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On Some Stability Notations for Fuzzy Three-level Fractional Programming Problem

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Abstract: In this paper, suggested a solution algorithm to a fuzzy three-level fractional programming problem (F-TLFP). This problem involving fuzzy parameters on the right side of the constraints. First, a non-fuzzy problem (α-TLFP) with a crisp set of constraints is established depending on the concept of the α-level set of fuzzy numbers. Second, problem (α-TLFP) could be converted into a real-valued three-level fractional programming problem (RV-TLFP) which could be transformed into a real-valued bi-level fractional programming problem (RV-BLFP) by the duality theorem of linear fractional programming. In the same way, the problem (RV-BLFP) is transformed into a real-valued single-level fractional programming problem (RV-SLFP) again by the duality theorem, which could be solved for obtaining an α-optimal solution of problem (F-TLFP). Also, some stability notions are characterized and defined for the problem of concern by extending the Karush-Kuhn-Tucker optimality conditions equivalent to the problem (RV-SLFP). An algorithm is a presentation of infinite steps to solve and investigate the stability of the solution of the problem (F-TLFP). An illustrative numerical example is provided to demonstrate the proposed solution algorithm.

Keywords: Fuzzy numbers, Three-level Programming, Fractional Programming, Stability.

1 Introduction

Three-level optimization is a type of multi-level optimization which is a technique developed to solve the decentralized problem with multiple decision-makers in the hierarchical organization [1, 2, 3]. Three-level programming problem is concerned with minimizing or maximizing some quantity represented by an objective function.

Fractional programming (FP), which has been used as an important planning tool function for the last four decades, is applied to different disciplines such as engineering, business, and economics. Fractional programming is generally used for modeling real-life problems with fractional objectives such as profit/cost, inventory/sales, actual cost/standard cost, and output/employee [4, 5, 6 and, 7].

The fuzzy set and numbers have been introduced the concept in 1965 by Zadeh [8]. Fuzzy set theory has been to deal with imprecise numerical quantities nearly. It is well applied and developed in a wide of real problems.

Stability of solutions becomes more and more attractive in the area of mathematical programming. Publications on this topic usually investigate the impact of parameter changes (in the right-hand side or/and in the objective functions or/and in the left-hand side or/and in the domination structure) on the solution in various models of vector optimization problems. Stability study allows the decision-maker to take on decisions under various charges keeping the solution of the problem under consideration the same in a specified solution domain, which has great importance in management decision making as well as outside of it.

M. Osman et al. [9], diseased the characterization of some basic stability notations is parametric bi-level multi-objective linear fractional programming problems in a rough environment. In their paper, some notions have been extended to investigate the stability set of the first kind to the three-level fractional programming problem involving fuzzy parameters in the right-hand side of the constraints (TLFP-FP).

This paper presented; the stability set of the first kind for the three-level fractional programming (TLFP-FP)

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problem. Some stability notations such as the solvability set and the stability set have been defined for such a problem, where fuzzy parameters in the constraints of the problem (TLFP-FP) are involved. The offered solution algorithm has been described as infinite steps to solve the problem (TLFP-FP). Also, the stability set of the first kind (SSK1) corresponding to the obtained-optimal solution has been determined.

According to our experience, it is believed that the stability in three-level fractional programming problems with fuzzy has not been handle and debate in the literature before, the structure of this paper as follows:

Section 2 contains the mathematical formulation of the fuzzy three-level fractional programming problem involving fuzzy parameters on the right-hand side of the constraints. The definition of the fuzzy number with its membership function is also given. The a-level set of this fuzzy number is then defined. In Section 3, an interval-valued three-level fractional programming problem is stated. Some basic concepts of stability for the problem of concern are defined in Section 4. The utilization of the Karush-Kuhn-Tucker necessary optimality conditions corresponding to the real-valued three-level fractional programming problem is developed in Section 5. Also, the outlines of the solution algorithm infinite steps are described in Section 6. An illustrative numerical example to clarify the theory and the solution algorithm is provided in Section 7. Some conclusions of the results in this paper are included in Section 8.

2 The Solution Concept and Problem Formularization

In this section, we consider the following three-level fractional programming problem involving m-vector of fuzzy parameters $(\tilde{\vartheta})$ in the constraints, (TLFP-FP):

(TLFP-FP):

$$\left.\begin{array}{l}
[1^{st}-\text{level}] \\
Max_{x_1} F_1(x) = \dfrac{c_1^T x + \rho_1}{d_1^T x + \beta_1}, \\
\text{where}\, x_2, x_3 \text{ solves} \\
[2^{nd}-\text{level}] \\
Max_{x_2} F_2(x) = \dfrac{c_2^T x + \rho_2}{d_2^T x + \beta_2}, \\
\text{where}\, x_3 \text{ solves} \\
[3^{rd}-\text{level}] \\
Max_{x_3} F_3(x) = \dfrac{c_3^T x + \rho_3}{d_3^T x + \beta_3}, \\
\text{Subject to} \\
x \in X(\tilde{\vartheta}) = \{x \in \Re^n | \sum_{j=1}^{n} a_{ij} x_j \le \tilde{\vartheta}_i, i = 1,2,\ldots,m, x \ge 0\}.
\end{array}\right\} \tag{1}$$

Here $\tilde{\vartheta}_i, i = 1,2,\ldots,m$ represent fuzzy parameters on the right side of the constraints of the problem (TLFP-FP) (1). These fuzzy parameters are assumed to the characterized as fuzzy numbers as introduced by Dubois and Prade in [10, 11, 12, 20].

Definition 1. A fuzzy number is represented with three points as $\tilde{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3)$, This representation is interpreted as a membership function and holds the following conditions, [13].

(i) ϑ_1 to ϑ_2 is an increasing function.

(ii) ϑ_2 to ϑ_3 is a decreasing function.

(iii) $\vartheta_1 \le \vartheta_2 \le \vartheta_3$.

$$\mu_{\tilde{\vartheta}}(\vartheta) = \begin{cases} 0 & for\ \vartheta < \vartheta_1 \\ \dfrac{\vartheta - \vartheta_1}{\vartheta_2 - \vartheta_1} & for\ \vartheta_1 \le \vartheta \le \vartheta_2, \\ \dfrac{\vartheta_3 - \vartheta}{\vartheta_3 - \vartheta_2} & for\ \vartheta_2 \le \vartheta \le \vartheta_3, \\ 0 & for\ \vartheta > \vartheta_3. \end{cases}$$

Figure.1 illustrates the graph of a possible shape of a membership function of a fuzzy number $\tilde{\vartheta}$.

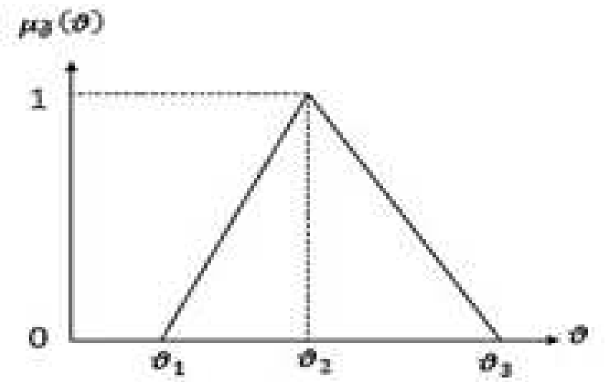

Fig. 1: Membership functions of triangular fuzzy number $\tilde{\vartheta}$

We now assume that $\tilde{\vartheta}_i, i = 1,2,\ldots,m$ in problem (TLFP-FP) (1) are fuzzy numbers whose membership function is $\mu\tilde{\vartheta}_i(\vartheta_i)$. Then, we could introduce the following α-level set or α-cut of the fuzzy parameters $\tilde{\vartheta}_i, i = 1,2,\ldots,m$.

Definition 2. (α-Level Set)The α-level set of the fuzzy numbers $\tilde{\vartheta}_i, i = 1,2,\ldots,m$ is defined as an ordinary set $L_\alpha(\tilde{\vartheta})$ for which the degree of their membership function exceeds the level α, (see [13]):

$$L_\alpha(\tilde{\vartheta}) = \{\vartheta | \mu_{\tilde{\vartheta}_i}(\vartheta_i) \ge \alpha, \qquad i = 1,2,\ldots,m\} \tag{2}$$

It is clear that the a-level sets have the following properties:

$$\alpha_1 \le \alpha_2 \text{ if and only if } L_{\alpha_1}(\tilde{\vartheta}) \supset L_{\alpha_2}(\tilde{\vartheta}).$$

For a certain degree α, then problem (TLFP-FP) (1) could be understood as the following non-fuzzy α-three-level fractional programming problem (α-TLFP):

(α-TLFP):

$$\left.\begin{array}{l} [1^{st}-\text{level}] \\ Max_{x_1} F_1(x) = \dfrac{c_1^T x + \rho_1}{d_1^T x + \beta_1}, \\ \text{where} x_2, x_3 \text{ solves} \\ [2^{nd}-\text{level}] \\ Max_{x_2} F_2(x) = \dfrac{c_2^T x + \rho_2}{d_2^T x + \beta_2}, \\ \text{where} x_3 \text{ solves} \\ [3^{rd}-\text{level}] \\ Max_{x_3} F_3(x) = \dfrac{c_3^T x + \rho_3}{d_3^T x + \beta_3}, \\ \text{Subject to} \\ x \in X(\tilde{\vartheta}) = \{x \in \Re^n | \sum_{j=1}^{n} a_{ij} x_j \leq \vartheta_i, i = 1,2,\ldots,m, x \geq 0\}. \end{array}\right\} \quad (3)$$

$\vartheta \in L_\alpha(\tilde{\vartheta})$.

It should be emphasized here that in (α-TLFP) (3), the parameters $\tilde{\vartheta}_i, i = 1,2,\ldots,m$ are treated as decision variables rather than constants.

Based on the definition of α-level set of the fuzzy number, we introduce the concept of α-optimal solution to the (α-TLFP) (3).

Definition 3. (α-Optimal Solution) $x^* \in X(\vartheta)$ is said to be an α-optimal solution to the problem (α-TLFP) (3), if and only if there does not exist another $x \in X(\vartheta), \vartheta \in L_\alpha(\tilde{\vartheta})$ such that $F_r(x) \geq F_r(x^*)$, for all $r = 1,2,3$ where the corresponding values of parameters $\vartheta^* = (\vartheta_1^*, \vartheta_2^*, \ldots, \vartheta_m^*)$ are called a-level optimal parameters.

3 Interval-valued Three-level Fractional Programming Problem

From the definition of the fuzzy number, $\tilde{\vartheta}_i, i = 1,2,\ldots,m$, it is significant to note that the α-level set of fuzzy number could be represented as the closed interval $[\vartheta_i^L, \vartheta_i^U], i = 1,2,\ldots,m$ which depends on interval-valued of α. Therefore, problem (α-TLFP) (3) is converted into an interval-valued three-level fractional programming problem (IV-TLFP) as follows:

(IV-TLFP):

$$\left.\begin{array}{l} [1^{st}-\text{level}] \\ Max_{x_1} F_1(x) = \dfrac{c_1^T x + \rho_1}{d_1^T x + \beta_1}, \\ \text{where} x_2, x_3 \text{ solves} \\ [2^{nd}-\text{level}] \\ Max_{x_2} F_2(x) = \dfrac{c_2^T x + \rho_2}{d_2^T x + \beta_2}, \\ \text{where} x_3 \text{ solves} \\ [3^{rd}-\text{level}] \\ Max_{x_3} F_3(x) = \dfrac{c_3^T x + \rho_3}{d_3^T x + \beta_3}, \\ \text{Subject to} \\ x \in X(\vartheta) = \{x \in \Re^n | \sum_{j=1}^{n} a_{ij} x_j \leq \vartheta_i, i = 1,2,\ldots,m, x \geq 0\}. \end{array}\right\} \quad (4)$$

$\vartheta_i^L \leq \vartheta_i \leq \vartheta_i^U, i = 1,2,\ldots,m.$

Depending on the concept of a convex linear combination method described in [3,10,14], the above problem (IV-TLFP) (4) could be written as follows:

(RV-TLFP):

$$\left.\begin{array}{l} [1^{st}-\text{level}] \\ Max_{x_1} F_1(x) = \dfrac{c_1^T x + \rho_1}{d_1^T x + \beta_1}, \\ \text{where} x_2, x_3 \text{ solves} \\ [2^{nd}-\text{level}] \\ Max_{x_2} F_2(x) = \dfrac{c_2^T x + \rho_2}{d_2^T x + \beta_2}, \\ \text{where} x_3 \text{ solves} \\ [3^{rd}-\text{level}] \\ Max_{x_3} F_3(x) = \dfrac{c_3^T x + \rho_3}{d_3^T x + \beta_3}, \\ \text{Subject to} \\ x \in X(\vartheta) = \{x \in \Re^n | \sum_{j=1}^{n} a_{ij} x_j \leq (\lambda \vartheta_i^L + (1-\lambda_i)\vartheta_i^U, \\ i = 1,2,\ldots,m, x \geq 0\}. \end{array}\right\} \quad (5)$$

where; $\lambda \in [0,1], i = 1,2,\ldots,m.$

To solve problem (5), first, we solve the $3^r d$-level decision-maker by using the LINGO software package [15]:

$$\left.\begin{array}{l} Max_{x_3} F_3(x) = \dfrac{c_3^T x + \rho_3}{d_3^T x + \beta_3}, \\ \text{Subject to} \\ x \in X(\vartheta), \end{array}\right\} \quad (6)$$

to obtain an a-optimal solution $X^{*T} = (x_1^*, x_2^*, x_3^*)$ with the corresponding optimum value function $F_3^* = F_3(x^*)$.

Now, the 3^{rd}-level decision maker problem is transformed by the dual linear fractional programming method described in [16] and could be written in the following manner:

$$\left.\begin{aligned}
&Min_{x_3}\,\psi(u) = u_0,\\
&\text{subject to}\\
&\beta_3 u_0 - \sum_{j=1}^{n}(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i \geq \rho_3,\\
&d_{3i}^T u_0 + \sum_{j=1}^{n} a_{ij}u_i \geq c_3^T,\\
&x_j \geq 0, u_i \geq 0, i = 1,2,\ldots,m.
\end{aligned}\right\} \quad (7)$$

u_0-Unrestricted.

Consequently, problem (RV-TLFP) (5) could be transformed into (RV-BLFP) as the following problem.

(RV-BLFP):

$$\left.\begin{aligned}
&[1^{st}-\text{level}]\\
&Max_{x_1} F_1(x) = \frac{c_1^T x + \rho_1}{d_1^T x + \beta_1},\\
&\text{where}\, x_2, x_3 \,\text{solves}\\
&[2^{nd}-\text{level}]\\
&Max_{x_2} F_2(x) = \frac{c_2^T x + \rho_2}{d_2^T x + \beta_2},\\
&\text{Subject to}\\
&x \in X(\vartheta) = \{x \in \Re^n, \vartheta \in \Re^m | \sum_{j=1}^{n} a_{ij}x_j \leq\\
&(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U), i = 1,2,\ldots,m, x \geq 0\}\\
&\beta_3 u_0 - \sum_{j=1}^{n}(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i \geq \rho_3,\\
&d_{3i}^T u_0 + \sum_{j=1}^{n} a_{ij}u_i \geq c_{3i}^T,\\
&x_j \geq 0, u_i \geq 0, i = 1,2,\ldots,m.\\
&u_0 - \text{Unrestricted.}
\end{aligned}\right\} \quad (8)$$

where; $\lambda_i \in [0,1],\ i = 1,2,\ldots,m.$

At this step, the following 2^{nd}-level decision maker problem is solved using the LINGO software package [15].

$$\left.\begin{aligned}
&Max_{x_2} F_2(x) = \frac{c_2^T x + \rho_2}{d_2^T x + \beta_2},\\
&\text{Subject to}\\
&x \in X(\vartheta) = \{x \in \Re^n, \vartheta \in \Re^m | \sum_{j=1}^{n} a_{ij}x_j \leq\\
&(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U), i = 1,2,\ldots,m, x \geq 0\}\\
&\beta_3 u_0 - \sum_{j=1}^{n}(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i \geq \rho_3,\\
&d_{3i}^T u_0 + \sum_{j=1}^{n} \bar{a}_{ij}u_i \geq c_{3i}^T,\\
&x_j \geq 0, u_i \geq 0, i = 1,2,\ldots,m.\\
&u_0 - \text{Unrestricted.}
\end{aligned}\right\} \quad (9)$$

where; $\lambda_i \in [0,1],\ i = 1,2,\ldots,m.$

to obtain the α-optimal solution $X^{*S} = (x^*, u_0^*, u_i^*)$ with the corresponding optimum value function $F_2^* = F_2(x^*)$. Again, the 2^{nd}-level decision maker problem is transformed into the following dual linear fractional problem by [16] as:

$$\left.\begin{aligned}
&Min_{x_2}\,\psi(v) = v_0,\\
&\text{Subject to}\\
&\beta_2 v_0 - \sum_{j=1}^{n}(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)v_i + \rho_3 v_i + c_{3i}^T v_i \geq \rho_2,\\
&d_{2i}^T v_0 + \sum_{j=1}^{n} a_{ij}v_i + \beta_3 v_i - \sum_{j=1}^{n}(\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)v_i +\\
&d_{3i}^T v_i + \sum_{j=1}^{n} \bar{a}_{ij} \geq c_{2i}^T,\\
&x_j \geq 0, v_i \geq 0, i = 1,2,\ldots,m.\\
&v_0 - \text{Unrestricted.}
\end{aligned}\right\} \quad (10)$$

Therefore, problem (RV-BLFP) (8) could be transformed into a problem (RV-SLFP) as the following:

(RV-SLFP):

$$\left.\begin{array}{l} Max_{x_1} F_1(x) = \dfrac{c_1^T x+\rho_1}{d_1^T x+\beta_1}, \\ \text{Subject to} \\ x \in X(\vartheta) = \{x \in \Re^n, \vartheta \in \Re^m | \sum_{j=1}^{n} a_{ij}x_j \leq \\ (\lambda_i \vartheta_i^L + (1-\lambda_i))\vartheta_i^U, i=1,2,\ldots,m, x \geq 0\}, \\ \beta_2 v_0 - \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i + \rho_3 u_i + c_{3i}^T u_i \geq \rho_2, \\ d_{2i}^T v_0 + \sum_{j=1}^{n} a_{ij}v_i + \beta_3 v_i - \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)v_i + \\ d_{3i}^T v_i + \sum_{j=1}^{n} \bar{a}_{ij} v_i \geq c_{2i}^T, \\ x_j \geq 0, v_i \geq 0, i=1,2,\ldots,m, \\ v_0 - \text{Unrestricted}. \end{array}\right\} \quad (11)$$

where; $\lambda_i \in [0,1],\ i=1,2,\ldots,m.$

to get finally an a-optimal solution $X^{*F} = (x^*, v_0^*, v_i^*)$ with the corresponding optimum value function $F_1^* = F_1(x^*)$.

4 The Basic Concept of Stability for Problem (TLFP-FP)

In this section, we give the definitions of the set of feasible parameters, the solvability set, and the stability set of the first kind (SSK1) for the problem (TLFP-FP) (1) via problem (RV-SLFP) (11),(see[17, 18, 19]).

Definition 4. (The set of Feasible Parameters) The set of feasible parameters of the problem (RV-SLFP) (11), which is denoted by A, is defined by:

$$A = \{\vartheta \in \Re^m | G_\alpha(x,\vartheta_i) \neq \phi \ and\ \alpha \in [0,1], (i=1,2,\ldots,m)\}$$

where $G_\alpha(x,\vartheta_i),\ i=1,2,\ldots,m$ is the feasible region in the decision space of problem (RV-SLFP)(11).

Definition 5. (The Solvability Set)The solvability set of the problem (RV-SLFP) (1), which is denoted by B, is defined by:

$$B = \{\vartheta \in \Re^m | \, problem(RV-SLFP)\,(11)\, has\, an\, \alpha - optimal\ solution(x^*, v_0^*, v_i^*)\}.$$

Definition 6. (The Stability Set of the First Kind (SSK1))Suppose that $X^{*F} = (x^*, v_0^*, v_i^*)$ be an optimal solution of problem (RV-SLFP) (11), then the stability set of the first kind $S_1(X^{*F},\alpha)$ corresponding to $X^{*F} = (x^*, v_0^*, v_i^*)$ is defined by:

$$S_1(X^{*F},\alpha) = \{\vartheta_i \in \Re^m | X^{*F}\, is\, an\, \alpha - optimal\, solution\ of\, problem\,(RV-SLFP),\,(11)\}.$$

5 Utilization of the Karush-Kuhn-Tucker optimality conditions corresponding to the(RV-SLFP) (11)

For the expansion that follows, later on, problem (RV-SLFP) (11) could be rewritten as:

$$\left.\begin{array}{l} Max_{x_1} F_1(x) = \dfrac{c_1^T x+\rho_1}{d_1^T x+\beta_1}, \\ \text{Subject to} \\ \sum_{i=1}^{m} a_{ij}x_j - (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U) \leq 0 \\ -\beta_2 v_0 + \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i - \rho_3 u_i - c_{3i}^T u_i + \rho_2 \leq 0, \\ -d_{2i}^T v_0 - \sum_{j=1}^{n} a_{ij}v_i - \beta_3 v_i + \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U) \\ v_i - d_{3i}^T v_i - \sum_{j=1}^{n} \bar{a}_{ij} v_i + c_{2i}^T \leq 0, \\ x_j \geq 0, v_i \geq 0,\ \vartheta_i^L, \vartheta_i^U \geq 0,\ i=1,2,\ldots,m \\ v_0 - \text{Unrestricted}. \end{array}\right\} \quad (12)$$

where; $\lambda_i \in [0,1],\ i=1,2,\ldots,m.$

The Lagrange function of problem (RV-SLFP) (12) is established as follows, (see [17, 18, 19]):

$$\begin{array}{l} L = F_1(x) - \tau_i(\sum_{j=i}^{m} a_{ij}x_j - (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U) - \\ \pi(-\beta_2 v_0 + \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + (1-\lambda_i)\vartheta_i^U)u_i - \rho_3 u_i - c_{3i}^T u_i + \rho_2) - \\ \gamma_i(-d_{2i}^T v_0 - \sum_{j=1}^{n} a_{ij}v_i - \beta_3 v_i + \sum_{j=1}^{n} (\lambda_i \vartheta_i^L + \\ (1-\lambda_i)\vartheta_i^U)v_i - d_{3i}^T v_i - \sum_{j=1}^{n} \bar{a}_{ij} v_i + c_{2i}^T) + \\ \sigma_i x_j + \varphi v_0 + \xi_i v_i + \eta_i \vartheta_i^L + \zeta_i \vartheta_i^U = 0. \end{array}$$

where, $\tau_i, \pi, \gamma_i, \sigma_i, \varphi, \xi_i, \eta_i$ and ζ_i are the Lagrange multipliers. Then the Karush-Kuhn-Tucker necessary

optimality conditions corresponding to (RV-SLFP) .
problem (12) would take the next form:

$$\left.\begin{array}{l}
\frac{\partial L}{\partial x_1}=\frac{\partial}{\partial x_1}F_1(x)-\frac{\partial}{\partial x_1}\tau_i\left(\sum_{i=1}^{m}a_{ij}x_j-(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)\right)+\frac{\partial}{\partial x_1}\sigma_i x_j.\\
\frac{\partial L}{\partial x_2}=\frac{\partial}{\partial x_2}F_1(x)-\frac{\partial}{\partial x_2}\tau_i\left(\sum_{i=1}^{m}a_{ij}x_j-(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)\right)+\frac{\partial}{\partial x_2}\sigma_i x_j.\\
\frac{\partial L}{\partial x_3}=\frac{\partial}{\partial x_3}F_1(x)-\frac{\partial}{\partial x_3}\tau_i\left(\sum_{i=1}^{m}a_{ij}x_j-(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)\right)+\frac{\partial}{\partial x_3}\sigma_i x_j.\\
\frac{\partial L}{\partial v_0}=-\frac{\partial L}{\partial v_0}\pi\left(-\beta_2 v_0+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)u_i-\rho_3 u_i-c_{3i}^T u_i+\rho_2\right)-\\
\frac{\partial L}{\partial v_0}\gamma_i\left(-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\right)+\frac{\partial L}{\partial v_0}\varphi v_0.\\
\frac{\partial L}{\partial v_i}=-\frac{\partial L}{\partial v_i}\gamma_i\left(-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\right)+\frac{\partial L}{\partial v_i}\xi_i v_i.\\
\frac{\partial L}{\partial \vartheta_i^L}=-\frac{\partial L}{\partial \vartheta_i^L}\pi\left(-\beta_2 v_0+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)u_i-\rho_3 u_i-c_{3i}^T u_i+\rho_2\right)-\\
\frac{\partial L}{\partial \vartheta_i^L}\gamma_i\left(-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\right)+\frac{\partial L}{\partial \vartheta_i^L}\eta_i\vartheta_i^L.\\
\frac{\partial L}{\partial \vartheta_i^U}=-\frac{\partial L}{\partial \vartheta_i^U}\pi_i\left(-\beta_2 v_0+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)u_i-\rho_3 u_i-c_{3i}^T u_i+\rho_2\right)-\\
\frac{\partial L}{\partial \vartheta_i^U}\gamma_i\left(-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\right)+\frac{\partial L}{\partial \vartheta_i^U}\zeta_i\vartheta_i^U.\\
\tau_i\left(\sum_{i=1}^{m}a_{ij}x_j-(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)\right)=0.\\
\pi\left(-\beta_2 v_0+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)u_i-\rho_3 u_i-c_{3i}^T u_i+\rho_2\right)=0.\\
\gamma_i\left(-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\right)=0.\\
\sigma_i x_j=0. \qquad \varphi v_0=0. \qquad \xi_i v_i=0. \qquad \eta_i\vartheta_i^L=0. \qquad \zeta_i\vartheta_i^U=0.\\
\sum_{i=1}^{m}a_{ij}x_j-\left(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U\right)\le 0.\\
-\beta_2 v_0+\sum_{j=1}^{n}\left(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U\right)u_i-\rho_3 u_i-c_{3i}^T u_i+\rho_2\le 0.\\
-d_{2i}^T v_0-\sum_{j=1}^{n}a_{ij}v_i-\beta_3 v_i+\sum_{j=1}^{n}\left(\lambda_i\vartheta_i^L+(1-\lambda_i)\vartheta_i^U\right)v_i-d_{3i}^T v_i-\sum_{j=1}^{n}\bar{a}_{ij}v_i+c_{2i}^T\le 0.\\
x_j\ge 0, v_i\ge 0, \vartheta_i^L, \vartheta_i^U\ge 0, \tau_i, \pi, \gamma_i, \sigma_i, \varphi, \xi_i, \eta_i, \zeta_i\ge 0, i=1,2,\ldots,m.\\
v_0-\text{Unrestricted.}
\end{array}\right\} \tag{13}$$

where; $\lambda_i\in[0,1],\ i=1,2,\ldots,m.$

where all the states of the Kuhn-Tucker conditions (13) are calculated at the α-optimal solution $X^{*F} = (x^*, v_0^*, v_i^*)$ of the problem (RV-SLFP).

Moreover, $\tau_i, \pi, \gamma_i, \sigma_i, \varphi, \xi_i, \eta_i$ and ζ_i are the Lagrange multipliers. Also, solving the system of equations (13), the fuzzy stability set of the first kind $S_1(X^{*F}, \alpha)$ for the fuzzy multi-level fractional programming problem with fuzziness in constraints (TLFP-FP) (1) would be characterized.

6 Solution Algorithm

Following the above discussion, an algorithm would be developed for obtaining the stability set of the first kind $S_1(X^{*F}, \alpha)$ for the parametric problem (TLFP-FP) is described in a series of steps. The suggested algorithm could be summarized in the following manner:

Stage I: Finding the a-optimal solution of problem (TLFP-FP).

Step (1): Start with an initial level set $\alpha = \alpha^* \in [0,1]$, acceptable for all decision-makers.

Step (2): Convert the problem (TLFP-FP) (1) into the form of problem (α-TLFP) (3).

Step (3): Rewrite problem (α-TLFP) (3) in the form of problem (IV-TLFP) (4).

Step (4): Convert the problem (IV-TLFP) (4) into the problem (RV-TLFP) (5) by applying the concept of convex linear combination on the constraint functions.

Step (5): Solve the third-level decision-maker problem (6).

Step (6): Transform the third-level decision-maker by the duality theory of linear fractional programming problem to the dual problem (7).

Step (7): Convert the problem (RV-TLFP) (5) into the problem (RV-BLFP) (8).

Step (8): Solve the second-level decision-maker problem (9).

Step (9): Transform the second-level decision-maker problem by the duality theory of linear fractional programming problem to dual problem (10).

Step (10): Convert the problem (RV-BLFP) (8) into the problem (RV-SLFP) (11).

Step (11): Solve the problem (RV-SLFP) (11).

Stage II: Determination of the stability set of the first kind $S_1(X^{*F}, \alpha)$, go to Step 12.

Step (12): Apply the Karush- Kuhn-Tucker optimality conditions to find, the stability set of the first kind, equations (12) for the problem (RV-SLFP) (11).

Step (13): Reduce and then solve the system of equations (13) to characterize the stability set of the first kind $S_1(X^{*F}, \alpha)$ and Stop.

7 An illustrative Numerical Example

In what follows we provide a numerical example to illustrate the solution algorithm described in the previous section, consider the following problem (F-TLFP):

[1^{st}-level]

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12}$,

where y,z solves

[2^{nd}-level]

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

where z solves

[3^{th}-level]

$Max_z F_3(x,y,z) = \frac{x+7y+2z+2}{6x+y-z+5}$,

Subject to

$4x+y+z \le \tilde{\vartheta}_1$,
$x+3y+z \le \tilde{\vartheta}_2$,
$2x+y+2z \le \tilde{\vartheta}_3$,
$x,y,z \ge 0$.

where $\tilde{\vartheta}_j, j = 1,2,3$ are fuzzy parameters and are characterized by the following triangular fuzzy numbers:

$$\tilde{\vartheta}_1 = (\vartheta_{11}, \vartheta_{12}, \vartheta_{13}) = (5,10,15).$$
$$\tilde{\vartheta}_2 = (\vartheta_{21}, \vartheta_{22}, \vartheta_{23}) = (2,6,10).$$
$$\tilde{\vartheta}_3 = (\vartheta_{31}, \vartheta_{32}, \vartheta_{33}) = (3,6,9).$$

Stage I: Finding an a-optimal solution to the parametric problem (α-TLFP).

Let $\alpha = 0.3$, then we get:

$$5+5\alpha \le \vartheta_1 \le 15-5\alpha, \ \ 2+4\alpha \le \vartheta_2 \le 10-4\alpha,$$
$$3+3\alpha \le \vartheta_2 \le 9-3\alpha.$$

The equivalent non-fuzzy problem (α-TLFP) takes the form:

[1^{st}-level]

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12}$,

where y,z solves

[2^{nd}-level]

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

where z solves

[3^{th}-level]

$Max_z F_3(x,y,z) = \frac{x+7y+2z+2}{6x+y-z+5}$,

Subject to

$4x+y+z \le \vartheta_1$,
$x+3y+z \le \vartheta_2$,
$2x+y+2z \le \vartheta_3$,
$6.5 \le \vartheta_1 \le 13.5$,
$3.2 \le \vartheta_2 \le 8.8$,
$3.9 \le \vartheta_3 \le 8.1$,
$x,y,z \ge 0$.

Problem (α-TLFP) could be written as an interval-valued three-level fractional programming problem (IV-TLFP) in the following form:

[1^{st}-level]

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12}$,

where y,z solves

[2^{nd}-level]

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

where z solve

[3^{th}-level]

$Max_z F_3(x,y,z) = \frac{x+7y+2z+2}{6x+y-z+5}$,

Subject to

$4x+y+z \leq [6.5, 13.5]$,
$x+3y+z \leq [3.2, 8.8]$,
$2x+y+2z \leq [3.9, 8.1]$,
$x,y,z \geq 0$.

Using the concept of convex linear combination on the constraints, then problem (IV-TLFP) is transformed into the problem (RV-TLFP) as follows:

[1^{st}-level]

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12}$,

where y,z solves

[2^{nd}-level]

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

where z solves

[3^{th}-level]

$Max_z F_3(x,y,z) = \frac{x+7y+2z+2}{6x+y-z+5}$,

Subject to

$4x+y+z \leq 10.7$,
$x+3y+z \leq 7.12$,
$2x+y+2z \leq 5.16$,
$x,y,z \geq 0$.

First; we solve the 3^{th}-level decision maker problem using LINGO [15]:

$Max_z F_3(x,y,z) = \frac{x+7y+2z+2}{6x+y-z+5}$,

Subject to

$4x+y+z \leq 10.7$,
$x+3y+z \leq 7.12$,
$2x+y+2z \leq 5.16$,
$x,y,z \geq 0$.

whose α-optimal solution is found:

$X^{*T} = (x^*, y^*, z^*) = (0, 1.8160, 1.6720)$ with the optimum objective value $F_3^{*T} = 3.510109$.

Transforming the 3^{th}-level decision maker problem using again the duality of linear fractional programming [16]. This problem could be written as:

$Min_z \psi(u) = u_0$,

Subject to

$5u_0 - 10.7u_1 - 7.12u_2 - 5.16u_3 \geq 2$,
$6u_0 + 4u_1 + u_2 + u_3 \geq 1$,
$u_0 + u_1 + 3u_2 + u_3 \geq 7$,
$-u_0 + 2u_1 + u_2 + 2u_3 \geq 2$,
$x,y,z \geq 0, u_i \geq 0, i = 1,2,3$.
u_0-Unrestricted.

Now, the problem (RV-TLFP) could be transformed into (RV-BLFP) as the following problem.

[1^{st}-level]

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12}$,

where y,z solves

[2^{nd}-level]

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

Subject to

$4x+y+z \leq 10.7$,
$x+3y+z \leq 7.12$,
$2x+y+2z \leq 5.16$,
$5u_0 - 10.7u_1 - 7.12u_2 - 5.16u_3 \geq 2$,
$6u_0 + 4u_1 + u_2 + u_3 \geq 1$,
$u_0 + u_1 + 3u_2 + u_3 \geq 7$,
$-u_0 + 2u_1 + u_2 + 2u_3 \geq 2$,
$x,y,z \geq 0, u_i \geq 0, i = 1,2,3$.
u_0-Unrestricted.

Secondly; we solve the 2^{nd}-level decision maker problem using LINGO [15] of linear fractional programming problem to the dual problem as the following:

$Max_y F_2(x,y,z) = \frac{x-4y+7z+12}{x-2y+z+6}$,

Subject to

$4x+y+z \leq 10.7$,
$x+3y+z \leq 7.12$,
$2x+y+2z \leq 5.16$,
$5u_0 - 10.7u_1 - 7.12u_2 - 5.16u_3 \geq 2$,
$6u_0 + 4u_1 + u_2 + u_3 \geq 1$,
$u_0 + u_1 + 3u_2 + u_3 \geq 7$,
$-u_0 + 2u_1 + u_2 + 2u_3 \geq 2$,
$x,y,z \geq 0, u_i \geq 0, i = 1,2,3$.
u_0-Unrestricted.

whose α-optimal solution is found:

$$X^{*S} = (x^*, y^*, z^*, u_0^*, u_1^*, u_2^*, u_3^*)$$
$$= (0, 1.816000, 1.67200, 5.784919, 1.234568, 0, 2.657892)$$

with the optimum objective value $F_2^{*S} = 4.069307$, Again, we transform the 2^{nd}-level decision maker problem of the dual problem which could be written as:

$Min_v \psi(v) = v_0$,

Subject to

$6v_0 - 10.7v_1 - 7.12v_2 - 5.16v_3 + 2v_4 + v_5 + 7v_6 + 2v_7 \geq 12$,
$v_0 + 4v_1 + v_2 + v_3 \geq 1$,
$-2v_0 + v_1 + 3v_2 + v_3 \geq -4$,
$v_0 + 2v_1 + v_2 + 2v_3 \geq 7$,
$-5v_4 + 10.7v_5 + 7.12v_6 + 5.16v_7 \geq 0$,
$-6v_4 - 4v_5 - v_6 - v_7 \geq 0$,
$-v_4 - v_5 - 3v_6 - v_7 \geq 0$,

$v_4 - 2v_5 - v_6 - 2v_7 \geq 0,$
v_0-Unrestricted.

Therefore, problem (RV-BLFP) could be transformed into (RV-SLFP) in the following form:

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12},$

Subject to

$4x+y+z \leq 10.7,$
$x+3y+z \leq 7.12,$
$2x+y+2z \leq 5.16,$
$6v_0 - 10.7v_1 - 7.12v_2 - 5.16v_3 + 2v_4 + v_5 + v_6 + 3v_7 \geq 12,$
$v_0 + 4v_1 + v_2 + v_3 \geq 1,$
$-2v_0 + v_1 + 3v_2 + v_3 \geq -4,$
$v_0 + 2v_1 + v_2 + 2v_3 \geq 7,$
$-5v_4 + 10.7v_5 + 7.12v_6 + 5.16v_7 \geq 0,$
$-6v_4 - 4v_5 - v_6 - v_7 \geq 0,$
$-v_4 - v_5 - 3v_6 - v_7 \geq 0,$
$v_4 - 2v_5 - v_6 - 2v_7 \geq 0,$
v_0-Unrestricted.

whose α-optimal solution is found:

$$\begin{aligned} X^{*F} &= (x^*, y^*, z^*, v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*, v_7^*) \\ &= (1.962716, 1.234568, 0, 4.607515, 0.3924843, \\ &\qquad 1.607516, 0, 0, 0, 0, 0). \end{aligned}$$

with the optimum objective value $F_1^* = 2.377622$.

Stage II: Determination of the stability set of the first kind

$$S_1(x^*, y^*, z^*, v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*, v_7^*, \alpha)$$

It is clear that the stability of the optimal solution of the problem (RV-SLFP) leads to the stability of the optimal solution of the given fuzzy problem (RV-TLFP). For this, let us begin with:

$Max_x F_1(x,y,z) = \frac{10x+5y-z+15}{2x+y-z+12},$

Subject to

$4x+y+z \leq 0.4\vartheta_1^L + 0.6\vartheta_1^U,$
$x+3y+z \leq 0.3\vartheta_2^L + 0.7\vartheta_2^U,$
$2x+y+2z \leq 0.7\vartheta_3^L + 0.3\vartheta_3^U,$
$6v_0 - (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_1 - (0.3\vartheta_2^L + 0.7\vartheta_2^U)v_2$
$-(0.7\vartheta_3^L + 0.3\vartheta_3^U)v_3 + 2v_4 + v_5 + 7v_6 + 2v_7 \geq 12,$
$v_0 + 4v_1 + v_2 + v_3 \geq 1,$
$-2v_0 + v_1 + 3v_2 + v_3 \geq -4,$
$v_0 + 2v_1 + v_2 + 2v_3 \geq 7,$
$-5v_4 + (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_5 + (0.3\vartheta_2^L + 0.7\vartheta_2^U)v_6$
$+(0.7\vartheta_3^L + 0.3\vartheta_3^U)v_7 \geq 0,$
$-6v_4 - 4v_5 - v_6 - v_7 \geq 0,$
$-v_4 - v_5 - 3v_6 - v_7 \geq 0,$
$v_4 - 2v_5 - v_6 - 2v_7 \geq 0,$
$x, y, z \geq 0, v_i \geq 0, i = 1, 2, \ldots, 7.$
$\vartheta_1^L, \vartheta_1^U, \vartheta_2^L, \vartheta_2^U, \vartheta_3^L, \vartheta_3^U \geq 0,$
v_0-Unrestricted.

The Lagrange function corresponding to the problem (RV-SLFP) is formulated as follows:

$$\begin{aligned} L = &\left(\frac{10x+5y-z+15}{2x+y-z+12}\right) - \tau_1(4x+y+z-0.4\vartheta_1^L - 0.6\vartheta_1^U) - \\ &\tau_2(x+3y+z-0.3\vartheta_2^L - 0.7\vartheta_2^U) - \\ &\tau_3(2x+y+2z-0.7\vartheta_3^L - 0.3\vartheta_3^U) - \\ &\pi(-6v_0 + (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_1 + \\ &(0.3\vartheta_2^L + 0.7\vartheta_2^U)v_2 + (0.7\vartheta_3^L + 0.3\vartheta_3^U)v_3 - \\ &2v_4 - v_5 - 7v_6 - 2v_7 - 12) - \\ &\gamma_1(-v_0 - 4v_1 - v_2 - v_3 + 1) - \gamma_2(2v_0 - v_1 - 3v_2 - v_3 + 4) - \\ &\gamma_3(-v_0 - 2v_1 - v_2 - 2v_3 + 7) - \gamma_4 \\ &(5v_4 - (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_5 - (0.3\vartheta_2^L + 0.7\vartheta_2^U)v_6 - \\ &(0.7\vartheta_3^L + 0.3\vartheta_3^U)v_7) - \gamma_5(6v_4 + 4v_5 + v_6 + v_7) - \\ &\gamma_6(v_4 + v_5 + 3v_6 + v_7) - \gamma_7(-v_4 + 2v_5 + v_6 + 2v_7) - \\ &\xi_1(-v_1) - \xi_2(-v_2) - \xi_3(-v_3) - \xi_4(-v_4) - \\ &\xi_5(-v_5) - \xi_6(-v_6) - \xi_7(-v_7) + \sigma_1(x) + \sigma_2(y) + \\ &\sigma_3(z) - \eta_1(-\vartheta_1^L) - \zeta_1(-\vartheta_1^U) - \eta_2(-\vartheta_2^L) - \\ &\zeta_2(-\vartheta_2^U) - \eta_3(-\vartheta_3^L) - \zeta_3(-\vartheta_3^U) - \varphi(-v_0) = 0. \end{aligned}$$

Where $\tau_i, \pi, \gamma_i, \sigma_i, \varphi, \xi_i, \eta_i$ and, $\zeta_i, (i \in I)$ are the Lagrange multipliers. Then the Karush-Kuhn-Tucker necessary optimality conditions (see [17, 18, 19])corresponding to the problem (RV-SLFP) would have the following form:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{10(2x+y-z+12) - 2(10x+5y-z+15)}{(2x+y-z+12)^2} \\ &\quad -4\tau_1 - \tau_2 - 2\tau_3 + \sigma_1 = 0, \\ \frac{\partial L}{\partial x_2} &= \frac{5(2x+y-z+12) - (10x+5y-z+15)}{(2x+y-z+12)^2} \\ &\quad -\tau_1 - 3\tau_2 - \tau_3 + \sigma_2 = 0, \\ \frac{\partial L}{\partial x_2} &= \frac{-(2x+y-z+12) - (10x+5y-z+15)}{(2x+y-z+12)^2} \\ &\quad -\tau_1 - \tau_2 - 2\tau_3 + \sigma_3 = 0, \\ \frac{\partial L}{\partial v_0} &= 6\pi + \gamma_1 - 2\gamma_2 + \gamma_3 + \tau_{28} = 0, \\ \frac{\partial L}{\partial v_1} &= -\pi(0.4\vartheta_1^L + 0.6\vartheta_1^U) + 4\gamma_1 + \gamma_2 + 2\gamma_3 + \xi_1 = 0, \\ \frac{\partial L}{\partial v_2} &= -\pi(0.3\vartheta_2^L + 0.7\vartheta_2^U) + \gamma_1 + 3\gamma_2 + \gamma_3 + \xi_2 = 0, \\ \frac{\partial L}{\partial v_3} &= -\pi(0.7\vartheta_3^L + 0.3\vartheta_3^U) + \gamma_1 + \gamma_2 + 2\gamma_3 + \xi_3 = 0, \\ \frac{\partial L}{\partial v_4} &= 2\pi - 5\gamma_4 - 6\gamma_5 - \gamma_6 + \gamma_7 + \xi_4 = 0, \\ \frac{\partial L}{\partial v_5} &= \pi + \gamma_4(0.4\vartheta_1^L + 0.6\vartheta_1^U) - 4\gamma_5 - \gamma_6 - 2\gamma_7 + \xi_5 = 0, \end{aligned}$$

$$\frac{\partial L}{\partial v_6} = 7\pi + \gamma_4(0.3\vartheta_2^L + 0.7\vartheta_2^U) - \gamma_5 - 3\gamma_6 - \gamma_7 + \xi_6 = 0,$$
$$\frac{\partial L}{\partial v_7} = 2\pi + \gamma_4(0.7\vartheta_3^L + 0.3\vartheta_3^U) - \gamma_5 - \gamma_6 - 2\gamma_7 + \xi_7 = 0,$$
$$\frac{\partial L}{\partial \vartheta_1^L} = 0.4\tau_1 - 0.4\pi v_1 + 0.4\gamma_4 v_5 + \eta_1 = 0,$$
$$\frac{\partial L}{\partial \vartheta_1^U} = 0.6\tau_1 - 0.6\pi v_1 + 0.6\gamma_4 v_5 + \zeta_1 = 0,$$
$$\frac{\partial L}{\partial \vartheta_2^L} = 0.3\tau_2 - 0.3\pi v_2 + 0.3\gamma_4 v_6 + \eta_2 = 0,$$
$$\frac{\partial L}{\partial \vartheta_2^U} = 0.7\tau_2 - 0.7\pi v_2 + 0.7\gamma_4 v_6 + \zeta_2 = 0,$$
$$\frac{\partial L}{\partial \vartheta_3^L} = 0.7\tau_3 - 0.3\pi v_3 + 0.3\gamma_4 v_7 + \eta_3 = 0,$$
$$\frac{\partial L}{\partial \vartheta_3^U} = 0.3\tau_3 - 0.4\pi v_3 + 0.3\gamma_4 v_7 + \zeta_3 = 0,$$

$\tau_1(4x + y + z - 0.4\vartheta_1^L - 0.6\vartheta_1^U) = 0,$
$tau_2(x + 3y + z - 0.3\vartheta_2^L - 0.7\vartheta_2^U) = 0,$
$\tau_3(2x + y + 2z - 0.7\vartheta_3^L - 0.3) = 0, \vartheta_3^U$
$\pi(-6v_0 + (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_1 + (0.3\vartheta_2^L + 0.7\vartheta_2^U)v_2 + (0.7\vartheta_3^L + 0.3\vartheta_3^U)v_3 - 2v_4 - v_5 - 7v_6 - 2v_7 + 12) = 0,$
$\gamma_1(-v_0 - 4v_1 - v_2 - v_3 + 1) = 0,$
$\gamma_2(-2v_0 + v_1 - 3v_2 - v_3 - 4) = 0,$
$\gamma_3(-v_0 - 2v_1 - v_2 - 2v_3 + 7) = 0,$
$\gamma_4(5v_4 - (0.4\vartheta_1^L + 0.6\vartheta_1^U)v_5 - (0.3\vartheta_2^L + 0.7\vartheta_2^U)v_6 - (0.7\vartheta_3^L + 0.3\vartheta_3^U)v_7) = 0,$
$\gamma_5(6v_4 + 4v_5 + v_6 + v_7) = 0,$
$\gamma_6(v_4 + v_5 + 3v_6 + v_7) = 0,$
$\gamma_7(-v_4 + 2v_5 + v_6 + 2v_7) = 0,$
$\xi_1 v_1 = 0, \xi_2 v_2 = 0, \xi_3 v_3 = 0, \xi_4 v_4 = 0, \xi_5 v_5 = 0, \xi_6 v_6 = 0, \xi_7 v_7 = 0, \sigma_1 x = 0, \sigma_2 y = 0, \sigma_3 z = 0,$
$\eta_1\vartheta_1^L = 0, \zeta_1\vartheta_1^U = 0, \eta_2\vartheta_2^L = 0, \zeta_2\vartheta_2^U = 0, \eta_3\vartheta_3^L = 0, \zeta_3\vartheta_3^U = 0, \varphi v_0 = 0,$
v_0-Unrestricted.

The above system of equations is reduced to the following equations:

$0.3056 - 4\tau_1\tau_2 - 2\tau_3 + \sigma_1 = 0,$
$0.1528 - \tau_1 - 3\tau_2 - \tau_3 + \sigma_2 = 0,$
$0.0802 - \tau_1 - \tau_2 - 2\tau_3 + \sigma_3 = 0,$
$6 + \gamma_1 - 2\gamma_2 + \gamma_3 + \varphi = 0,$
$-\pi(0.4\vartheta_1^L + 0.6\vartheta_1^U) + 4\gamma_1 + \gamma_2 + 2\gamma_3 + \xi_1 = 0,$
$-\pi(0.3\vartheta_2^L + 0.7\vartheta_2^U) + \gamma_1 + 3\gamma_2 + \gamma_3 + \xi_2 = 0,$
$-\pi(0.7\vartheta_3^L + 0.3\vartheta_3^U) + \gamma_1 + \gamma_2 + 2\gamma_3 + \xi_3 = 0,$
$2\pi - 5\gamma_4 - 6\gamma_5 - \gamma_6 + \gamma_7 + \xi_4 = 0,$
$\pi + \gamma_4(0.4\vartheta_1^L + 0.6\vartheta_1^U) - 4\gamma_5 - \gamma_6 - 2\gamma_7 + \xi_5 = 0,$
$7\pi + \gamma_4(0.3\vartheta_2^L + 0.7\vartheta_2^U) - \gamma_5 - \gamma_6 - \gamma_7 + \xi_6 = 0,$
$2\pi + \gamma_4(0.7\vartheta_3^L + 0.3\vartheta_3^U) - \gamma_5 - \gamma_6 - 2\gamma_7 + \xi_7 = 0,$
$0.4\tau_1 - 0.15696\pi + \eta_1 = 0,$
$0.6\tau_1 - 0.23544\pi + \zeta_1 = 0,$
$0.3\tau_2 + 0.48225\pi + \eta_2 = 0,$
$0.7\tau_2 - 1.12525\pi + \zeta_2 = 0,$
$0.7\tau_3 + \eta_3 = 0,$
$0.3\tau_3 + \zeta_3 = 0,$
$\tau_1(9.0853 - 0.4\vartheta_1^L - 0.6\vartheta_1^U) = 0,$
$\tau_2(5.6662 - 0.3\vartheta_2^L - 0.7\vartheta_2^U) = 0,$
$\tau_3(5.1585 - 0.7\vartheta_3^L - 0.3\vartheta_3^U) = 0,$
$\pi(-15.645 + 0.3924(0.4\vartheta\vartheta_1^L + 0.6\vartheta_1^U) + 1.6075(0.3\vartheta_2^L + 0.7\vartheta_2^U)) = 0,$
$\gamma_1(-6.7846) = 0,$
$\gamma_2(-17.6451) = 0,$
$\gamma_3(0.0002) = 0,$
$0.3924\xi_1 = 0,$
$1.6075\xi_2 = 0,$
$1.9627\sigma_1 = 0,$
$1.2345\sigma_2 = 0,$
$\eta_1\vartheta_1^L = 0,$
$\zeta_1\vartheta_1^U = 0,$
$\eta_2\vartheta_2^L = 0,$
$\zeta_2\vartheta_2^U = 0,$
$\eta_3\vartheta_3^L = 0,$
$\zeta_3\vartheta_3^U = 0,$
$4.6075\varphi = 0,$
$9.0853 = 0.4\vartheta_1^L + 0.6\vartheta_1^U,$
$5.6662 = 0.3\vartheta_2^L + 0.7\vartheta_2^U,$
$5.1599 = 0.7\vartheta_3^L + 0.3\vartheta_3^U,$
$27.645 - 0.3924(0.4\vartheta_1^L + 0.6\vartheta_1^U) - 1.6075(0.3\vartheta_2^L + 0.7\vartheta_2^U) \geq 12,$

Therefore, the stability set of the first kind of problem (RV-TLFP) of the numerical example is given by:

$$S_1(1.962716, 1.234568, 0, 4.607515, 0.3924843, 1.607516, 0, 0, 0, 0, 0, 0.3) = \{\alpha \in [0,1] | \tau_1 = \tau_2 = \tau_3 = \pi = \gamma_1 = \gamma_2 = \gamma_3 = \xi_2 = \eta_1 = \zeta_1 = \eta_2 = \zeta_2 = \eta_3 = \zeta_3 = \varphi = 0,$$
$$-5\gamma_4 - 6\gamma_5 - \gamma_6 - \gamma_7 + \xi_4 = 0,$$
$$\sigma_1 = -0.3056, \sigma_2 = -0.1528, \sigma_3 = 0.0802,$$
$$9.0853\gamma_4 - 4\gamma_5 - \gamma_6 - 2\gamma_7 + \xi_5 = 0,$$
$$5.6662\gamma_4 - \gamma_5 - 3\gamma_6 - \gamma_7 + \xi_6 = 0,$$
$$5.1585\gamma_4 - \gamma_5 - 3\gamma_6 - 2\gamma_7 + \xi_7 = 0,$$
$$0.3924(0.4\vartheta_1^L + 0.6\vartheta_1^U) + 1.6075(0.3\vartheta_2^L + 0.7\vartheta_2^U) \le 15.645, 0.4\vartheta_1^L + 0.6\vartheta_1^U \ge 9.0853, 0.3\vartheta_2^L + 0.7\vartheta_2^U \ge 5.6662, 0.7\vartheta_3^L + 0.3\vartheta_3^U \ge 5.1599\}$$

8 Concluding Remarks

In this paper, the stability set of the first kind for multi-level fractional programming (TLFP-FP) problem has been characterized and determined. Some stability notations such as the solvability set and the stability set have been defined for such a problem, where fuzzy parameters on the right side of the constraints of the problem (TLFP-FP) are involved. A suggested algorithm has been described as infinite steps to solve the problem (TLFP-FP). Also, the stability set of the first kind (SSK1) corresponding to the obtained a-optimal solution has been determined.

Future open points for research in the area of the parametric problem (F-TLFP) needed to be studied in the future. Some of these points are given in the following:

I. Some stability notations for fuzzy Three-objective multi-level quadratic integer programming problem must be discussed.

II. Some stability notations for fuzzy Three-objective fractional integer programming problem should be investigated.

III. Some stability notations for fuzzy Three-level quadratic integer programming problem could be tackled.

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