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Characterizations of Continuous Uniform Distribution by Some Functional Relations of Characteristic Functions

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Abstract: Characteristic function is an important property of statistical distribution. The characteristic function uniquely determines a probability distribution. In this article, we have characterized the continuous uniform distribution by the characteristic function and observed that these characterizations may serve as a basis for possible applications in parameter estimation, goodness-of-fit tests, efficiency of a particular hypothesis test, and other applied problems of real life.

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1 Introduction

Many authors and researchers have investigated the characterizations of continuous probability distributions. Before we fit a particular probability distribution to the real-world data, it becomes necessary to justify whether the given probability distribution satisfies the underlying requirements by its characterization. For details on characterizations of continuous probability distributions, the interested readers are referred to Ahsanullah [1], Ahsanullah et al. [2, 3, 4], Galambos and Kotz [5], Kotz and Shanbhag [6], and Nagaraja [7], among others. The characterization of the uniform distribution is also one of the most important research areas of research of both pure and applied problems. In recent years, many researchers have studied the characterizations of the uniform distribution. See, for example, Ahsanullah [8, 9], Hossain and Ahsanullah [10], Arslan et al. [11], Hamedani and Volkmer [12], Arnold et al. [13], Arslan [14], Huang et al. [15], Nadarajah et al. [16], and Lee [17], among others. In this paper, we have considered the characterization of the uniform distribution, $X \sim U(a, b)$, defined for a positive continuous random variable X, by the characteristic function. Since the characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions, these characterizations may serve as a basis for parameter estimation, see Glänzel [18, 19] and Glänzel et al. [20]. According to Glänzel [19], the characterizations may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. These conditions are used by various authors to test goodness of fit, efficiency of a particular test of hypothesis and the power of a particular estimating, etc. As pointed out by Volkova et al. [21], "At present, tests based on characterizations and efficiency of such tests are not well studied; this field contains relatively few publications". For details, the interested readers are referred to Volkova et al. [21] and the references therein. The general problem of testing uniformity is stated as follows: Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables (i.i.d.r.v.'s) having a continuous distribution function (d.f.) F. Then, the test of the uniformity hypothesis is given by $H_0: F(x) = x, x \in [0,1]$. For example, Volkova et al. [21] used a well-known characterization result of Ahsanullah [6] to test the uniformity of a distribution. Similarly, Volkova and Nikitin [22] used a well-known characterization result of Ahsanullah [23] to test exponentiality of a distribution. As such, motivated by the importance of and possible applications of the characterizations of probability distributions in parameter estimation and goodness-of-fit tests, and other applied problems of real life, in this paper we have established some new characterization results of the uniform distribution, $X \sim U(a, b)$, by the characteristic function.

The organization of this paper is as follows: In Section 2, the characteristic function of a continuous probability

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distribution is defined, and some of its basic essential properties for the purpose of our proposed characterizations are presented. Based on these properties, we have established some new characterization results of the uniform distribution by the characteristic function in Section 3. The concluding remarks are manifested in Section 4.

2. Characteristic Function and Some Properties of the Uniform Distribution

In this section, we first briefly discuss the characteristic function of a continuous probability distribution and its properties. For details on the characteristic function, the interested readers are referred to Polya [24], Lukacs and Laha [25], Ramchandran [26], Lukacs [27, 28], and Askey [29], among others.

Definition 2.1. Characteristic Function:
The characteristic function (cf), $\Phi_X(t)$ of a continuous probability density function (pdf), $(x), -\infty < x < \infty$, is defined as

$$\Phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_{-\infty}^{\infty} e^{itx} f(x) dx, \quad i = \sqrt{-1}.$$

Properties:
(i) The cf, $\Phi_X(t)$ of a pdf, $f(x), -\infty < x < \infty$, is a real valued function that uniquely determines the probability distribution.
(ii) If the characteristic function of a random variable is integrable, then its cumulative distribution function (cdf), $F(x)$ is absolutely continuous and its pdf, $f(x)$, is given by
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \Phi_X(t)\, dt.$$
(iii) A convex combination of a finite or countable number of characteristic functions is also a characteristic function.

(iv) The product of a finite number of characteristic functions is also a characteristic function.

(v) Polya [24] showed that if, for a non-negative continuous random variable X, the cf, $\Phi_X(t)$ is such that

(a) $\Phi_X(0) = 1$,

(b) $\Phi_X(t)$ is convex for $t > 0$, and

(c) $\Phi_X(\infty) = 0$,

then $\Phi_X(t)$ is the cf of an absolutely continuous cdf, $F(x)$ symmetric about 0.

(vi) The cf can also be used to find the moments. If a random variable, X, has moments up to the $n\ (> 0)th$ order, then the cf, $\Phi_X(t)$, is n times continuously differentiable on $-\infty < x < \infty$, and

$$E(X^n) = i^{-n} \left. \frac{d^n \Phi_X(t)}{dt^n} \right|_{t=0}.$$
(vii) Using the definition of the moment generating function (MGF), the cf, $\Phi_X(t)$, is also given by
$$\Phi_X(t) = M_X(it).$$

3. Characterizations of the Uniform Distribution by Characteristic Functions

In this section, we present the proposed characterizations of the continuous uniform distribution, $X \sim U(a, b)$, by its characteristic function. For the sake of completeness, the definition and some basic essential distributional properties of the uniform distribution are also briefly provided in this section. For details on the uniform distribution, see, for example, Patel et al. [30], Johnson et al. [31], Balakrishnan and Nevzorov [32], and Forbes et al. [33], among others.

3.1. Uniform Distribution:

Definition: A continuous random variable X is said to have a uniform distribution over the interval $[a, b]$, denoted as $X \sim U(a, b)$, if its pdf, $f_X(x)$, and cumulative distribution function, $F_X(x)$, are respectively given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, a \le x \le b \\ 0, x < a \; or \; x > b \end{cases},$$

and

$$F_X(x) = \begin{cases} 0, < a \\ \frac{x-a}{b-a}, a \le x \le b \\ 1, x > b \end{cases},$$

where a is the location parameter, $b - a$ s the scale parameter, and $-\infty < a < b < \infty$.
When $a = 0$ and $b = 1$, it is called the standard uniform distribution, with its pdf given by

$$f_X(x) = 1, for\ 0 \le x \le 1.$$

Properties:
Moments: Some of these are given as follows: \

(i) nth Moment: $E(X^n) = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$

(ii) Expected Value (or 1st Moment): $E(X) = \frac{a+b}{2}$

(i) Variance (or 2nd Central Moment): $V(X) = \frac{(b-a)^2}{12}$

Moment Generating Function: $M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)}, & t \ne 0 \\ 1, & t = 0 \end{cases}.$

Characteristic Function: $\Phi_X(t) = E(e^{itX}) = M_X(it) = \begin{cases} \frac{e^{itb} - e^{ita}}{it(b-a)}, & t \ne 0 \\ 1, & t = 0 \end{cases}.$

3.2. Characterizations

In what follows, we establish the proposed characterization results of the continuous uniform distribution by the characteristic function. Klebanov [34] proved that if $\varphi(t)$ is the characteristic function of an absolutely continues random variable symmetric about 0, then the relation

$$(1/2)(\varphi(t) + \varphi(t/2))(\varphi(t/4))^2 = (\varphi(t/2))^3$$

characterizes a uniform distribution. We will present here a slight difference of the Klebanov's theorem [34] and an alternative proof.

3.2.1. Main Results

3.2.1.1. Results Based on the Characteristic Function

Theorem 3.1. Suppose a random variable X is absolutely continuous with cdf, $F(x)$, where $F(-1) = 0$ and $F(1) = 1$. Suppose X has a uniform distribution symmetric about 0 and $\varphi(t)$ is the cf of X, that is,

$$\Phi_X(t) = E(e^{itX}) = \int_{-1}^{1} \frac{e^{itx}}{2} dx = \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}, \quad i = \sqrt{-1}.$$

Then

$$(1/2)(\varphi(t) + \varphi(t/2))(\varphi(t/4))^2 = (\varphi(t/2))^3$$

if and only if

$$F(x) = ((x+1)/2), -1 \le x \le 1,$$

which is a uniform distribution symmetric about 0, with its cf as given above.

Proof. Suppose that the random variable X has a uniform distribution symmetric about 0. Then its cf is given by

$$\varphi(t) = (sin\, t)/t. \tag{3.1}$$

Replacing t by $\frac{t}{2}$ in the equation (3.1), we have

$$\varphi(\tfrac{t}{2}) = 2(sin\, \tfrac{t}{2})/t. \tag{3.2}$$

Let $m(t/2) = (\varphi(t)/\varphi(t/2)$. From the equations (3.1) and (3.2), it is easily seen that

$$m(t/2) = ((\varphi(t))/(\varphi(t/2))) = cos\,(t/2). \tag{3.3}$$

Replacing t by $\frac{t}{2}$ in the equation (3.3), we have

$$m(t/4) = ((\varphi(t/2))/(\varphi(t/4))) = cos\,(t/4). \tag{3.4}$$

From the equations (3.3) and (3.4), and using the trigonometric identity, $cos\,(2\theta) = 2\,cos^2\theta - 1$, it is easily seen that

$$\frac{1}{2}(m(t/2) + 1) = (m(t/4))^2. \tag{3.5}$$

Thus, in view of the equations (3.3) and (3.4), the equation (3.5) easily reduces to

$$(1/2)(\varphi(t) + \varphi(t/2))(\varphi(t/4))^2 = (\varphi(t/2))^3\,.$$

This proves the "if condition".

Now, we prove the "only if condition". Suppose that the relation

$$(1/2)(\varphi(t) + \varphi(t/2))(\varphi(t/4))^2 = (\varphi(t/2))^3\,. \tag{3.6}$$

holds. Dividing both sides of the equation (3.6) by $(\varphi(t/4))^2$, and substituting $m(t/2) = (\varphi(t)/\varphi(t/2)$, we obtain

$$\frac{1}{2}(m(t/2) + 1) = (m(t/4))^2. \tag{3.7}$$

From the equations (3.3) and (3.4), and using the trigonometric identity, $cos\,(2\theta) = 2\,cos^2\theta - 1$, it is easily seen that the solution of the equation (3.7) is given by

$$m(t/2) = cos\,(t/2). \tag{3.8}$$

Now, using (3.8), we have

$$\varphi(t) = m(t/2)\varphi(t/2) = cos\,(t/2)\,\varphi(t/2) = cos\,(t/2)\,cos\,(t/4)\,\varphi(t/4) = \ldots\,,$$

or, proceeding in the same manner, and using the formula 38.8, $\frac{sin\,z}{z} = \Pi_{j=1}^{\infty} \cos\left(\frac{z}{2^j}\right)$, page 207, Spiegel et al. [35], and $\varphi(0) = 1$, we have

$$\varphi(t) = \Pi_{j=1}^{\infty} \cos\left(\frac{t}{2^j}\right) \varphi(0) = \frac{sint}{t},$$

which is the required cf of the uniform distribution on $[-1,1]$. This completes the proof of the Theorem 3.1.

3.2.1.2. Results Based on Functional Relations of the Characteristic Functions

We will present here two theorems using functional relations of the characteristic functions.

Theorem 3.2. Suppose a random variable X has an absolutely continuous cdf, $F(x)$, with $F(-1) = 0$ and $F(1) = 1$. We assume the random variable X is symmetric about 0. Let the corresponding cf be $\varphi(t)$.Then

$$t\varphi''(t) + 2\varphi'(t) + t\varphi(t) = 0,$$

if and only if

$$F(x) = ((x+1)/2), -1 \leq x \leq 1.$$

which is a uniform distribution symmetric about 0, with its cf as defined in Theorem 3.1.

Proof. For the uniform distribution on $[-1, 1]$, we have its cf given by

$$\varphi(t) = (sin\, t)/t. \qquad (3.9)$$

It is easy to see, from the equation (3.9), that

$$t\varphi''(t) + 2\varphi'(t) + t\varphi(t) = 0.$$

This proves the “if condition”.

Now, we shall prove the “only if condition”. Suppose that

$$t\varphi''(t) + 2\varphi'(t) + t\varphi(t) = 0. \qquad (3.10)$$

We can write the equation (3.10) as

$$t\varphi''(t) + \varphi'(t) + \varphi'(t) = -t\varphi(t).$$

That is,

$$((d^2)/(dt^2))(t\varphi(t)) = -t\varphi(t). \qquad (3.11)$$

Let

$$\Psi(t) = t\varphi(t). \qquad (3.12)$$

Then, substituting (3.12) in the equation (3.11), we have

$$\Psi''(t) + \Psi(t) = 0\ . \qquad (3.13)$$

The solution of the (3.13) is easily given by

$$\Psi(t) = ae^{it} + be^{-it}\ , \qquad (3.14)$$

where a and b are arbitrary constants. Now, from the equation (3.14), using the definition of the cf, we have $\Psi(0) = 0$ and $\Psi'(0) = 1$. Using these conditions in the equation (3.14) and solving for a and b, we must have

$$\Psi(t) = \frac{e^{it} - e^{-it}}{2\,i} = sin\, t. \qquad (3.15)$$

Thus, from the equations (3.12) and (3.15), it follows that

$$\varphi(t) = (sin\, t)/t,$$

which is the cf of the uniform distribution function on $[-1, 1]$. This completes the proof of the Theorem 3.2.

Theorem 3.3. Suppose a random variable X has an absolutely continuous pdf with cdf, $F(x)$, where $F(-1) = 0$ and $F(1) = 1$. We assume the random variable X is symmetric about 0. Let the corresponding cf be $\varphi(t)$.Then

$$(\varphi(2t))^2 = (\varphi(t))^2 - t^2(\varphi(t))^4 ,$$

if and only if

$$F(x) = ((x+1)/2), -1 \leq x \leq 1.$$

Proof. It is easy to prove the necessary condition. We will prove here the sufficient condition. Suppose that

$$(\varphi(2t))^2 = (\varphi(t))^2 - t^2(\varphi(t))^4. \tag{3.16}$$

Let

$$t\varphi(t) = \Psi(t). \tag{3.17}$$

Then, from the equations (3.17) and (3.18), after simplifications, we easily have

$$(\Psi(2t))^2 = 4(\Psi(t))^2 - 4(\Psi(t))^4. \tag{3.18}$$

Following the similar arguments as in the Theorem 3.2, the solution of the equation (3.18) under the condition $\Psi(0) = 0$ and $\Psi'(0) = 1$ is easily given by

$$\Psi(t) = ae^{it} + be^{-it}. \tag{3.19}$$

Since $\Psi(0) = 0$ and $\Psi'(0) = 1$, Using these conditions in the equation (3.19) and solving for a and b, we must have

$$\Psi(t) = \frac{e^{it} - e^{-it}}{2i} = sin\, t. \tag{3.20}$$

Thus, from the equations (3.17) and (3.20), it follows that

$$\varphi(t) = (sin\, t)/t,$$

which is the cf of the uniform distribution function on $[-1, 1]$. This completes the proof of the Theorem 3.3.

4. Concluding Remarks

As stated in the introduction, the characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions, and hence these characterizations may serve as a basis for parameter estimation. As such, motivated by the importance of and possible applications of the characterizations of probability distributions in parameter estimation and goodness-of-fit tests, and other applied problems of real life, in this paper, we have established some new characterization results of the uniform distribution, $X{\sim}U(a, b)$, by the characteristic function based on some of its basic essential distributional properties discussed in the paper. We sincerely believe that this paper will be further helpful to test the uniformity of a distribution analogously to Volkova and Nikitin (2013) and Volkova et al. (2020). Moreover, we hope that the paper will be quite useful for other researchers and practitioners in the fields of probability, statistics, and other applied sciences.

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