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Bulk viscous plane-symmetric cosmological model in Lyra's Geometry and General Relativity Theory

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Abstract: In relativity theory and Lyra manifold, we present the viscosity's effect on the universe's thermodynamic functions. We model the universe using by plane-symmetric model. The Einstein field equations' solution is obtained by considering that the expansion scalar θ of the space-time is proportional to the component σ_1^1 of the shear tenor σ_i^j. For the universe, we investigate and studied the thermodynamic functions of it. The time component introduced by Lyra in the displacement vector is defined as a viscosity term to restore the consistency of the conservation equation of the momentum tensor [$T_{i;j}^j = 0$]. The geometrical and physical properties are studied for the obtained models.

Keywords: Thermodynamic functions; plane-symmetric cosmological model; Lyra geometry, General relativity theory, Entropy; Viscosity

1 Introduction

In 1916, a geometrizing of the gravitational theory was proposed by Einstein [1]. He identifies the potential with the metric tensor g_{ij}. In 1918, Weyl [2] tried to introduce a geometrizing of gravitation and electromagnetism both. He introduced a scalar field ,ϕ, to be identified with field of the electromagnetic . This generalization is not satisfactory because the length of the vector is not invariant through the parallel transport (non-integrability). Einstein [3] criticized non-integrability of the length transfer .

Lyra in 1951, [4] introduced a modification of Riemannian geometry by entering a function $x^0(x^i)$ (Gauge Function)in its manifold. In a modification of Lyra, the metric tensor g_{ij} is preserved and the length of the vector is integrable. When $x^0(x^i) = 1$ (normal gauge), the scalar curvature of Lyra and Weyl is equal. Sen [5] in 1957 proposed a gravitational theory depending on the modification theory of Lyra. He introduced the notation of the displacement vector with only a nonvanishing component (The time component). In more investigations proposed in Lyra geometry under the field equations introduced by Sen [5] the component introduced in the displacement vector has unclear physical meaning.

Hegazy and Farook [6] tried to introduce a meaning to the Lyra term (physical meaning) by study its effect on the entropy of the universe. They observed that it has no impact on entropy because it introduced from geometry and it's not a component of the energy tensor (T_{ij}). Hegazy [7] began with the result obtained in Hegazy and Farook [6] and identify the Lyra term with a viscosity term of the momentum tensor. The present study is consistent with the results obtained by Hegazy [7] since we must define the Lyra term as viscosity to restore the consistency of the conservation equation of the momentum tensor.

In Lyra's manifold and the presence of viscosity, some space-times were studied. Bianchi type (BT) III was investigated by Yadav and Yadav [8] with barotropic perfect fluid and Patra and Sethi [9] studied it with time-varying Λ and Chaplygin gas. Accelerating BT VI_0 was studied by Asgar and Ansari [10]. An LRS BT I was investigated by Pradhan and Pandey [11]. Singh et al. [12] investigated FRW models when the deceleration parameter consider as a function of the time t.

In relativity theory in the existence of viscosity, BT II was investigated by Sharma [13]. In the existence of the magnetic field, Tyagi and Sharma [14] investigated some BT II bulk viscous string space time and Yadav et al. [15]

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introduced some BT I viscous fluid string cosmological models. By considering Λ and q as time dependent, Ali [16] studied axially symmetric BT I cosmological models. Mak and Harko [17] presented BT I with causal viscosity. With cosmological term Λ as a function of t, Singh and Baghel [18] presented bulk viscous BT V space time.

In the relativity theory and its modified theories, the plane-symmetric cosmological model (PSCM) with different forms of matter distribution was studied.

Bayaskar et al. [19] studied PSCM with interacting fields. in the scalar-tensor theory, Ali et al. [20] investigated some inhomogeneous PSCM. Tiwari [21] investigated PSCM in self-creation cosmology. Pawar et al. [22] presented PSCM in Brans-Dicke's theory. In the self-creation theory with deceleration parameter as negative and constant, PSCM was investigated by Katore et al. [23] and Panigrahi and Sahu [24] constructed macro PSCM. With shear and vanishing acceleration, Pradhan et al. [25] introduced a class of solutions of the field equations of PSCM in the existence of perfect fluid. Venkateswarlu et al. [26] presented non-static solutions of PSCM in the scalar-tensor theory. In zero mass scalar fields, Venkateswarlu and Kumar [27] studied string PSCM. Katore et al.[28] presented PSCM with dark energy and perfect fluid. In the Bimetric gravitational theory, Mohanty and Sahoo [29] presented non-static PSCM. Pradhan et al. [30] studied inhomogeneous PSCM with viscous fluid and electromagnetic field. The inhomogeneous plane-symmetric magnetized viscous fluid with time dependent Λ was investigated by Pradhan and Pandey [31]. Pawar et al. [32] presented PSCM dust magnetized string cosmological model in the existence of viscosity . In $f(R,T)$ theory of gravitation, Agrawal and Pawar [33] presented PSCM with strange quark and quark matter and Chirde and Shekh [34] studied PSCM dark energy in the form of wet dark fluid. By considering G and Λ are varying. Singh and Ram [35] obtained the exact non-static solutions to Einstein equations for PSCM with perfect fluid in the presence of attractive scalar fields. By using the Riccati equation with the cosmological constant, Vuille and Dunn [36] studied the exact PSCM solutions for a massless scalar field. Raddy [37] obtained static and non-static PSCM string in Lyra geometry. Reddy and Innaiah [38] obtained a non-static PSCM in Lyra's geometry for stiff fluid. Pradhan and Pandey [39] discussed inhomogeneous bulk viscous PSCM with time dependent Λ. Pradhan and Kumhar [40] obtained inhomogeneous PSCM with perfect fluid and electromagnetic field in Lyra geometry. PSCM with Λ in $f(R,T)$ theory was analyze by Shaikh and Bhoyar [41]. Pradhan [42] investigated PSCM with viscosity and cosmological term Λ as time dependent. Pawar and Deshmukh [43] studied string PSCM with bulk viscous fluid.

In the following paper, we study PSCM in the presence of viscous fluid. In section 2, we present the metric and the field equations with its solution with the physical and geometrical properties of the obtained models . The entropy and other thermodynamics function in the existence of viscosity is given in section 3. In the absence of Lyra term, PSCM is studied in section 4. Conclusion is indicated in section 5.

2 The space-time and equations of the field with its solution

For PSCM, The line element ds^2 reads as:

$$ds^2 = A^2dt^2 - A^2dx^2 - B^2dy^2 - C^2dz^2, \quad (1)$$

Here A, B and C depend on t only.
In the notation of the normal gauge $(x^0(x^i) = 1)$, Einstein equations read as [5].

$$G^i_j + \frac{3}{2}\phi^i\phi_j - \frac{3}{4}g^i_j\phi_k\phi^k = -\chi T^i_j, \quad (2)$$

Here, χ is a constant, G^i_j is the Einstein tensor and ϕ_i is a time like vector in the form:

$$\phi_i = (\beta(t), 0, 0, 0). \quad (3)$$

T^j_i is the momentum tensor which reads as;

$$T^j_i = (p+\rho)u_iu^j - pg^j_i - \xi\theta[g^j_i - u_iu^j], \quad (4)$$

here ξ is the coefficient of the bulk viscosity, θ the expansion scalar, ρ energy density and p isotropic pressure. In the coordinates system which is co-moving $u^0 = \frac{1}{A}, u_0 = A, u_iu^j = 1, (u^1 = u^2 = u^3 = 0)$. Equation(4) leads to:

$$T^1_1 = -[p+\xi\theta] = T^2_2 = T^3_3, \quad T^0_0 = \rho, \quad (5)$$

From (1), the field equations (2) read as:

$$\frac{1}{A^2}[\frac{\ddot{C}}{C} + \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2] = -\chi[p+\xi\theta], \quad (6)$$

$$\frac{1}{A^2}[\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} + \frac{\ddot{C}}{C} + \frac{3}{4}\beta^2] = -\chi[p+\xi\theta], \quad (7)$$

$$\frac{1}{A^2}[\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} + \frac{\ddot{B}}{B} + \frac{3}{4}\beta^2] = -\chi[p+\xi\theta], \quad (8)$$

$$\frac{1}{A^2}[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3}{4}\beta^2] = \chi\rho, \quad (9)$$

where dots mean differentiation according to t.
From $T^i_{j;i} = 0$ we get:

$$\dot{\rho} + (p+\xi\theta+\rho)(A\theta) = 0, \quad (10)$$

Conservation of the Lyra term (L.H.S. Eq (2)) leads to:

$$\frac{3}{2A^2}\beta(\dot{\beta} + \beta[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}]) = 0. \quad (11)$$

Equation (11) has a solution in the form

$$\beta = 0, \qquad , \qquad \beta = \frac{N_1}{ABC}, \tag{12}$$

where N_1 is a constant.

For $\beta = 0$, we obtain solutions in the relativity theory which will be presented in section 5.

Equations (6)- (11) are a system of high order non linear differential equations. The system consist of 8 unknowns $A, B, C, \rho, p, \beta, \xi$ with six equations. Equation (11) reduces the number of unknowns to seven since we obtained β as a function of A, B and C. To obtain the complete solution we add the following physical constrains.

We consider θ is proportional to the component σ_1^1 of σ_i^j (the shear tensor), hence:

$$A = (BC)^L, \tag{13}$$

where L is a constant.
From equations (6)-(8) with (13) we obtain:

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 0, \tag{14}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} = 0. \tag{15}$$

Let $BC = \mu$ and $\frac{B}{C} = \nu$ equations (14) and (15) become:

$$\frac{\ddot{\nu}}{\nu} - \frac{\dot{\nu}^2}{\nu^2} + \frac{\dot{\nu}\dot{\mu}}{\nu\mu} = 0, \tag{16}$$

$$\frac{\ddot{\nu}}{\nu} + \frac{\ddot{\mu}}{\mu} - \frac{\dot{\nu}^2}{\nu^2} + \frac{\dot{\nu}\dot{\mu}}{\nu\mu} = 0. \tag{17}$$

From (16) and (17) we get:

$$\ddot{\mu} = 0, \tag{18}$$

Equation (18) has a solution in the form:

$$\mu(t) = c_1 t + c_2, \tag{19}$$

where c_1 and c_2 are constants. Equation (16) can be rewritten as:

$$\frac{1}{\mu}\left(\frac{\dot{\nu}\mu}{\nu}\right)^{\cdot} = 0, \tag{20}$$

then $\frac{\dot{\nu}\mu}{\nu} = c_3$ (constant) which give:

$$\nu(t) = [c_4(c_1 t + c_2)]^{\frac{c_3}{c_1}}, \tag{21}$$

where c_4 is a constants. From (19), (21) and (13), we get:

$$B(t) = \sqrt{\alpha T^{\frac{c_3}{c_1}+1}}, \quad A(t) = T^L, \qquad C(t) = \frac{\sqrt{T^{1-\frac{c_3}{c_1}}}}{\sqrt{\alpha}}, \tag{22}$$

where $\alpha = c_4^{\frac{c_3}{c_1}}$ is a constant and $T = (c_1 t + c_2)$.

The line element (1) reads as:

$$ds^2 = T^{2L} dt^2 - T^{2L} dx^2 - \alpha T^{\frac{c_3}{c_1}+1} dy^2 - \frac{T^{1-\frac{c_3}{c_1}}}{\alpha} dz^2,$$
$$T = (c_1 t + c_2). \tag{23}$$

From (12) the Lyra term ($\beta(t)$) reads as:

$$\beta(t) = N_1 T^{-(1+L)}. \tag{24}$$

From (9), ρ reads as:

$$\rho = \frac{T^{-4L-2}\left(\left(c_1^2(4L+1) - c_3^2\right) T^{2L} - 3N_1^2\right)}{4\chi}. \tag{25}$$

From (6)-(8), the term $(p + \xi\theta)$ given by:

$$p + \xi\theta = \frac{T^{-2-4L}[T^{2L}\left((1+4L)c_1^2 - c_3^2\right) - 3N_1^2]}{4\chi}. \tag{26}$$

From (10) we get:

$$\dot{\rho} + (\rho + p + \xi\theta)(A\theta) = \frac{3c_1 L N_1^2 T^{-4L-3}}{2\chi}, \tag{27}$$

which implies to:

$$\dot{\rho} + (\rho + p + \xi\theta)(A\theta) \neq 0. \tag{28}$$

To restore the consistency of (28) ($T^i_{j;i} = 0$), we choice:

$$\xi\theta = -\frac{3N_1^2 T^{-4L-2}}{4\chi}, \tag{29}$$

Hence:

$$\xi = -\frac{3N_1^2 T^{-3L-1}}{4c_1(L+1)\chi}. \tag{30}$$

The choice of the ξ in form of Lyra term is consistence with the results obtained by Hegazy [7] From (26), the pressure p reads as:

$$p = \frac{\left(c_1^2(4L+1) - c_3^2\right) T^{-2(L+1)}}{4\chi}. \tag{31}$$

At $t = 0$, p, ρ and ξ begin with large values, decreasing toward a finite quantities at $t = t_0$ and at the end of evolution p, ρ and ξ approach to zero. For the (23), the volume $V = T^{(1+2L)}$, the deceleration parameter $q = -1 + \frac{3}{1+2L}$, the expansion scalar $\theta = \frac{c_1(L+1)}{T^{L+1}}$, the non vanishing components of (σ_i^j) read as $\sigma_1^1 = \frac{c_1(2L-1)}{3T^{L+1}}$, $\sigma_2^2 = \frac{c_1(1-2L)+3c_3}{6T^{L+1}}$, $\sigma_3^3 = \frac{c_1(1-2L)-3c_3}{6T^{L+1}}$ and the shear $\sigma = \sqrt{\frac{1}{12}\left(c_1^2(1-2L)^2 + 3c_3^2\right)} T^{-(L+1)}$.

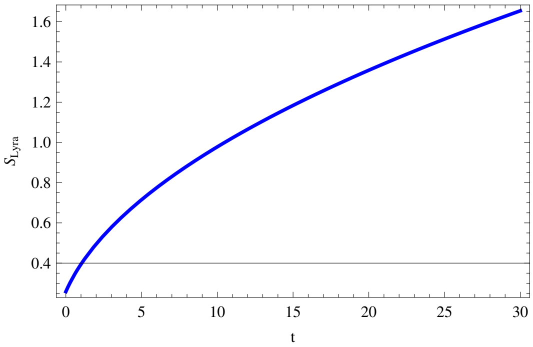

Fig. 1: The entropy $\mathbf{S}_{Lyra}$vs. time, $0 < t < 30$.

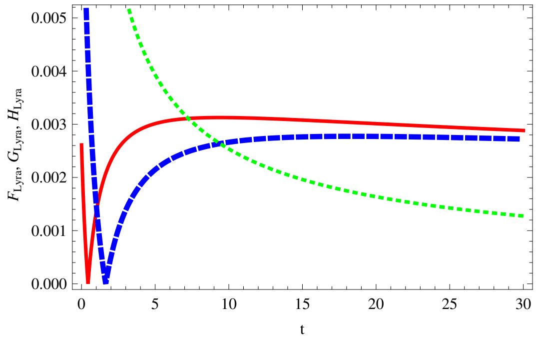

Fig. 2: $\mathbf{F}_{Lyra}$ (Red thick line), $\mathbf{G}_{Lyra}$ (Blue dashed line) and $\mathbf{H}_{Lyra}$ (Green dotted line) vs. t, $0 < t < 30$.

3 Study thermodynamic functions in the existence of viscosity

If internal energy U of the universe is defined by ρV and the temperature $\mathbf{T} = \frac{H\,(mean\ Hubble\ parameter)}{2\pi}$ [44]. In existence of viscosity [45], [46], [47], [48], [6], [7], [49] and [50] we get:

$$\frac{d\mathbf{S}_{Lyra}}{dt} = \frac{2\pi V[\xi\theta^2]}{H} = -\frac{9\pi(L+1)N_1^2 T^{-(2L+1)}}{2\chi(2L+1)}, \quad (32)$$

where S_{Lyra} is the entropy in Lyra geometry. by integration (32) we get:

$$\mathbf{S}_{Lyra} = \frac{9\pi(L+1)N_1^2 T^{-2L}}{4\chi L c_1(2L+1)}. \quad (33)$$

The functions **H** (Enthalpy), **F** (Helmholtz) and **G** (Gibbs) in Lyra geometry take the form:

$$\mathbf{H}_{Lyra} = PV + U = (\rho + P)V = \frac{T^{-(2L+1)}\left(2\left(c_1^2(4L+1) - c_3^2\right)(c_1 t + c_2)^{2L} - 3N_1^2\right)}{4\chi}, \quad (34)$$

$$\mathbf{F}_{Lyra} = U - \mathbf{T}\mathbf{S}_{Lyra} = \frac{T^{-3L-1}(-6LN_1^2(c_1 t + c_2)^L)}{8L\chi} + \frac{T^{-3L-1}(2L(c_1^2(4L+1) - c_3^2)(c_1 t + c_2)^{3L} - 3(L+1)N_1^2)}{8L\chi} \quad (35)$$

$$\mathbf{G}_{Lyra} = \mathbf{H}_{Lyra} - \mathbf{T}\mathbf{S}_{Lyra} = \frac{T^{-3L-1}(-6LN_1^2\,(c_1 t + c_2)^L)}{8L\chi} + \frac{T^{-3L-1}(4L\left(c_1^2(4L+1) - c_3^2\right)(c_1 t + c_2)^{3L} - 3(L+1)N_1^2)}{8L\chi} \quad (36)$$

The behavior of **H, F and G** with t can be given as follow. The constants are considered as: $\alpha = 1, c_2 = 1.5, c_1 = 2, c_3 = 1, L = -0.25, N_1 = 0.5, \chi = -8\pi$

The entropy begins with a small value (nearly zero) at the beginning of evolution and increases uniformly with time to reach a large value at the end of the evolution (See Fig. 1).

For $0 < t \leq 3$, $\mathbf{H}_{Lyra} = 0$ and as $t > 3$, $\mathbf{H}_{Lyra}$ decreases uniformly to a zero value at the end of the evolution. For $0 < t < 0.5$, $\mathbf{F}_{Lyra}$ decrease to reaches zero value and as $t > 0.5$ increases again to a constant value at the end of the evolution. when $0 < t < 1.5$, $\mathbf{G}_{Lyra}$ decreases to reaches zero value and increases again to reaches the value of $\mathbf{F}_{Lyra}$ at the end of the evolution (See Fig. 2).

4 Plane symmetric model without Lyra term

In absence of Lyra term (theory of relativity) ($\beta = 0$), metric coefficients A, B *and* C have the values given by (22) since they do not depend on β. Physical quantities: ρ_{GR} (the density), p_{GR} (the pressure) and ξ_{GR} (bulk viscosity coefficient) will changes and read as:

$$\rho_{GR} = \frac{\left(c_1^2(4L+1) - c_3^2\right)T^{-2(L+1)}}{4\chi}, \quad (37)$$

$$p_{GR} + \xi_{GR}\theta = \frac{\left(c_1^2(4L+1) - c_3^2\right)T^{-2(L+1)}}{4\chi}. \quad (38)$$

Equation(38) can be divided as:

$$\xi_{GR}\theta = -\frac{c_3^2 T^{-2(L+1)}}{4\chi}, \quad (39)$$

and

$$p_{GR} = \frac{c_1^2(4L+1)T^{-2(L+1)}}{4\chi}. \quad (40)$$

From (32), the entropy $\mathbf{S}_{GR}$ is governed by the equation:

$$\frac{d\mathbf{S}_{GR}}{dt} = \frac{2\pi V[\xi_{GR}\theta^2]}{\mathbf{H}} = -\frac{3\pi c_3^2(L+1)T^{-1}}{2\chi(2L+1)}, \quad (41)$$

by integration we have:

$$\mathbf{S}_{GR} = -\frac{3\pi c_3^2(L+1)\ln T}{2c_1\chi(2L+1)}. \quad (42)$$

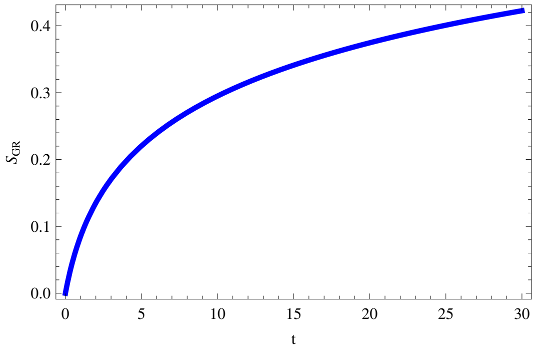

Fig. 3: The entropy $\mathbf{S}_{GR}$ vs. time t, $0 < t < 30$.

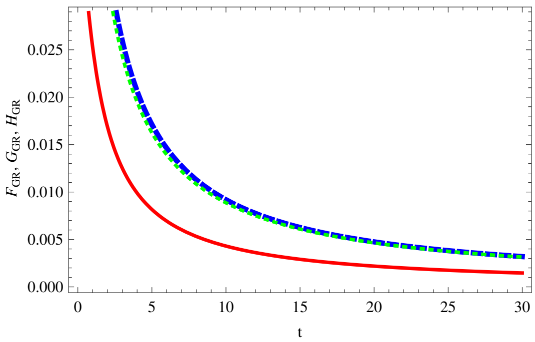

Fig. 4: The Helmholtz $\mathbf{F}_{GR}$ (Red line), $\mathbf{G}_{GR}$ (Blue line) and $\mathbf{H}_{GR}$ (Green line) vs. time t, $0 < t < 30$.

Other thermodynamics functions ($\mathbf{H}_{GR}$, $\mathbf{G}_{GR}$) and $\mathbf{F}_{GR}$ in general relativity read as:

$$\mathbf{H}_{GR} = \frac{\left(c_1^2(8L+2) - c_3^2\right) T^{-1}}{4\chi}, \tag{43}$$

$$\mathbf{G}_{GR} = \frac{T^{-(L+1)}\left(\left(c_1^2(8L+2) - c_3^2\right) T^L + c_3^2(L+1)\ln T\right)}{4\chi}, \tag{44}$$

$$\mathbf{F}_{GR} = \frac{T^{-(L+1)}\left(\left(c_1^2(4L+1) - c_3^2\right) T^L + c_3^2(L+1)\ln T\right)}{4\chi}. \tag{45}$$

In the following we show the behaviors of $\mathbf{S}_{GR}$, $\mathbf{H}_{GR}$, $\mathbf{G}_{GR}$ and $\mathbf{F}_{GR}$ with the time. the constants considered as: $\alpha = 1, c_2 = 1, c_1 = 1, c_3 = 1, L = 1.1, N_1 = 5, \chi = -8\pi$

At $t = 0$ $\mathbf{S}_{GR} = 0$, as the time elapsed the entropy is increases and at the end of the evolution its has a large value (See Fig. 3).

$\mathbf{F}_{GR}$ begins with large value after one unit of the beginning of evolution and decreases to zero value at the end of the present stage. $\mathbf{G}_{GR}$ and $\mathbf{H}_{GR}$ have zeros value in the interval $0 < t < 2.2$. As $t > 2.2$, $\mathbf{G}_{GR}$ and $\mathbf{H}_{GR}$ are decreases to reach small values at the end of present stage (See Fig. 4).

5 Conclusions

In this paper, we have studied a PSCM in the existence of viscous fluid in Lyra manifold and general relativity. For the solution introduced on Lyra geometry, to obtain an accelerating universe we need $q = -1 + \frac{3}{1+2L} < 0$ that is $L > 1$. In this case, p, ρ and the coefficients of the bulk viscosity ξ begin with infinite value at $T = 0$, decreases toward constant values as $T = T_\circ$ and reach zero value at the end of the present stage this is in agreement with Big Bang theory. The volume element V is increasing with time. The expansion θ decreases with time. Unfortunately, we obtain the entropy as a function decreasing with time t which is not accepted (second law of thermodynamics). Hence, a plane-symmetric cosmological model in the existence of viscous fluid as an accelerating is not suitable to explain the thermodynamic functions of the universe. For the period of evolution of the universe with $q > 0$ we need $L < 1$ and to obtain the volume element as an increasing function of t we need $L > -\frac{1}{2}$ so the suitable range of L is $(-\frac{1}{2}, 1)$. In this case, p , ρ and the coefficients of the bulk viscosity ξ begin with infinite value at $T = 0$, decreases to reach a constant values as $T = T_0$ and reach zero value at the end of stage this is in agreement with Big Bang theory. Interestingly, for $L = 2.83333$ we obtain $q = -0.55$ (the deceleration parameter today). The entropy $\mathbf{S}$ is an increasing function of t. Solutions obtained in general relativity represent an accelerating universe that begins with an infinite rate of pressure, density, and the coefficients of the bulk viscosity. The entropy is obtained as an increasing function of the time t. It is noted that the Lyra term has a vital role in the changes of the behaviors of the thermodynamic functions in Lyra manifold about the relativity theory. Comparison the results obtained in the two theories are not possible because each theory explains a different stage of evolution.

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