Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110119

Auxiliary Principle Technique for Variational Inequalities

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Received: 2 Aug. 2016, Revised: 5 Sep. 2016, Accepted: 8 Sep. 2016
Published online: 1 Jan. 2017

Abstract: In this paper, some implicit type methods for solving general variational inequalities are suggested and investigated using the auxiliary principle technique. It is shown that the convergence of the implicit method requires only the g-pseudomonotonicity, which is a weaker condition than g-monotonicity. Our results can be viewed as important refinement and improvement of the known results. The technique and ideas of this paper may be extended for other classes of variational inequalities and related optimization problems. Some special cases are also discussed.

Keywords: Variational inequalities, nonconvex functions, fixed-point problem, Projection operator, convergence.
2010 AMS Subject Classification: 49J40, 90C33

1 Introduction

Variational inequalities, which were introduced by Stampacchia [14] can be considered as a natural and important extension of the variational principles. Several unrelated problems, which arise in various fields of pure and applied sciences can be studied in the general and unified framework of variational inequalities, see [1–14] and the references therein. It is well known that the optimality conditions of the differentiable and nondifferentiable convex functions can be characterized by Variational inequalities. In the recent years, the concept of convexity has been generalized in several directions, see, for example, [2] and the references therein. A significant generalization of the convex set is the introduction of the g-convex set [2] and g-convex function [11]. We would like to emphasize that g-convex sets and g-convex functions may not be convex sets and convex functions. It has been shown [11] that the minimum of a differentiable g-convex function on the g-convex set can be characterized by a class of variational inequalities. This fact has been used to introduce and consider a new class of variational inequalities, which is called the general nonlinear variational. Some iterative methods have been suggested for general variational inequalities uisng the fixed point approach. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. These are very strict conditions to verify. This fact motivated to modify the projection method or to develop other methods. The extragradient-type methods [5] overcome this difficulty by performing an additional forward step and a projection at each iteration according to the double projection. Their convergence requires only that a solution exists and the monotone operator is Lipschitz continuous. To overcomes these difficulties, several modified projection and extragradient-type methods have been suggested and developed for solving variational inequalities. We would to point out that the projection technique can't be extended and generalized for solving some classes of variational inequalities involving the nonlinear (non)differentiable functions. These facts motivated us to use the auxiliary principle technique, which is mainly due to Glowinski et al. [4]. Noor [9, 10] used this technique to suggest some iterative methods for solving various classes of variational inequalities. This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique, see [9, 10].

In this paper, we use the auxiliary principle technique to suggest an implicit iterative method for solving the general variational inequalities, see Algorithm 3.3. Using the predictor-corrector technique, it is shown that this

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implicit method is equivalent to the extragradient method, see Algorithm 3.4 and Algorithm 3.5. In particular, for $g = I$, the identity operator, we obtain the extragradient method of Korpelevich [5] for solving the variational inequalities. We have proved that the convergence of the implicit iterative method only requires the g-pseudomonotonicity, which is a weaker condition than g-monotonicity. Consequently, we have improved the convergence criteria of the extragradient method of Korpelevich [5]. Some special cases are also considered. The ideas and techniques of this paper may be starting point for a wide range of novel and innovative applications of the general variational inequalities in various fields.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g : H \to H$, we consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{1}$$

The inequality of the type (1) is called the general variational inequality involving two operators, which was introduced and studied by Noor [11]. For the numerical analysis, applications and other aspects of these variational inequalities, see [7, 8, 12, 13] and the references therein. It has been shown [11] that the minimum of a differentiable nonconvex function on a nonconvex set K in H can be characterized by the general variational inequality (1).

For $g = I$, the identity operator, the general variational inequality (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \forall v \in K, \tag{2}$$

which is known as the classical variational inequality and was introduced by Stampacchia [14]. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see [1–12, 12, 14] and the references therein.

Lemma 2.1 [3, 4]. Let K be a closed convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if, $\quad u = P_K z, \quad$ where P_K is the projection of H onto the closed convex set K in H. It is well known that the projection operator P_K is a nonexpansive operator.

We now define a new concept of g-pseudomonotonicity.

Definition 2.3. An operator $T : H \to H$ said to be *g-pseudomonotone* with respect to an arbitrary operator g, if and only if,

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \text{implies} \quad \langle Tv, v - g(u)\rangle \geq 0, \quad \forall u, v \in H.$$

If $g = I$, the identity operator, then Definition 2.3 reduces to the usual definition of pseudmonotonicity.

3 Main Results

In this Section, we use the auxiliary principle technique of Glowinski et al [4], as developed by Noor [9, 10] to suggest and investigate some implicit type methods for solving the general variational inequality (1).

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding $w \in H : g(w) \in K$ such that

$$\langle \rho Tu + w - g(u), g(v) - w\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3}$$

which is called the auxiliary general variational inequality. Clearly, if $w = g(u)$, then $w \in H; g(w) \in K$ is the solution of (1). This observation enables us to suggest the following iterative method for solving the general variational inequality (1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho Tu_n + u_{n+1} - g(u_n), g(v) - u_{n+1}\rangle \geq 0, \quad \forall v \in H; g(v) \in K,$$

which is equivalent to the following iterative method, using Lemma 2.1.

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = P_K[g(u_n) - \rho Tu_n], \quad n = 0, 1, 2...$$

Algorithm 3.2 was suggested and investigated by Noor [11] for solving (1). For the convergence analysis of Algorithm 3.1, see Noor [10, 11], where its has been that the convergence analysis of Algorithm 3.1 requires that the operator T must be strongly monotone and Lipschitz continuous. These are very strict conditions and rule out its applications in many problems. To overcome these drawbacks, we again use the fixed point formulation (3) to suggest an other iterative method using the auxiliary principle technique.

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding $w \in H : g(w) \in K$ such that

$$\langle \rho Tw + w - g(u), g(v) - w\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{4}$$

which is called the auxiliary general variational inequality.
We point out that problem (3) and (4) are quite different. If $g(w) = u$, then clearly $w \in H; g(w) \in K$ is a solution of (1). This enables to suggest the following iterative method for solving (1).

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T u_{n+1} + u_{n+1} - g(u_n), v - u_{n+1} \rangle \geq 0, \quad \forall v \in H; g(v) \in K. \tag{5}$$

Using Lemma 2.1, Algorithm 3.3 is equivalent to the following iterative method.

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = P_K[g(u_n) - \rho T u_{n+1}], \quad n = 0, 1, \ldots$$

Algorithm 3.3 is an implicit method. To implement Algorithm 3.1, one usually uses the predictor-corrector technique. We use Algorithm 3.1 as predictor and Algorithm 3.3 as corrector. Consequently, Algorithm 3.3 is equivalent to the following iterative method.

Algorithm 3.5. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T u_n + y_n - g(u_n), g(v) - y_n \rangle \geq 0, \quad \forall v \in H : g(v) \in K$$
$$\langle \rho T y_n + u_{n+1} - g(u_n), g(v) - u_{n+1} \rangle \geq 0, \quad \forall v \in H; g(v) \in K.$$

Using Lemma 2.1, Algorithm 3.5 is equivalent to the following predictor-corrector method for solving (1).

Algorithm 3.6. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = P_K[g(u_n) - \rho T u_n]$$
$$u_{n+1} = P_K[g(u_n) - \rho T y_n],$$
$$n = 0, 1, \ldots.$$

Algorithm 3.3 is the the extragradient method for solving general variational inequality (1) in the sense of Korpelevich [5].

We remark that if $g = I$, the identity operator, then Algorithm 3.6 reduces to:

Algorithm 3.7. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[u_n - \rho T y_n], \quad n = 0, 1, \ldots.$$

Algorithm 3.7 is known as an extragradient method for solving variational inequalities (2)and is mainly due to Korpelevich [5].

We now again use the auxiliary problem (4) to suggest the following predictor-corrector method for solving the problem (1).

Algorithm 3.8. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T u_n + y_n - g(u_n), g(v) - y_n \rangle \geq 0, \quad \forall v \in H : g(v) \in K$$
$$\langle \rho T y_n + u_{n+1} - g(y_n), g(v) - u_{n+1} \rangle \geq 0, \quad \forall v \in H; g(v) \in K.$$

Using Lemma 2.1, Algorithm 3.8 is equivalent to the following predictor-corrector method for solving (1).

Algorithm 3.9. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = P_K[g(u_n) - \rho T u_n]$$
$$u_{n+1} = P_K[g(y_n) - \rho T y_n], \quad n = 0, 1, \ldots.$$

Algorithm 3.9 is the the extragradient method for solving general variational inequality (1) in the sense of Noor [10].

We remark that if $g = I$, the identity operator, then Algorithm 3.6 reduces to:
Algorithm 3.10. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[y_n - \rho T y_n], \quad n = 0, 1, \ldots.$$

Algorithm 3.10 is known as modified extragradient method for solving variational inequalities (2), which is mainly due to Noor [9, 10].

It is important to note that extragradient method of Korpelevich [4] and modified double projection method of Noor[9,10] are quite different from each other and their convergent analysis need different techniques. The auxiliary principle technique can be used to construct several iterative methods for solving variational inequalities.

We now consider the convergence analysis of Algorithm 3.3 under some suitable mild conditions.

Theorem 3.1. Let $u \in H : g(u) \in K$ be a solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If the operator T is g-pseudomonotone, then

$$\|g(u) - u_{n+1}\|^2 \leq \|g(u) - g(u_n)\|^2 - \|u_{n+1} - g(u_n)\|^2, \tag{6}$$

Proof. Let $u \in H : g(u) \in K$ be solution of (1). Then, using the g-pseudomonotonicity of T, we have

$$\langle Tv, v - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{7}$$

Take $v = u_{n+1}$ in (7), we have

$$\langle Tu_{n+1}, u_{n+1} - g(u)\rangle \geq 0. \quad (8)$$

Taking $v = u$ in (5), we have

$$\langle \rho Tu_{n+1} + u_{n+1} - g(u_n), g(u) - u_{n+1}\rangle \geq 0. \quad (9)$$

From (8) and (9), we have

$$\langle u_{n+1} - g(u_n), g(u) - u_{n+1}\rangle \geq 0,$$

which implies, using $2\langle u,v\rangle = \|u-v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall u,v \in H$, that

$$\|g(u) - u_{n+1}\|^2 \leq \|g(u) - g(u_n)\|^2 - \|u_{n+1} - g(u_n)\|^2,$$

the required result (6). □

Theorem 3.2. Let $u \in H : g(u) \in K$ be a solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If the operator T is g-pseudomonotone and g-inverse eixts, then $\lim_{n\to\infty} g(u_n) = u$.

Proof. Let $\bar{u} \in\in H : g(\bar{u}) \in K$ be a solution of (1). Then, the sequences $\{\|g(u_n) - \bar{u}\|\}$ is nonincreasing and bounded and

$$\sum_{n=0}^{\infty} \|u_{n+1} - g(u_n)\|^2 \leq \|g(u_0) - gu\|^2,$$

which implies

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0, \quad (10)$$

that is,

$$\lim_{n\to\infty} u_n = u$$

since g^{-1} exits.

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (5) and taking the limits and using (10), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle \geq 0, \quad \forall v \in K.$$

This shows that $\hat{u} \in H : g(\hat{u}) \in K$ solves (1) and

$$\|u_{n+1} - g(\hat{u})\|^2 \leq \|g(u_n) - g(\hat{u})\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$, is the solution of (1), the required result. □.

Acknowledgement.

The authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research facilities.

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