Appl. Math. Inf. Sci. **9**, No. 5, 2303-2307 (2015) 2303

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090511

Necessity of the Third Condition from the Definition of Omega Chaos

Marek Lampart[*]

Department of Applied Mathematics & IT4Innovations. VŠB - Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic

Received: 12 Jan. 2015, Revised: 13 Apr. 2015, Accepted: 14 Apr. 2015
Published online: 1 Sep. 2015

Abstract: The notion of omega chaos was introduced by S. Li in 1993 for continuous maps of compact metric spaces by three conditions: 1. the set difference of omega limit sets is uncountable, 2. intersection of omega limit sets is nonempty and 3. each omega limit set of the point from the omega scrambled set is not contained in the set of all periodic points. It was also pointed that the third condition is superfluous for continuous maps of the compact interval. As a main result of this paper it will be shown that the third condition is essential even in one dimension by construction of two examples of homeomorphisms on one dimensional arcwise connected space having two point set or infinite respectively that satisfies first and second condition but the third condition is not fulfilled from the definition of omega chaos.

Keywords: omega chaos, Warsaw circle

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

Within the last forty years numerous papers and books have been devoted to the research of discrete dynamic systems. The main aim of the theory of discrete dynamic systems is focused the understanding of what the trajectories of all points from the state space look like. Mostly the periodic structures and asymptotic properties of the orbit were studied. Many authors were fascinated by those motions which are not only periodic but also are not quasi-periodic. These movings were assumed to be *unpredictable* or *sensitive to initial conditions*, later named as *chaotic*. There appeared many notions of chaos, starting with the famous paper by T.Y. Li and J. Yorke [14] in 1975. Later on, several notions of chaos motivated by diverse aspects were introduced (for more see e.g. [4] and references therein). Many natural questions arose. Which notion of chaos is the best one, or stronger then others (see e.g. [6,7,15] and references therein)? Which condition from each notion of chaos is the best one and which one is superfluous (see e.g. discussion on conditions of Devaney's chaos [3,1,17])? For this purpose, the following conventions are recalled.

Let (X,d) be a metric space with metric d and $C(X,X)$ the set of all continuous maps $f : X \to X$. Let $f \in C(X,X)$, $x \in X$ and n be a positive integer. The n-th iteration of x under f is denoted by f^n, the set of all fixed points of f by $\mathrm{Fix}(f)$, the set of periodic points of f by $\mathrm{Per}(f)$. The sequence $\{f^n(x)\}_{n=0}^{\infty}$ is the trajectory of x, and the set $\omega_f(x)$ of all limit points of the trajectory is the *ω-limit set* of x.

The main aim of the is paper is focused on the notion of omega chaos, and necessity of its third condition, introduced by S. Li in 1993 [13]:

Definition 1.*Let $f \in C(X,X)$ and $S \subset X$ contain uncountably many points. We say that f is ω-chaotic and S is ω-scrambled set for f if for any distinct $x,y \in S$:*

1.$\omega_f(x) \setminus \omega_f(y)$ is uncountable,
2.$\omega_f(x) \cap \omega_f(y)$ is nonempty,
3.$\omega_f(x)$ is not contained in the set of periodic points.

This notion of chaos is not well understand today, many open questions remains unsolved since this problem is related the understanding of the set of all omega limit

[*] Corresponding author e-mail: marek.lampart@vsb.cz

sets. In [15] authors discussed omega chaos for the circle maps. [11,10] studied cardinality of omega scrambled sets and relations to the chaos in the sense of T.Y. Li and J. Yorke. Later in [16,2] authors discussed Lebesgue measure of omega scrambled sets for continuous maps on the interval. Recently in [12] the specification property was compared to the omega chaos and in [8] the notion of omega chaos was transferred to from individual dynamics to collective one. It is worthy to note that there is no omega chaos for each minimal map, that is each omega limit set equals to the whole space, since minimal maps form a huge class of dynamic systems, see e.g. [9] and references therein. Other papers related to the notion of omega chaos are not known to the author.

As it is noted in [13] the third condition from the Definition 1 is superfluous for continuous maps on the unit closed interval (compact). Motivated by questions and remarks above, the goal is to study the third condition of omega chaos and discuss its necessity on spaces that are not homeomorphic to the interval. The following definition will be used for simplicity (the third condition from Definition 1 was excluded):

Definition 2.*Let $f \in C(X,X)$ and $S \subset X$ contain at least two points. We say that f is $\omega^\star$-chaotic and S is $\omega^\star$-scrambled set for f if for any distinct $x, y \in S$:*

1.$\omega_f(x) \setminus \omega_f(y)$ is uncountable,
2.$\omega_f(x) \cap \omega_f(y)$ is nonempty,

In particular, f is $\omega_2^\star$-chaotic or $\omega_\infty^\star$-chaotic if there is an $\omega^\star$-scrambled set containing two or infinitely many points.

As a main result it will be shown that the third condition from the Definition 1 could not be omitted in a general one dimensional spaces, two $\omega^\star$-chaotic examples will be constructed in Theorem 1 and Theorem 2.

2 Main results

The construction in the proof of the following theorem joins two shifted copies of Warsaw circles (see Figure 1) and define map on it in such a way that each point from the principal part is tending to the limit line having it as its omega limit set and the map is identical on the limit line. That shows that the third condition from the Definition 1 of omega chaos is essential in general. The idea of the proof was extended in Theorem 2 to get infinite $\omega^\star$-scrambled set in one dimension where again the third condition from the omega chaos is not fullfiled.

Theorem 1.*There is one dimensional arcwise connected space W and homeomorphism P on W such that P is $\omega^\star$ chaotic. Moreover each $\omega^\star$-scrambled set contains two points and the third condition from the Definition 1 is not satisfied.*

*Proof.*Let $W = A_1 \cup A_2 \cup C$ where

$$A_1 = \{(x, \sin(\pi/x) + 0.5) \in \mathbb{R}^2 \ : \ x \in (0,1)\},$$
$$A_2 = \{(-x, \sin(\pi/x) - 0.5) \in \mathbb{R}^2 \ : \ x \in (0,1)\},$$
$$C = [-1,1] \times \{0\} \cup \{-1\} \times [-2,-0.5] \cup \cup \{0\} \times [-2,1.5] \cup \{1\} \times [-2,0.5],$$

see Figure 1. Now define homeomorphism $P : W \to W$ by

$$P(x) = \begin{cases} p_{1_3} \circ p_{1_2} \circ p_{1_1}(x), & \text{if } x \in A_1, \\ p_{2_3} \circ p_{2_2} \circ p_{2_1}(x), & \text{if } x \in A_2, \\ x, & \text{if } x \in C \end{cases} \quad (1)$$

where

$$p_{1_3} : A_1 \to (1,\infty)$$

such that

$$(x, \sin(\pi/x) + 0.5) \mapsto (1 - \tan((\pi/2)x - \pi/2)),$$

$$p_{1_2} : (1,\infty) \to (1,\infty)$$

such that

$$x \mapsto (x-1)/(1+(x-1)^2) + x,$$

$$p_{1_1} : (1,\infty) \to A_1$$

such that

$$x \mapsto (1/x, \sin(\pi x) + 0.5)$$

and

$$p_{2_3} : A_2 \to (1,\infty)$$

such that

$$(-x, \sin(\pi/x) - 0.5) \mapsto (1 - \tan((\pi/2)x - \pi/2)),$$

$$p_{2_2} : (1,\infty) \to (1,\infty)$$

such that

$$x \mapsto (x-1)/(1+(x-1)^2) + x,$$

$$p_{2_1} : (1,\infty) \to A_2$$

such that

$$x \mapsto (-1/x, \sin(\pi x) - 0.5).$$

It is easy to see that for any $x_1 \in A_1$ and that for any $x_2 \in A_2$ is

$$\begin{aligned} \omega_W(x_1) &= \{0\} \times [-0.5, 1.5], \\ \omega_W(x_2) &= \{0\} \times [-1.5, 0.5]. \end{aligned} \quad (2)$$

Next put $S = \{x_1, x_2\}$ where $x_1 \in A_1$ and $x_2 \in A_2$ are arbitrarily chosen points. Since

1.$\omega_W(x_1) \setminus \omega_W(x_2) = \{0\} \times [0.5, 1.5]$ and $\omega_W(x_2) \setminus \omega_W(x_1) = \{0\} \times [-1.5, -0.5]$ is uncountable,
2.$\omega_W(x_1) \cap \omega_W(x_2) = \{0\} \times [-0.5, 0.5]$ is nonempty,
3.$\omega_W(x_1)$ and $\omega_W(x_2)$ is contained in the set of periodic points since $\omega_W(x_1) \cup \omega_W(x_2) \subset \mathrm{Fix}(W)$,

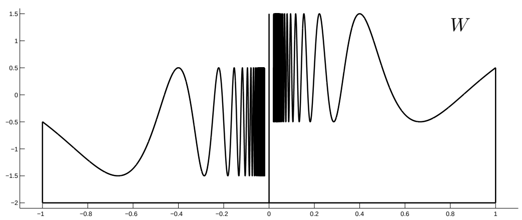


Fig. 1: Warsaw circle W.

the set S is $\omega^\star$-scrambled. Hence the map W is $\omega_2^\star$-chaotic and due to (2) and the fact that $\omega_W(c) = \{c\}$ for any $c \in C$, any $\omega^\star$-scrambled set contains exactly two points.

*Remark.*Let us note that analogous construction could be done in two dimensional space $[0,1]\times[0,1]$ motivated by the Crooked Warsaw circle. It could be defined a triangular map $F : [0,1]^2 \to [0,1]^2$ where $F(x,y) = (f(x), g(x,y))$ in such a way that $F(A_i) = A_{i+1}$ where $A_i = [(2^2-1)/2^{i+2}, 1/2^i]\times[0,2/3]$, $i \in \mathbb{N}\cup\{0\}$, $F(B_j) = B_{j+1}$ where $B_j = [1/2^{j+1} + 1/2^{3(j+1)}, (2^2-1)/2^{j+2} - 1/2^{3j}]\times[2/3,1]$, $j \in \mathbb{N}$ and F will be unimodal on the rest of $[0,1]\times[0,1]$, hence F will be identical on $\{0\}\times[0,1]$ (for more about triangular maps see e.g. [5] and references therein). Then for $a \in A_1$ and $b \in B_1$ it is $\omega_F(a) = \{0\}\times[0,2/3]$ and $\omega_F(b) = \{0\}\times[1/3,1]$ showing that F is $\omega_2^\star$ chaotic and the third condition from the Definition 1 is not satisfied.

Theorem 2.*There is one dimensional arcwise connected space $\mathbb{X}$ and a homeomorphism $\mathbb{H}$ on $\mathbb{X}$ such that $\mathbb{H}$ is $\omega_\infty^\star$-chaotic and the third condition from the Definition 1 is not satisfied.*

*Proof.*Let X_i be the one dimensional arcwise connected space defined as follows for any $i \in \mathbb{N}$ (so called Warsaw circle, see Figure 2): $X_i = A_i \cup B_i \cup [-3,1]\times\{0\}$ were

$$A_i = \left\{ \left(x, \sin(\pi/x) - 1 + \frac{2^i - 1}{2^i}\right) \in \mathbb{R}^2 \ : \ x \in (0,1) \right\},$$

$$B_i = \{1\}\times\left[-3, -1 + \frac{2^i-1}{2^i}\right],$$

$$C_i = \{0\}\times\left[-3, \frac{2^i-1}{2^i}\right].$$

Now, homeomorphisms h_i on each X_i will be defined by

$$h_i(x) = \begin{cases} h_{i_3}\circ h_{i_2}\circ h_{i_1}(x), & \text{if } x \in A_i \\ x, & \text{otherwise} \end{cases} \tag{3}$$

where

$$h_{i_3} : A_i \to (1,\infty)$$

such that

$$\left(x, \sin(\pi/x) - 1 + \frac{2^i-1}{2^i}\right) \mapsto (1 - \tan((\pi/2)x - \pi/2)),$$

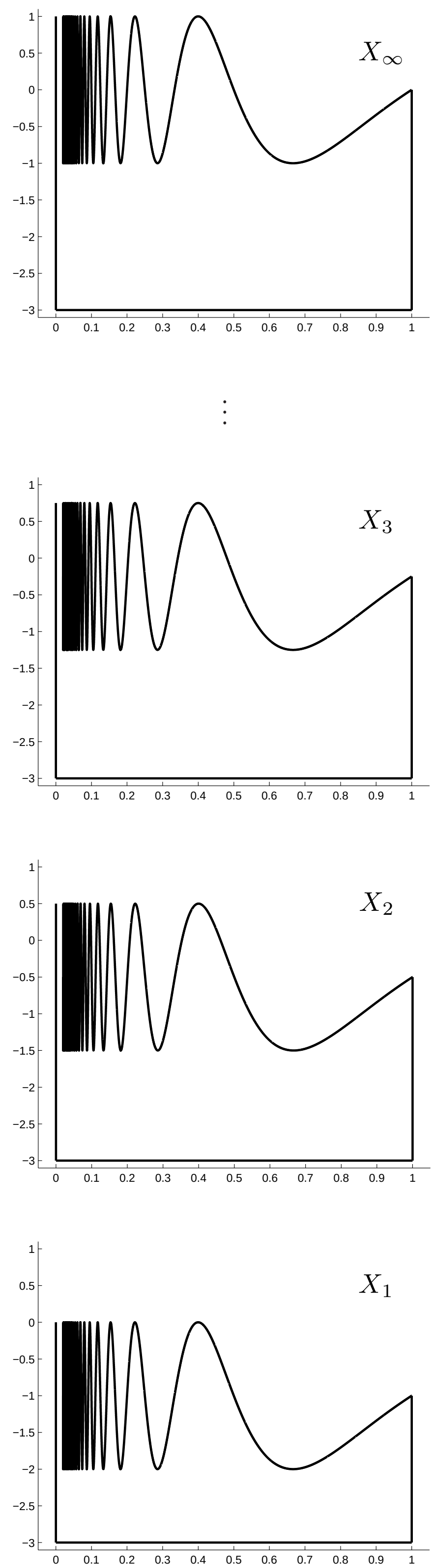


Fig. 2: Warsaw circles X_j.

$$h_{i_2} : (1,\infty) \to (1,\infty)$$

such that

$$x \mapsto (x-1)/(1+(x-1)^2)+x$$

and

$$h_{i_1} : (1,\infty) \to A_i$$

such that

$$x \mapsto \left(\frac{1}{x}, \sin(\pi x) - 1 + \frac{2^i - 1}{2^i}\right).$$

Let X_∞ be the one dimensional space arcwise connected space defined as follows: $X_\infty = A_\infty \cup B_\infty \cup [-3,1] \times \{0\}$ were

$$A_\infty = \{(x, \sin(\pi/x)) \in \mathbb{R}^2 \ : \ x \in (0,1)\},$$
$$B_\infty = \{1\} \times [-3,0],$$
$$C_\infty = \{0\} \times [-3,1].$$

Analogously, homeomorphism h_∞ on X_∞ will be defined by

$$h_\infty(x) = \begin{cases} h_{\infty_3} \circ h_{\infty_2} \circ h_{\infty_1}(x), & \text{if } x \in A_\infty \\ x, & \text{otherwise} \end{cases} \tag{4}$$

where

$$h_{\infty_3} : A_\infty \to (1,\infty)$$

such that

$$(x, \sin(\pi/x)) \mapsto (1 - \tan((\pi/2)x - \pi/2)),$$

$$h_{\infty_2} : (1,\infty) \to (1,\infty)$$

such that

$$x \mapsto (x-1)/(1+(x-1)^2)+x$$

and

$$h_{\infty_1} : (1,\infty) \to A_\infty$$

such that

$$x \mapsto (1/x, \sin(\pi x)).$$

Next, by collapsing

$$\left(\bigcup_{n=1}^{\infty} \{0\} \times \left[-3, \frac{2^n-1}{2^n}\right] \times \left\{\frac{1}{n}\right\}\right) \cup \{0\} \times [-3,1] \times \{0\}$$

into $\{0\} \times [-3,1] \times \{0\}$ we get a metrizable space. Denote it by $\mathbb{X}$. One can think about it as a subspace of $\mathbb{R}^3$, see Figure 3. The topology on $\mathbb{X}$ is given by the metric inherited from $\mathbb{R}^3$. The space $\mathbb{X}$ can be imagined as a union of slices X_j with common interval $\{0\} \times [-3,1] \times \{0\}$, here $j \in \mathbb{N} \cup \{\infty\}$. The space $\mathbb{X}$ is one dimensional arcwise connected since each slice X_j is.

Now the map $\mathbb{H}$ on $\mathbb{X}$ will be defined in such a way that $\mathbb{H}$ restricted on each X_j equals to h_j. Obviously, $\mathbb{H}$ is homeomorphism since each h_j is.

Finally, put $S = \cup_{i=1}^{\infty} x_i$ where $x_i \in A_i$ is arbitrarily chosen point for any i. The set S is $\omega_\infty^\star$-scrambled since:

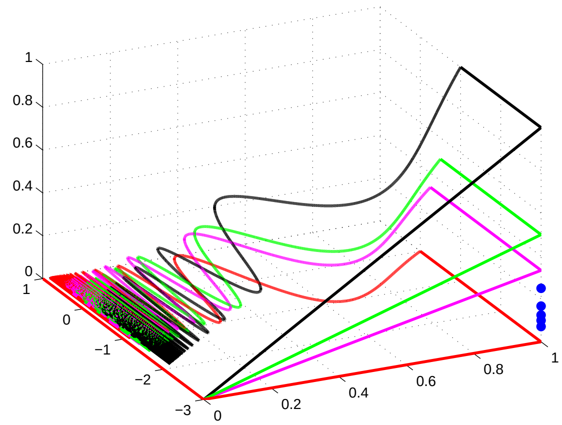


Fig. 3: The space $\mathbb{X}$.

1. if $n > m$ then $\omega_{\mathbb{H}}(x_n) \setminus \omega_{\mathbb{H}}(x_m) = \{0\} \times [a,c] \times \{0\}$ is uncountable as well as $\omega_{\mathbb{H}}(x_m) \setminus \omega_{\mathbb{H}}(x_n) = \{0\} \times [b,d] \times \{0\}$, here $a = (2^n-1)/2^n$, $b = -2+a$, $c = (2^m-1)/2^m$ and $d = -2+c$,
2. if $n > m$ then $\omega_{\mathbb{H}}(x_n) \cap \omega_{\mathbb{H}}(x_m) = \{0\} \times [c,b] \times \{0\}$ is nonempty,
3. $\omega_{\mathbb{H}}(x_i)$ is contained in the set of all periodic points since each $\omega_{\mathbb{H}}(x_i) \subset \{0\} \times [-3,1] \times \{0\} \subset \mathrm{Fix}(\mathbb{H})$.

Ending the proof.

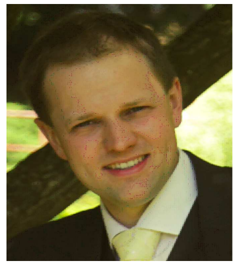
Acknowledgement

This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence Project (CZ.1.05/1.1.00/02.0070). The work was also supported by the Grant Agency of the Czech Republic, Grant No. P201/10/0887.

References

[1] J. Banks, J. Brooks, G. Cairns, G., Davis and P. Stacey, Amer. Math. Monthly **99**, 332-334 (1992).
[2] J. Bobok, Int. J. Bif. and Chaos **16**, 737-740 (2006).
[3] R.L. Devaney, *An introduction to chaotic dynamical systems*. Boulder: Westview Press, 2003. ISBN 0-8133-4085-3.
[4] G.L. Forti, Aeq. Math. **70**, 1-13 (2005).
[5] G.L. Forti, L. Paganoni, J. Smítal, Bull. Aust. Math. Soc. **51**, 395-415 (1995).
[6] J.L.G. Guirao and M. Lampart, Chaos, Solitons & Fractals **24**, 1203-1206 (2005).
[7] J.L.G. Guirao and M. Lampart, Chaos, Solitons & Fractals **28**, 788-792 (2006).
[8] J.L.G. Guirao, D. Kwietniak, M. Lampart, P. Oprocha and A. Peris, Nonli. Anal.: Theo., Meth. and Appl. **71**, 1-8 (2009).

[9] S. Kolyada and L. Snoha, Scholarpedia, 4(11): 5803 (2009).
[10] M. Lampart, Real Anal. Ex. **27**, 801-808 (2001).
[11] M. Lampart, Acta Math. Univ. Com. **72**, 119-129 (2003).
[12] M. Lampart and P. Oprocha, Topol. and its Appl. **156**, 2979-2985 (2009).
[13] S. Li, Trans. Amer. Math. Soc. **339**, 243-249 (1993).
[14] T. Y. Li and J. A. Yorke, Amer. Math. Monthly **82**, 985-992 (1975).
[15] M. Miyazawa, Tokyo J. Math. **25**, 453-458 (2002).
[16] J. Smítal and M. Štefánková, Disc. and Cont. Dynam. Sys. **9**, 1323-1327 (2003).
[17] M. Vellekoop and R. Berglund, Amer. Math. Monthly **101**, 353-355 (1994).



Marek Lampart is Associate Professor at Department of Applied Mathematics and Junior Researcher at IT4Innovations VŠB-Technical University of Ostrava, Czech Republic. He received the PhD degree in “Pure Mathematics” at the Silesian University of Opava (Czech Republic). He is referee of several international journals in the frame of pure and applied mathematics. His main research interests are: dynamical systems on compact metric spaces, applied dynamical systems in economy and mechanics.