Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040103

α-ψ-Contractive Mapping on S-Metric Space

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Received: 20 Aug. 2014, Revised: 24 Oct. 2014, Accepted: 25 Sep. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, we introduce α-ψ-contractive mapping in S-metric space and we prove the existence of a fixed point for such mapping under some conditions.

Keywords: fixed point theory, S-metric space

1 Introduction

Throughout this paper denote all natural numbers by $\mathbf{N}$ and all real number by $\mathbf{R}$. The work in this paper is inspired by Samet's generalization of Banach's contraction principles in a metric space by introducing α-ψ-contraction in [1]. In this paper study the existence of a fixed point for an α-ψ-contractive self mapping T on an S-metric space. Many recent results in the past few years showing the existence of a fixed point for a contractive self mapping in deferent types of metric spaces, see [2],[4],[5],[6], [7],[8],[9],[10]. In this paper, we give a generalization of the results of [3] in the S-metric space. First, we start by giving a few definitions.

Definition 1. Let X be a nonempty set. An S-metric space on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for all $x,y,z,t \in X$:
(i) $S(x,y,z) \geq 0$,
(ii) $S(x,y,z) = 0$ if and only if $x = y = z$,
(iii) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$
The pair (X,S) is called an S-metric space.

Here some examples of such space which were presented in [3].
1)Let $X = \mathbf{R}^n$ and $||\,.\,||$ a norm on X, then $S(x,y,z) = ||yz - 2x|| + ||x + y||$ is an S-metric space.
2)Let $X = \mathbf{R}^n$ and $||\,.\,||$ a norm on X, then $S(x,y,z) = ||x - z|| + ||y - z||$ is an S-metric space.
3)Let X be a nonempty set, d the ordinary metric space on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric space.

Definition 2.[3] Let (X,S) be an S-metric space.
1)A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
2)A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists a natural number n_0 such that for all $n \geq n_0$, we have $S(x_n,x_n,x) < \varepsilon$ and we donate this by $\lim_{n\to\infty} x_n = x$.
3)A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number n_0 such that for all $n,m \geq n_0$, we have $S(x_n,x_n,x_m) < \varepsilon$.
4)An S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

These next two lemmas are very useful for our purpose.

Lemma 3.[3] In an S-metric space, we have

$$S(x,x,y) = S(y,y,x)$$

for all $x,y \in X$.

Lemma 4.[3] Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n,x_n,y_n) \to S(x,x,y)$.

Definition 5. [1] Denote by Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Also, this next lemma is very useful for our purpose.

Lemma 6.[1] For every function $\psi : [0,+\infty) \to [0,+\infty)$ the following holds:
if ψ is nondecreasing, then for each $t > 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ implies that $\psi(t) < t$.

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Now, we define the α-ψ-contractive self mapping in S-metric space.

Definition 7. Let T be a self mapping on a complete S-metric space (X,S). We say that T is α-ψ-contractive self mapping if there exists a function $\alpha : X\times X\times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y \in X$ we have

$$\alpha(x,x,y)S(Tx,Tx,Ty) \leq \psi(S(x,x,y)).$$

Definition 8. Let (X,S) be a S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is $\alpha-$admissible if $x,y,z \in X$, $\alpha(x,y,z) \geq 1$ implies that $\alpha(Tx,Ty,Tz) \geq 1$.

Example:
Let $X = [0,\infty)$, d the ordinary metric space on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric space. Let $\alpha : X\times X\times X \longrightarrow [0,\infty)$ define T by:

$$Tx = \sqrt{x},$$

and define α by

$$\alpha(x,y,z) = e^{max\{x,y\}-z} \quad if \quad max\{x,y\} \geq z$$

and

$$\alpha(x,y,z) = 0 \quad if \quad max\{x,y\} < z.$$

It is easy to see that T is $\alpha-$admissible.

2 Fixed point of α-ψ-contractive self mapping in S-metric space

In this section we prove the existence of a fixed point for an α-ψ-contractive self mapping.

Theorem 1.1. Let T be an α-ψ-contractive self mapping on a complete S-metric space (X,S), where $\psi \in \Psi$, satisfying the following conditions:
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$;
(iii) T is continuous.
Then, T has a fixed point.

Proof. Consider the sequence $\{x_n\}$ defined by $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_n = Tx_{n-1} = T^nx_0, \cdots$. By assumption we know that $\alpha(x_0,x_0,Tx_0) \geq 1$, hence since T is α-admissible, therefore, $\alpha(x_1,x_1,x_2) \geq 1$. So, using the fact that T is α-admissible and by induction on n we conclude that

$$\alpha(x_n,x_n,x_{n+1}) \geq 1.$$

Now, since for $n \in \mathbf{N}$ we have $\alpha(x_n,x_n,x_{n+1}) \geq 1$ and T be an α-ψ-contractive we deduce,

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &= S(Tx_{n-1},Tx_{n-1},Tx_n) \\ &\leq \alpha(x_{n-1},x_{n-1},x_n)S(Tx_{n-1},Tx_{n-1},Tx_n) \\ &\leq \psi(S(x_{n-1},x_{n-1},x_n)). \end{aligned} \tag{1}$$

Hence, by induction on n we get,

$$S(x_n,x_n,x_{n+1}) \leq \psi^n(S(x_0,x_0,x_1)) \quad for \ all \quad n \in \mathbf{N}.$$

Fix $\varepsilon > 0$, let $n(\varepsilon) \in \mathbf{N}$ such that $\sum_{n\geq n(\varepsilon)} \psi^n(S(x_0,x_0,x_1)) < \frac{\varepsilon}{2}$. Now, let $n,m \in \mathbf{N}$ with $m > n > n(\varepsilon)$, by the triangle inequality property of the S-metric space we deduce,

$$\begin{aligned} S(x_n,x_n,x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i,x_i,x_{i+1}) + S(x_{m-1},x_{m-1},x_m) \qquad (2) \\ &\leq 2\sum_{k=n}^{m-1} \psi^k(S(x_0,x_0,x_1)) + \psi^{m-1}(S(x_0,x_0,x_1)) \\ &\leq 2\sum_{n\geq n(\varepsilon)} \psi^n(S(x_0,x_0,x_1)) < 2\times\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since (X,S) is a complete, there exist $a \in X$ such that $\lim_{x\to+\infty} x_n = a$. Also, since T is continuous we have

$$a = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Ta.$$

Thus, T has a fixed point as desired.

In our next theorem we omit the continuity T hypothesis.

Theorem 1.2. Let T be an α-ψ-contractive self mapping on a complete S-metric space (X,S), and $\psi \in \Psi$, satisfying the following conditions:
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_n,x_{n+1}) \geq 1$ for all $n \in \mathbf{N}$ and x_n converge to x, then $\alpha(x_n,x_n,x) \geq 1$ for all $n \in \mathbf{N}$.
Then, T has a fixed point.

Proof. Using all the notations in the proof of Theorem2, and by that proof, we know that $\{x_n\}$ converges say to $a \in X$. and for all $n \in \mathbf{N}$ we have,

$$\alpha(x_n,x_n,a) \geq 1.$$

So, by using Lemma 1, we deduce that,

$$\begin{aligned} S(Ta,Ta,a) &\leq 2S(Ta,Ta,Tx_n) + S(a,a,x_{n+1}) \\ &\leq 2S(Tx_n,Tx_n,Ta) + S(a,a,x_{n+1}) \\ &\leq 2\alpha(x_n,x_n,a)S(Tx_n,Tx_n,Ta) + S(a,a,x_{n+1}) \\ &\leq 2\psi(S(x_n,x_n,a)) + S(a,a,x_{n+1}). \end{aligned} \tag{3}$$

Since ψ is continuous at 0 and when we take the limit as $n \to +\infty$ we obtain $S(Ta,Ta,a) = 0$. Hence, $Ta = a$. Hence, T has a fixed point as required.

Next, we prove the following corollary.

Corollary 1.3. Let T be a self mapping on a complete S-metric space (X,S), T is α-admissible, and there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$ and there exists $L \in [0,1)$ such that for all $x,y \in X$ we have

$$\alpha(x,x,y)S(Tx,Tx,Ty) \leq LS(x,x,y),$$

then T has a fixed point.

Proof. Consider $\psi(t) = Lt$, it is not difficult to see that $\psi \in \Psi$. Also, by the remark in section 3 of [3], we know that T is continuous. Thus, all the conditions of Theorem 2 are satisfied. Therefore, T has a fixed point.

To have uniqueness, we need have some restrictions on α.

Theorem 1.4. Let T be an α-ψ-contractive self mapping on an S-metric space that satisfies all the hypothesis of Theorem 2, and assume that for every two fixed points x,y of T, there exists $z \in X$ such that $\alpha(x,x,z) \geq 1$ and $\alpha(y,y,z) \geq 1$. Then the fixed point of T is unique.

Proof. Let x,y be two fixed points of T, we know by the hypothesis of the theorem that there exists $z \in X$ such that $\alpha(x,x,z) \geq 1$ and $\alpha(y,y,z) \geq 1$. Since T is α-admissible and by induction on n, we obtain for all n $\alpha(x,x,T^nz) \geq 1$ and $\alpha(y,y,T^nz) \geq 1$. Thus,

$$\begin{aligned} S(x,x,T^nz) &= S(Tx,Tx,T(T^{n-1}z) \\ &\leq \alpha(x,x,T^{n-1}z)S(Tx,Tx,T(T^{n-1}z) \\ &\leq \psi(S(x,x,T^{n-1}z). \end{aligned} \quad (4)$$

So, by induction on n we get,

$$S(x,x,T^nz) \leq \psi^n(S(x,x,z)).$$

Hence, as $n \to +\infty$ we have $T^nz \to x$. Similarly, as $n \to +\infty$ we have $T^nz \to y$. By the uniqueness of the limit we obtain $x = y$ as desired.

Example:
Let $X = [0,1] \cup [2,3]$, and define the S-metric space by $S: X^3 \longrightarrow (-\infty,+\infty)$ by $S(x,y,z) = max\{x,y,z\}$ if $\{x,y,z\} \cap [2,3] \neq \emptyset$ and $S(x,y,z) = |x-z| + |y-z|$ if $\{x,y,z\} \subset [0,1]$. Now define $T: X \longrightarrow X$ and $\alpha: X \times X \times X \longrightarrow X$ by: $Tx = \dfrac{x+1}{2}$ if $0 \leq x \leq 1$, $T2 = 1.5$, and $Tx = \dfrac{x+2}{2}$ if $2 \leq x \leq 3$. Also, define α as follows:

$$\alpha(x,y,z) = e^{max\{x,y\}-z} \quad if \quad max\{x,y\} \geq z$$

and

$$\alpha(x,y,z) = 0 \quad if \quad max\{x,y\} < z.$$

It is easy to see that T is $\alpha-$admissible. Note that, we can always pick our x and y such that $x > y$. Also T is an increasing function. So, for every $x \geq y \in X$ we have:

$$S(Tx,Tx,Ty) \leq \alpha(x,x,y)S(Tx,Tx,Ty) \leq \frac{1}{2}S(x,x,y), \{x,y\} \subset [0,1]$$

and similarly,

$$S(Tx,Tx,Ty) \leq \alpha(x,x,y)S(Tx,Tx,Ty) \leq \frac{1}{2}S(x,x,y), \{x,y\} \cap [2,3] \neq \emptyset.$$

Note that in this case our fixed point is 1, and $L = \dfrac{1}{2}$.

Remark:
In closing, we want to bring to the reader's attention that α does not have to be defined on X^3, it should be enough defining α on X^2.

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