{\sigma^4}-\frac{4A^3}{n\sigma^6}-\frac{4nA}{\sigma^2}\right] \qquad \textit{where, } A=u'u \\
&= n^2+12n+44+\frac{48}{n}-4n^2-32-24n+2n^2+12n+n^2 = 12\left(1+\frac{4}{n}\right)
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
E(w^5) =& E\left[\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right]^5 = \frac{E(u'u)^5}{n^{5/2}\sigma^{10}}-n^{5/2}-\frac{5E(u'u)^4}{\sigma^8 n^{3/2}}+\frac{10E(u'u)^3}{\sigma^6 n^{1/2}}-\frac{10E(u'u)^2 n^{1/2}}{\sigma^4}+\frac{5n^{3/2}E(u'u)}{\sigma^2} \\
=& n^{5/2}+20n^{3/2}+140n^{1/2}+\frac{400}{n^{1/2}}+\frac{384}{n^{3/2}}-5n^{5/2}-60n^{3/2}-220n^{1/2}+10n^{5/2}-\frac{240}{n^{1/2}}+80n1/2+60n^{3/2} \\
&-10n^{5/2}-20n^{3/2}+5n^{5/2}-n^{5/2}=\frac{32}{n^{1/2}}\left(5+\frac{12}{n}\right)
\end{aligned} \tag{53}
$$

3.7 Expectations of various combination of $\mathbf{z}$ & $\mathbf{w}$

By using, the expectations of combinations of z and w based on results at (42)

$$
E(zw) = E\left[\left(\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right)\left(\frac{u'AX\beta}{n^{1/2}\sigma^2}\right)\right] = E\left[\frac{u'uu'AX\beta}{n\sigma^4}-\frac{u'AX\beta}{\sigma^2}\right] = \frac{E(u'uu'AX\beta)}{n\sigma^4}-\frac{E(u'AX\beta)}{\sigma^2}=0 \tag{54}
$$

$$
\begin{aligned}
E(z^2w) =& E\left[\frac{u'll'uu'u}{n^{3/2}\sigma^6}-n^{1/2}\frac{u'll'u}{n\sigma^4}\right] = \frac{E(u'll'uu'u)}{n^{3/2}\sigma^6}-n^{1/2}\frac{E(u'll'u)}{n\sigma^4} \\
=& \frac{1}{\sigma^2 n^{3/2}}\left[n(n+2)\sigma^2\left(\frac{\theta}{1-\theta}\right)\right]-n^{1/2}\left(\frac{\theta}{1-\theta}\right)=\left(\frac{\theta}{1-\theta}\right)\left(\frac{2+n}{n^{1/2}}-n^{1/2}\right)=\frac{2}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
E(z^2w^2) =& \frac{1}{n^2\sigma^8}E(u'll'uu'u)+\frac{1}{\sigma^4}E(u'll'u)-\frac{2}{n\sigma^6}E(u'll'uu'u) \\
=& \frac{1}{n}\left[\left(\frac{\theta}{1-\theta}\right)n^2+6\left(\frac{\theta}{1-\theta}\right)n+8\left(\frac{\theta}{1-\theta}\right)\right]+n\left(\frac{\theta}{1-\theta}\right)-2(2+n)\left(\frac{\theta}{1-\theta}\right) \\
=& \left(\frac{\theta}{1-\theta}\right)\left(\frac{n^2+6n+8+n^2-4n-2n^2}{n}\right)=2\left(\frac{\theta}{1-\theta}\right)\left(1+\frac{4}{n}\right)
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
E(z^2w^3) =& E\left[\left(\frac{u'll'u}{n\sigma^4}\right)\left(\frac{u'uu'uu'u}{n^{3/2}\sigma^6}-n^{3/2}-3\frac{u'uu'u}{\sigma^4 n^{1/2}}+3n^{1/2}\frac{u'u}{\sigma^2}\right)\right] \\
=& E\left[\frac{u'll'uu'uu'uu'u}{n^{5/2}\sigma^{10}}-n^{1/2}\frac{u'll'u}{\sigma^4}-3\frac{u'll'uu'uu'u}{\sigma^8 n^{3/2}}+3n^{1/2}\frac{u'll'uu'u}{n\sigma^6}\right] \\
=& \left(\frac{\theta}{1-\theta}\right)\left(n^{3/2}+12n^{1/2}+\frac{44}{n^{1/2}}+\frac{48}{n^{3/2}}-n^{3/2}-18n^{1/2}-\frac{24}{n^{3/2}}-3n^{3/2}+6n^{1/2}\right) \\
=& \left(\frac{\theta}{1-\theta}\right)\left(\frac{20}{n^{1/2}}+\frac{48}{n^{3/2}}\right)=\frac{4}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)\left(5+\frac{12}{n}\right)
\end{aligned} \tag{57}
$$

$$
E(z^4w) = E\left[\frac{u'll'uu'll'uu'u}{n^{5/2}\sigma^{10}}-n^{1/2}\frac{u'll'uu'll'u}{n^2\sigma^8}\right]=\frac{1}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)(n+4+2n+8-3n)=\frac{12}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)^2 \tag{58}
$$

3.8 Expectations of various combinations of $\mathbf{M}, \mathbf{e}$ & $\mathbf{u}$

Similarly the expectations of combinations of M, e & u are as under

$$E\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 w = \frac{1}{n^{3/2}\sigma^4}E(u'uu'ee'u - u'ee'u) = \frac{1}{n^{3/2}}(n+2n-n) = \frac{2}{n^{1/2}} \tag{59}$$

$$E\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 w^2 = E\left[\left(\frac{e'u}{n^{1/2}\sigma}\right)^2\left(\frac{u'uu'u}{n\sigma^4}+n-\frac{u'u}{\sigma^2}\right)\right] = E\left[\left(\frac{u'ee'uu'uu'u}{n^2\sigma^6}+\frac{u'ee'u}{\sigma^2}-\frac{u'uu'ee'u}{n\sigma^4}\right)\right] = 2\left(1+\frac{4}{n}\right) \tag{60}$$

$$E\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 Z^2 = E\left[\left(\frac{e'u}{n^{1/2}\sigma}\right)^2\left(\frac{u'll'u}{n\sigma^4}\right)\right] = \frac{1}{n^2\sigma^6}E[u'll'u)(u'ee'u)] = \frac{1}{n^2\sigma^2}n\sigma^2\left(\frac{\theta}{1-\theta}\right)(1+2) = \frac{3}{n}\left(\frac{\theta}{1-\theta}\right) \tag{61}$$

$$\begin{aligned} E\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 Z^2 w =& E\left[\left(\frac{e'u}{n^{1/2}\sigma}\right)^2\left(\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right)\left(\frac{u'll'u}{n\sigma^4}\right)\right] = E\left[\left(\frac{1}{n^{5/2}\sigma^8}\right)^2 u'ee'uu'll'uu'u + \frac{n^{1/2}}{n\sigma^4}u'll'u\right] \\ =& \frac{2}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)\left(1+\frac{5}{n}\right) \end{aligned} \tag{62}$$

$$\begin{aligned} E\left(\frac{e'u}{n^{1/2}\sigma}\right)^2 w^3 =& E\left[\frac{u'ee'u}{n\sigma^2}\left(\frac{u'uu'u}{n\sigma^4}+n-2\frac{u'u}{\sigma^2}\right)\left(\frac{u'u}{n^{1/2}\sigma^2}-n^{1/2}\right)\right] \\ =& E\left[\frac{(u'ee'uu'uu'uu'u)}{n^{5/2}\sigma^8}-\frac{(u'ee'uu'uu'u)}{n^{3/2}\sigma^6}+\frac{(u'ee'uu'u)}{n^{1/2}\sigma^4}-\frac{n^{1/2}(u'ee'u)}{\sigma^2}-2\frac{(u'ee'uu'uu'u)}{n^{3/2}\sigma^6}+2\frac{(u'ee'uu'u)}{n^{1/2}\sigma^4}\right] \\ =& n^{1/2}+\frac{12}{n^{1/2}}+\frac{44}{n^{3/2}}+\frac{48}{n^{5/2}}-3n^{1/2}-\frac{18}{n^{1/2}}-\frac{24}{n^{3/2}}+\frac{6}{n^{1/2}}+3n^{1/2}-n^{1/2} \\ =& \frac{1}{n^{3/2}}\left(20+\frac{48}{n}\right) \end{aligned} \tag{63}$$

$$\begin{aligned} E(u'Mu)^2 = E(u'Muu'Mu) &= \sigma^4[(trM)(trM)+2tr(MM)] = \sigma^4[(p+\theta)(p+\theta)+2p+\theta)] \\ &= \sigma^4[(p+\theta)^2+2(p+\theta)] = \sigma^4(p+\theta)(p+\theta+2) \end{aligned} \tag{64}$$

$$\begin{aligned} E(u'Muw) &= \frac{1}{n^{1/2}\sigma^2}E(u'Muu'u-n\sigma^2u'Mu) = \frac{1}{n^{1/2}\sigma^2}E(u'Muu'u)-n^{1/2}E(u'Mu) \\ &= \frac{1}{n^{1/2}}\sigma^2[(p+\theta)+2(p+\theta)]-n^{1/2}\sigma^2(p+\theta) = \frac{2\sigma^2}{n^{1/2}}(p+\theta) \end{aligned} \tag{65}$$

$$\begin{aligned} E(u'Muw^2) =& 2\sigma^2E\left[\frac{u'Muu'uu'u}{n\sigma^6}+n\frac{u'Mu}{\sigma^2}-2\frac{u'uu'Mu}{\sigma^4}\right] \\ =& \sigma^2\left[n(p+\theta)+6(p+\theta)+\frac{8(p+\theta)}{n}+n(p+\theta)-4(p+\theta)-2n(p+\theta)\right] = 2\sigma^2(p+\theta)\left(1+\frac{4}{n}\right) \end{aligned} \tag{66}$$

$$E(u'Mu) = \sigma^2 trM = \sigma^2(p+\theta) \tag{67}$$

$$\begin{aligned} E(u'Muw^3) =& E\left[u'Mu\left(\frac{u'uu'uu'u}{n^{3/2}\sigma^8}-n^{3/2}-3\frac{u'uu'u}{n^{1/2}\sigma^4}+3n^{1/2}\frac{u'u}{\sigma^2}\right)\right] \\ =& \sigma^2E\left[\frac{u'Muu'uu'uu'u}{n^{3/2}\sigma^8}-n^{3/2}\frac{u'Mu}{\sigma^2}-3\frac{u'uu'uu'Mu}{n^{1/2}\sigma^6}+3n^{1/2}\frac{u'uu'Mu}{\sigma^4}\right] \\ =& \sigma^2\left[\frac{(n^3+12n^2+44n+48)(p+\theta)}{n^{3/2}}-n^{3/2}(p+\theta)-\frac{3}{n^{1/2}}(p+\theta)(n^2+6n+8)+3n^{1/2}(n+2)(p+\theta)\right] \\ =& \frac{4\sigma^2}{n^{1/2}}(p+\theta)\left(5+\frac{12}{n}\right) \end{aligned} \tag{68}$$

$$E(u'Muw^4) = \frac{1}{n^2\sigma^8}E(u'Muu'uu'uu'uu'u) - \frac{4}{\sigma^6}E(u'Muu'uu'uu'u) + \frac{6}{\sigma^4}E(u'Muu'uu'u) - \frac{4n}{\sigma^2}E(u'Muu'u) + n^2E(u'Mu)$$
$$= \sigma^2\left[140(p+\theta) + \frac{400}{n}(p+\theta) - 176(p+\theta) - \frac{192}{n}(p+\theta) + \frac{384}{n^2}(p+\theta) + 48(p+\theta)\right]$$
$$= 4\sigma^2(p+\theta)\left(3 + \frac{52}{n} + \frac{96}{n^2}\right) \tag{69}$$

$$E(u'Muz^4) = \frac{1}{n^2\sigma^8}E(u'll'uu'll'uu'Mu) = \frac{1}{n^2\sigma^2}\left[\left(n\sigma^2\frac{\theta}{1-\theta}\right)^2(p+\theta) + 2(p+\theta)n^2\left(\sigma^2\frac{\theta}{1-\theta}\right)^2 + 12n^2\sigma^4\left(\frac{\theta}{1-\theta}\right)^2\right]$$
$$= \left(\frac{\theta}{1-\theta}\right)^2\sigma^2[(p+\theta) + 2(p+\theta) + 4 + 8] = 3\sigma^2\left(\frac{\theta}{1-\theta}\right)^2(p+\theta+4) \tag{70}$$

$$E((u'Mu)^2z^2) = \frac{1}{n\sigma^4}E(u'Muu'Muu'll'u) = \left[(p+\theta)^2n\sigma^2\left(\frac{\theta}{1-\theta}\right) + 6(p+\theta)n\sigma^2\left(\frac{\theta}{1-\theta}\right) + 8n\sigma^2\left(\frac{\theta}{1-\theta}\right)\right]$$
$$= \sigma^4\left(\frac{\theta}{1-\theta}\right)(p+\theta+2)(p+\theta+4) \tag{71}$$

$$E((u'Mu)^2w) = \frac{1}{\sigma^2n^{1/2}}[E(u'Muu'Muu'u) - n\sigma^2E(u'Muu'Mu)]$$
$$= \sigma^4\left[\frac{1}{n^{1/2}}\{(p+\theta)^2n + 4(p+\theta)^2 + 2n(p+\theta) + 8(p+\theta)\} - n^{1/2}\{(p+\theta)^2 + 2(p+\theta)\}\right] \tag{72}$$
$$= \frac{4\sigma^4}{n^{1/2}}(p+\theta)(p+\theta+2)$$

$$E(u'Muz^2) = \frac{1}{n\sigma^4}E(u'Muu'll'u) = \sigma^2\left[(p+\theta)\left(\frac{\theta}{1-\theta}\right) + 2\left(\frac{\theta}{1-\theta}\right)\right] = \sigma^2\left(\frac{\theta}{1-\theta}\right)(p+\theta+2) \tag{73}$$

$$E[(u'Mu)z^2w] = E\left[u'Mu\frac{u'll'u}{n\sigma^4}\left(\frac{u'u}{n^{1/2}\sigma^6} - n^{1/2}\right)\right]$$
$$= E\left[\frac{(u'Muu'll'uu'u)}{n^{3/2}\sigma^6} - \frac{(u'Muu'll'u)}{n^{1/2}\sigma^4}\right]$$
$$= n\sigma^2\left(\frac{\theta}{1-\theta}\right)\left[\frac{n(p+\theta)}{n^{3/2}} + \frac{2}{n^{1/2}} + \frac{4(p+\theta)}{n^{3/2}} + \frac{8}{n^{3/2}}\right] - \left[\frac{1}{n^{1/2}}\sigma^2(p+\theta+2)\left(\frac{\theta}{1-\theta}\right)\right] \tag{74}$$
$$= n\sigma^2\left(\frac{\theta}{1-\theta}\right)\frac{4}{n^{3/2}}(2+p+\theta) = \frac{4\sigma^2}{n^{1/2}}\left(\frac{\theta}{1-\theta}\right)(p+\theta+2)$$

Utilizing these results (43) – (74) repeatedly, Thus, we obtain the results of equations at (41) of the **Theorem 4**.