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Matrix Representation for the Beta Type Polynomials

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Abstract: The aim of this paper is to study and investigate some new properties of the beta polynomials. Taking derivative of the generating functions for beta type polynomials, we give two partial differential equations (PDEs). By using these PDEs, we derive derivative formulas of the beta type polynomials. In order to construct a matrix representation for the beta polynomials, we firstly show that the set of beta polynomials is linearly independent. By using linearly independent properties, we prove that any polynomial of degree less than and equal n are written as a linearly combination of the beta polynomials. Therefore, we define matrix representation for the beta polynomials. Moreover, we provide the simulation of the beta polynomials with some their graphs. We also give remarks and examples and comments on the beta polynomials and their matrix representation.

Keywords: Polynomials; Generating functions; Bernstein polynomials; Beta polynomials; Matrix representation; Linearly independent; Partial differential equations; Simulation of the polynomials

1 Introduction

Polynomials are widely used in other areas of mathematics and information science as well as the other sciences: they are important in the study of continued fractions, operator theory, graph theory, discrete group theory, Computer Aided Geometric Design (CAGD), analytic functions, interpolation, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory, combinatorics, stochastic analysis, data compression, etc. It is known that by using generating functions of the special polynomials, many properties of the applied mathematics which focuses on the formulation of mathematical models, differential equations, representations theory, numerical analysis etc., are studied (*cf*. [1]-[7]).
Some special polynomials are also related to Applied Probability Theory. For example, the Bernstein polynomials are associated with the binomial distribution and the Poisson distribution in Probability Theory (*cf*. [1]-[7]).
The beta polynomials are also related to distribution function in Probability Theory. It is known that the Bernstein polynomials and the beta polynomials are used many branches of Mathematics and Applied Mathematics (*cf*. [7]). The Bernstein polynomials play a central role in the theory of Bezier curves and surfaces and also in CAGD. In [7], we gave relation between the beta polynomials and the Bernstein polynomials. We also gave relation between these polynomials and distribution function. In order to investigate some fundamental properties of the beta polynomials, we constructed a novel collection of generating functions which are used to derive identities and relations for the beta polynomials. In [6]-[7], we studied on the (q-) beta type polynomials.

In [1], Bhandari and Vignat have also studied on the beta polynomials. By using these polynomials, they gave a probabilistic representation of the multidimensional p-adic Volkenborn integral.
Similarly, matrices are widely used not only in applied mathematics, but also in the information science. For example, in the optimization of a dynamic programming equation, in all areas of positive and social sciences, life sciences, etc.
Throughout this paper, we use the following standard notations:

$\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,3,\ldots\} = \mathbb{N} \cup \{0\}$.
Here, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.
The beta polynomials are defined as follows:

Definition 1.1. Let $x \in [-1,0]$. Let n and k be nonnegative integers. Then we define

$$\mathfrak{B}_{k,n}(x) = x^k (x+1)^{n-k}. \tag{1}$$

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where $k = 0, 1, 2, \ldots, n$. We usually set

$$\mathfrak{B}_{k,n}(x) = 0$$

if $k < 0$ or $k > n$ (*cf.* [1], [6]).

In [7], we constructed generating functions for the functions, $\mathfrak{M}_{k,n}(x)$ as follows:

$$\mathfrak{h}_k(t,x) = \left(\frac{x}{1+x}\right)^k e^{t(1+x)} = \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^n}{n!}, \quad (2)$$

where $k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and for all $t \in \mathbb{C}$. There is one generating function for each value of k.

By using (2), we have

$$\mathfrak{M}_{k,n}(x) = x^k(1+x)^{n-k}, \quad (3)$$

where $n, k \in \mathbb{N}_0$. From (2), if $n \geq k$, we get the beta polynomials:

$$\mathfrak{M}_{k,n}(x) = \mathfrak{B}_{k,n}(x)$$

and if $n < k$, we define a rational function, $\mathfrak{b}_{k,n}(x)$ as follows:

$$\mathfrak{M}_{k,n}(x) = \mathfrak{b}_{k,n}(x),$$

where

$$\mathfrak{b}_{k,n}(x) = \frac{x^k}{(1+x)^{k-n}}$$

and $n \in \{0, 1, 2, \cdots, k-1\}$.

Hence, we set

$$\mathfrak{M}_{k,n}(x) = \mathfrak{B}_{k,n}(x) + \mathfrak{b}_{k,n}(x). \quad (4)$$

The beta polynomials are defined by means of the following generating functions (*cf.* [7]):

$$\mathfrak{F}_k(t,x) = \sum_{n=k}^{\infty} \mathfrak{B}_{k,n}(x)\frac{t^n}{n!},$$

where for all $t \in \mathbb{C}$ and

$$\mathfrak{F}_k(t,x) = \mathfrak{h}_k(t,x) - \sum_{n=0}^{k-1} \mathfrak{b}_{k,n}(x)\frac{t^n}{n!}.$$

Hence

$$\mathfrak{B}_{k,1}(x) = \mathfrak{B}_{k,2}(x) = \cdots = \mathfrak{B}_{k,k-1}(x) = 0$$

(cf. [6]-[7]).

We summarize our paper as follows:

In Section 2, we give some PDEs for the generating functions for the beta type polynomials. By using these equations, we find two derivative formulas of the beta-type polynomials.

In Section 3, we prove that a set of beta polynomials is linearly independent. We also give a matrix representation for these polynomials. Finally, we give two examples for this matrix representation of the beta type polynomials.

In Section 4, the simulation of the Beta type polynomials are demonstrated. We also give some graphics of the beta type polynomials.

In Section 5, we give some remarks, comments and applications on the matrix representation and graphics of the beta type polynomials.

2 PDEs for the generating functions

We know that applied mathematics consisted principally of applied analysis, most notably differential equations. Hence, in this section, we give PDEs for the generating functions. By using these equations, we derive two derivative formulas of the beta-type polynomials. These formulas are used to compute derivatives of the beta-type polynomials.

Taking derivative of (2), with respect to t, we obtain the following PDE for the generating functions:

$$\frac{\partial^2 \mathfrak{h}_k(x,t)}{\partial t^2} = (x+1)^2 \mathfrak{h}_k(x,t) \quad (5)$$

or

$$\frac{\partial^3 \mathfrak{h}_k(x,t)}{\partial t^2 \partial x} = 2(1+x)\mathfrak{h}_k(x,t) + k\mathfrak{h}_{k-1}(x,t) + (1+x)^2 t\mathfrak{h}_k(x,t). \quad (6)$$

Theorem 1.*Let $n \geq 1$ and $k \geq 1$. Then we have*

$$\frac{d}{dx}\mathfrak{M}_{k,n+2}(x) = 2(1+x)\mathfrak{M}_{k,n}(x) + n(1+x)^2 \mathfrak{M}_{k,n-1}(x) + k\mathfrak{M}_{k-1,n}(x).$$

*Proof.*By combining (2) with (5), we obtain

$$\sum_{n=2}^{\infty} \frac{d}{dx}\mathfrak{M}_{k,n}(x)\frac{t^{n-2}}{(n-2)!} = 2(1+x)\sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^n}{n!} + (1+x)^2 \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^{n+1}}{n!} + k\sum_{n=0}^{\infty} \mathfrak{M}_{k-1,n}(x)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{d}{dx}\mathfrak{M}_{k,n+2}(x)\frac{t^n}{n!} = 2(1+x)\sum_{n=0}^{\infty} \mathfrak{M}_{k+1,n}(x)\frac{t^n}{n!} + (1+x)^2 \sum_{n=0}^{\infty} n\mathfrak{M}_{k,n-1}(x)\frac{t^n}{n!} + k\sum_{n=0}^{\infty} \mathfrak{M}_{k-1,n}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result.

By using (5), we obtain

$$\frac{\partial^2 \mathfrak{h}_k(x,t)}{\partial t^2} = x^2 \mathfrak{h}_{k-2}(x,t).$$

By using the above PDE, we get

$$\frac{\partial^3 \mathfrak{h}_k(x,t)}{\partial t^2 \partial x} = 2x\mathfrak{h}_{k-2}(x,t) + (k-2)\mathfrak{h}_{k-1}(x,t) + x^2 t\mathfrak{h}_{k-2}(x,t).$$

By combining (2) with the above PDE, we easily arrive at the following theorem:

Theorem 2.*Let $n \geq 1$ and $k \geq 2$. Then we have*

$$\frac{d}{dx}\mathfrak{M}_{k,n+2}(x) = 2x\mathfrak{M}_{k-2,n}(x) + (k-2)\mathfrak{M}_{k-1,n}(x) + nx^2 \mathfrak{M}_{k-2,n-1}(x).$$

3 A Matrix representation for the polynomials $\mathfrak{B}_{k,n}(x)$

In this section, we show that the beta polynomials of order n form a basis for the space of polynomials of degree $\leq n$. Thus we see that the power basis spans the space of polynomials and any element of power basis can be represented as a linear combination of the beta polynomials.

That is if there exist constants α_1, α_2,...,α_n such that

$$\sum_{k=0}^{n} \alpha_k \mathfrak{B}_{k,n}(x) = 0 \tag{7}$$

holds for all x then all α_1, α_2,...,α_n is must be zero. We assume that (7) is true. Then we write

$$\begin{aligned} 0 = {} & \alpha_0 (1+x)^n + \alpha_1 x (1+x)^{n-1} + \alpha_2 x^2 (1+x)^{n-2} \\ & + \ldots + \alpha_{n-2} x^{n-2} (1+x)^2 + \alpha_{n-1} x^{n-1} (1+x) + \alpha_n x^n \end{aligned}$$

From the above we get

$$\begin{aligned} 0 = {} & \alpha_0 \sum_{j=0}^{n} \binom{n}{j} x^j + \alpha_1 \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} + \alpha_2 \sum_{j=0}^{n-2} \binom{n-2}{j} x^{j+2} \\ & + \ldots + \alpha_{n-2} \sum_{j=0}^{2} \binom{2}{j} x^{j+n-2} + \alpha_{n-1} \sum_{j=0}^{1} \binom{1}{j} x^{j+n-1} + \alpha_n x^n \end{aligned}$$

Hence

$$\begin{aligned} 0 = {} & \left(\alpha_0 \binom{n}{n} + \alpha_1 \binom{n-1}{n-1} + \ldots \alpha_1 \binom{0}{0} \right) x^n \\ & + \left(\alpha_0 \binom{n}{n-1} + \alpha_1 \binom{n-1}{n-2} + \ldots \alpha_{n-1} \binom{1}{0} \right) x^{n-1} \\ & + \left(\alpha_0 \binom{n}{n-2} + \alpha_1 \binom{n-1}{n-3} + \ldots \alpha_{n-2} \binom{2}{0} \right) x^{n-2} \\ & + \ldots + \alpha_0 \binom{n}{1} + \alpha_1 \binom{n-1}{0} + \alpha_0 \binom{n}{n} \end{aligned}$$

Thus we get the following equations

$$\begin{aligned} \binom{n}{n} \alpha_0 &= 0 \\ \binom{n}{1} \alpha_0 + \binom{n-1}{0} \alpha_1 &= 0 \\ \binom{n}{2} \alpha_0 + \binom{n-1}{1} \alpha_1 + \binom{n-2}{0} \alpha_2 &= 0 \\ & \ldots \\ \binom{n}{n} \alpha_0 + \binom{n-1}{n-1} \alpha_1 + \ldots + \binom{n}{n} \alpha_n &= 0. \end{aligned}$$

From the above equation we easily see that

$$\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0.$$

Consequently, $\{\mathfrak{B}_{0,n}(x), \mathfrak{B}_{1,n}(x), \ldots, \mathfrak{B}_{n,n}(x)\}$ is a linearly independent set. Therefore, any polynomial of degree less than and equal n can be written as a linearly combination of the beta polynomials. That is,

$$P_n(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} b_k \mathfrak{B}_{k,n}(x).$$

Here, $P_n(x)$ is called a polynomial in the polynomials $\mathfrak{B}_{k,n}(x)$ form. This form gives us many advantages if one wishes to analyze polynomials over a finite interval. We now ready to write the following matrix form:

$$P_n(x) = \left[\mathfrak{B}_{n,n}(x) \ \mathfrak{B}_{n-1,n}(x) \ \ldots \ \mathfrak{B}_{0,n}(x) \right] \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Applying the method of mathematical induction in the above equation, we get the following matrix representation of the polynomials $P_n(x)$ by the following theorem:

Theorem 3.

$$P_n(x) = XA^t B \tag{8}$$

where A^t denotes transpose of the matrix A and

$$X_{1\times n} = \left[1 \ x \ \ldots \ x^n \right],$$

$$B_{n\times 1} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

and

$$A_{n\times n} = [a_{ij}],$$

where A=

$$\begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \binom{1}{1} & \binom{1}{0} & 0 & 0 & \cdots & 0 & 0 \\ \binom{2}{2} & \binom{2}{1} & \binom{2}{0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n-1}{n-1} & \binom{n-1}{n-2} & \binom{n-1}{n-3} & \binom{n-1}{n-4} & \cdots & \binom{n-1}{0} & 0 \\ \binom{n}{n} & \binom{n}{n-1} & \binom{n}{n-2} & \binom{n}{n-3} & \cdots & \binom{n}{1} & \binom{n}{0} \end{bmatrix}.$$

and the a_{ij} elements are related to the coefficients of the beta polynomials $\mathfrak{B}_{k,n}(x)$.

We also note that the matrix in (8) is *upper triangular*.

By using (8), we compute some examples for the matrix representation of the polynomials $P_n(x)$ as follows:

Example 1. Substituting $n = 2$ into (8), the matrix representation of the *polynomial* $P_2(x)$ is given as follows:

$$P_2(x) = \left[1 \ x \ x^2 \right] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}.$$

From the above matrix representation, we get

$$P_2(x) = (b_0 + b_1 + b_2) + (b_1 + 2b_2)x + b_2x^2.$$

Example 2. Substituting $n = 2$ into (8), the matrix representation of the *polynomial* $P_3(x)$ is given as follows:

$$P_3(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

From the above matrix representation, we get

$$P_3(x) = (b_0 + b_1 + b_2 + b_3) + (b_1 + 2b_2 + 3b_3)x + (b_2 + 3b_3)x^2 + b_3x^3.$$

Remark 2.1.
Joy [4] gave a matrix representation of the Bernstein polynomials with their applications.

4 Simulation of the Beta polynomials

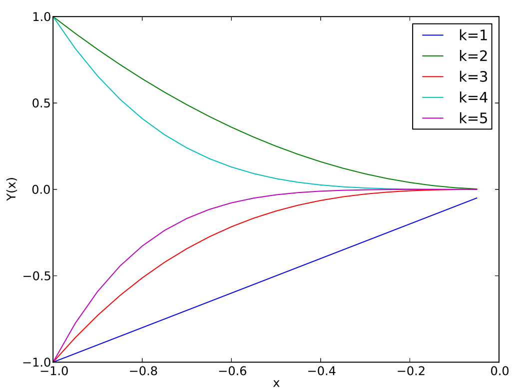

Fig. 1 Varying k values for n = k

$$y(x) = \mathfrak{B}_{k,n}(x) \tag{9}$$

where $k = 0, 1, 2, \ldots, n$ and x varies between [-1,0]. Below are the plots of y(x) with respect to k with varying offset values as given by:

$$n = k + offset \tag{10}$$

When offset is 0 then the plot shown in Figure 1 is obtained. As demonstrated in plots, increase in offset shifts the curve to 0 while sequezes them. Meantime, as k increases the resulting curves narrows.

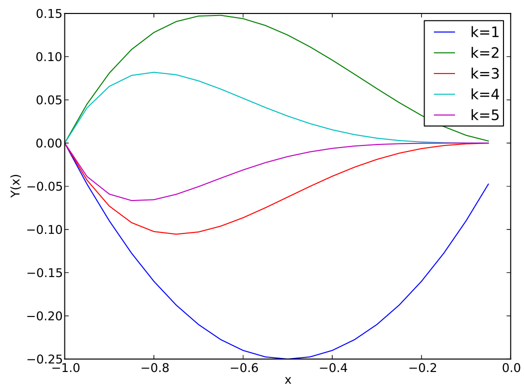

Fig. 2 Varying k values for n = k + 1

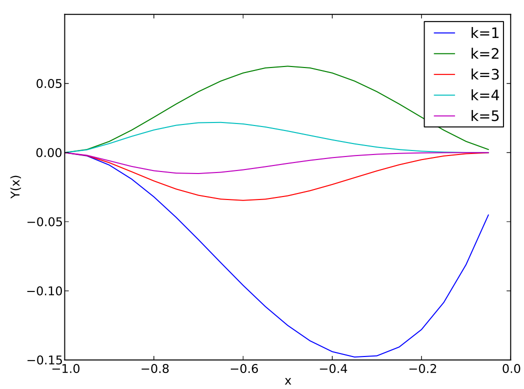

Fig. 3 Varying k values for n = k + 2

5 Conculusion

Applying Equation (8), one can easily see that every polynomials with degree n may be expressed in matrix form that relates to the beta polynomials. Therefore, any polynomial of degree less than and equal to n is written as the linear combination of the beta polynomials.

Graphics of the beta polynomials are provided to visualize the shape of polynomials on finite domain. The effects of k and n on the shape of the curve are demonstrated for the given range. These graphics may be used not only in Bezier type curves and surfaces, in Computer Aided Geometric Design (CAGD) but also in other areas.

In the work of Farouki [3], we know that the monomial form in Equation (8) of a polynomial $P_n(x)$ is frequently used. Thus, in the above graphics we show that the beta

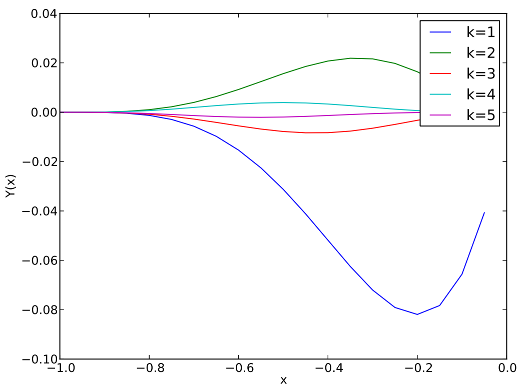

Fig. 4 Varying k values for n = k + 4

polynomials $\mathfrak{B}_{k,n}(x)$ form may also be used to manipulate the graph of a polynomial over a finite domain.
Consequently, we also think that a Bezier type curve may be constructed by the beta polynomials.

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