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Some Properties of Graph Related to Conjugacy Classes of Special Linear Group $SL_2(F)$

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Abstract: Suppose that G is a finite group. The graph $\Gamma(G)$ is related to conjugacy classes of G. Its vertices are the non-central conjugacy classes of G and there is an edge between each two distinct vertices of $\Gamma(G)$, if and only if their class sizes have a common prime divisor.
In this paper, some properties of graph $\Gamma(G)$ such as chromatic polynomial, chromatic number, clique number and independence number are studied for $G \cong SL_2(F)$, where F is a finite field.

Keywords: Conjugacy class, special linear group, independence number, chromatic number, clique number

1 Introduction

Let G be a finite group. $\Gamma(G)$ is the attached graph related to its conjugacy classes. The vertices of $\Gamma(G)$ are the non-central conjugacy classes of G and two distinct vertices are connected by an edge, if their class sizes have a common prime divisor.
This graph has been widely studied. See, for instance [1, 5]. In [1] Bertram, Herzog and Mann showed that $n(\Gamma(G)) \leqslant 2$ for all finite groups where $n(\Gamma(G))$ is the number of the connected components of $\Gamma(G)$. Also they proved that, the graph is complete for all non-abelian finite simple groups. Results are proved for infinite FC-groups. In [5] the authors proved that, the symmetric group S_3, the dihedral group D_5, the three pairwise non-isomorphic non-abelian groups of order 12, and the non-abelian group T_{21} of order 21, are the complete list of all G such that $\Gamma(G)$ contains no triangles.
The notation we use is standard. All groups considered in this paper are finite. Let G be a finite group, x an element of G. x^G denotes the conjugacy class containing x that is the set of all elements conjugate to x. $|x^G|$ denotes the size of x^G. A subgroup N of G is called a normal subgroup if it is invariant under conjugation. Let $\frac{G}{N}$ be a quotient group, gN an element of $\frac{G}{N}$. $(gN)^{\frac{G}{N}}$ denotes the conjugacy class containing gN and $|(gN)^{\frac{G}{N}}|$ denotes the size of $(gN)^{\frac{G}{N}}$. We denote the center of G by $Z(G)$, and the number of conjugacy classes of G by $k(G)$. Let Γ be a graph. The degree of a vertex v of Γ denoted by $d(v)$ and the number of vertices of graph Γ denoted by $|V(\Gamma)|$. Also the number of edges of graph Γ denoted by $|E(\Gamma)|$. The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The diameter of Γ is the maximum distance between two vertices of Γ and denoted by $diam(\Gamma)$. A complete graph is a graph in which every pair of distinct vertices are adjacent. An independent set in a graph Γ is a set of pairwise nonadjacent vertices. The independence number of a graph Γ is the maximum size of an independent set of vertices and denoted by $\alpha(\Gamma)$. A vertex cover of a graph Γ is a set $Q \subseteq V(\Gamma)$ that contains at least one endpoint of every edge. $\beta(\Gamma)$ is the minimum size of vertex cover. Let k be a positive integer. A k-vertex coloring of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a k-vertex coloring. A subset C of the vertices of Γ is called a clique if the induced subgraph on C is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. A Hamiltonian cycle is a path that visits each vertex of Γ exactly just once. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph. A graph is Hamiltonian-connected if for every pair of vertices u, v there is a Hamiltonian path from u to v.

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In this paper, we consider the graph $\Gamma(G)$ for $G \cong SL_2(F)$, where F is a finite field. We study some properties of this graph.

2 Preliminaries

We need the following lemmas which will be used later:
Lemma 2.1. [6] Let G be a non-abelian finite simple group. Then $\Gamma(G)$ is a complete graph.
Lemma 2.2. ([3], Theorem 1.1) Suppose that Γ is a graph, then:

$$\sum_{\varepsilon=1}^{|V(\Gamma)|} d(v_\varepsilon) = 2|E(\Gamma)|.$$

Lemma 2.3. ([8], Lemma 3.1.21) In a graph $\Gamma, S \subseteq V(\Gamma)$ is an independent set if and only if $\bar{S} = V(\Gamma) - S$ is a vertex cover and therefore:

$$\alpha(\Gamma) + \beta(\Gamma) = |V(\Gamma)|.$$

Lemma 2.4. [4] Let G be a finite group and N is a normal subgroup. Then:
i) $|g^N| \Big| |g^G|$; $g \in N$.
ii) $|(gN)^{\frac{G}{N}}| \Big| |g^G|$; $g \in G$.
Lemma 2.5. ([8], Theorem 5.2.16) Every k-critical graph is $(k-1)$-edge-connected.
Lemma 2.6. ([8], Proposition 5.2.18) If Γ is k-critical, then Γ has no cutset consisting of pairwise adjacent vertices.
Lemma 2.7. [5] Let G be a non-abelian finite group. Then $\Gamma(G)$ is a graph without triangles, if and only if G is isomorphic to one of the following solvable groups:
the symmetric group S_3;
the dihedral groups D_5 and D_6;
the alternating group A_4;
the group T_{12} of order 12 given by
$T_{12} = \langle a, b : a^6 = 1, b^2 = a^3, ba = a^{-1}b \rangle$;
the group T_{21} of order 21 given by
$T_{21} = \langle a, b : a^3 = b^7 = 1, ba = ab^2 \rangle$.

3 Main Results

Suppose that $G \cong SL_2(F)$, $F = GF(q)$ and q is a prime power.

Theorem 3.1. Let $G \cong SL_2(q)$:
i) If $q = 2^m, m \geq 1$, then $k(G) = 2^m + 1, |V(\Gamma(G))| = 2^m$.

ii) If $q = p^m$, where p is an odd prime number, $m \geq 1$, then $k(G) = p^m + 4, |V(\Gamma(G))| = p^m + 2$.

Proof.
Let $G \cong SL_2(q)$. Suppose that C_n is the number of conjugacy classes in $GL_n(q)$. Now by [7] we have:

$$C_n = q^n - (q^a + q^{a-1} + \cdots + q^{b+1} + q^b) + \cdots$$
$$; a = [\frac{1}{2}(n-1)], b = [\frac{1}{3}n].$$

Thus for $n = 0, 1, 2$ we have:

$$C_0 = 1, \qquad C_1 = q - 1, \qquad C_2 = q^2 - 1.$$

Also by [7] the number of conjugacy classes in $G \cong SL_2(F)$ is :

$$k(G) = (q-1)^{-1} \sum_{d|(2,q-1)} \varphi_2(d) C_{\frac{2}{d}} \tag{1}$$

where

$$\varphi_r(n) = n^r \Pi_{p|n}(1 - p^{-r}).$$

(product over the primes dividing n).
Now for (i), since $(2, q-1) = 1$, then $d = 1$. If we put $d = 1$ in (1), we have:

$$k(G) = (q-1)^{-1} \sum_{d=1} \varphi_2(d) C_{\frac{2}{d}}$$
$$= (q-1)^{-1}(\varphi_2(1) C_2) = 2^m + 1.$$

For (ii), q is a power of an odd prime number, then $(2, q-1) = 2$, therefore $d = 1, 2$.
So, we have:

$$k(G) = (q-1)^{-1} \sum_{d=1,2} \varphi_2(d) C_{\frac{2}{d}}$$
$$= (q-1)^{-1}(\varphi_2(1) C_2 + \varphi_2(2) C_1) = q + 4 = p^m + 4.$$

Since $Z(G) = \{\lambda I | \lambda \in F^*, \lambda^n = 1\}$ and $|Z(G)| = (n, q-1)$, thus for (i), we have $|Z(G)| = 1$ and then $|V(\Gamma(G))| = k(G) - |Z(G)| = 2^m$.

For (ii), $|Z(G)| = 2$ then

$$|V(\Gamma(G))| = k(G) - |Z(G)| = p^m + 2. \square$$

Theorem 3.2. Let $G \cong SL_2(q)$:

i) If $q = 2$, then the graph $\Gamma(G)$ is a non-complete graph, $|E(\Gamma(G))| = 0$, $\alpha(\Gamma(G)) = 2$, $\beta(\Gamma(G)) = 0$.
ii) If $q = 2^m, m \geq 2$, then the graph $\Gamma(G)$ is a complete graph, $|E(\Gamma(G))| = 2^{m-1}(2^m - 1), \alpha(\Gamma(G)) = 1$, $\beta(\Gamma(G)) = 2^m - 1$.
iii) If $q = p^m$, where p is an odd prime number, $m \geq 1$, then the graph, $\Gamma(G)$ is a complete graph, $|E(\Gamma(G))| = 2^{-1}(p^{2m} + 3p^m + 2), \alpha(\Gamma(G)) = 1$, $\beta(\Gamma(G)) = p^m + 1$.

Proof.

i) It is clear that if $q = 2$, then $G \cong S_3$ and the proof follows from Lemma 2.7.

ii)Since $q = 2^m (m \geq 2)$, then $PSL_2(2^m) \cong SL_2(2^m)$, therefore $SL_2(2^m)$ is a non-abelian finite simple group. Now by Lemma 2.1 the graph $\Gamma(G)$ is a complete graph. Thus for each $v \in V(\Gamma(G))$,
$d(v) = |V(\Gamma(G))| - 1 = 2^m - 1$.
By Lemma 2.2 we have:

$$\sum_{\varepsilon=1}^{|V(\Gamma(G))|} d(v_\varepsilon) = 2|E(\Gamma(G))|.$$

Thus

$$\sum_{\varepsilon=1}^{|V(\Gamma(G))|} d(v_\varepsilon) = \sum_{\varepsilon=1}^{2^m} (2^m - 1) = 2^m(2^m - 1) = 2|E(\Gamma(G))|.$$

So we have

$$|E(\Gamma(G))| = 2^{m-1}(2^m - 1).$$

Since $\Gamma(G)$ is a complete graph, therefore every independent set includes only one vertex.
Thus the independence number of graph $\Gamma(G)$ equals 1. Therefore $\alpha(\Gamma(G)) = 1$.
On the other hand, by Lemma 2.3:

$$\alpha(\Gamma(G)) + \beta(\Gamma(G)) = |V(\Gamma(G))|.$$

So we have:

$$\beta(\Gamma(G)) = 2^m - 1.$$

iii)Suppose that $p = 3$ and $m = 1$, then the set of conjugacy class sizes of G is
$\{1, 1, 4, 4, 4, 4, 6\}$.
According to the definition of graph $\Gamma(G)$, it is a complete graph with 5 vertices. Hence
$\alpha(\Gamma(G)) = 1, \beta(\Gamma(G)) = 4$.
Now suppose $G \cong SL_2(q)$ and $N = Z(SL_2(q))$ where $q = p^m, q \neq 3$, p is an odd prime number and $m \geq 1$, then $N \lhd G$ and also $PSL_2(q) \cong \frac{G}{N}$, since $PSL_2(q)$ is a non-abelian finite simple group, therefore by Lemma 2.1 $\Gamma(\frac{G}{N})$ is a complete graph. For every two arbitrary vertices of graph $\Gamma(\frac{G}{N})$ like $|(xN)^{\frac{G}{N}}|$ and $|(yN)^{\frac{G}{N}}|$ as $x, y \in G - Z(G)$, we have $(|(xN)^{\frac{G}{N}}|, |(yN)^{\frac{G}{N}}|) \neq 1$ and by Lemma 2.4 we have $|(xN)^{\frac{G}{N}}| \Big| |x^G|$ and $|(yN)^{\frac{G}{N}}| \Big| |y^G|$, thus $(|x^G|, |y^G|) \neq 1$.
Then every pair of distinct vertices of graph $\Gamma(G)$ is connected by an edge, so it is a complete graph. Then we have the following relation for every arbitrary vertex of $\Gamma(G)$ like v :

$$d(v) = |V(\Gamma(G))| - 1 = p^m + 1.$$

By Lemma 2.2:

$$\sum_{\varepsilon=1}^{|V(\Gamma(G))|} d(v_\varepsilon) = 2|E(\Gamma(G))|.$$

Thus

$$2|E(\Gamma(G))| = \sum_{\varepsilon=1}^{|V(\Gamma(G))|} d(v_\varepsilon) = \sum_{\varepsilon=1}^{p^m+2} (p^m + 1)$$
$$= (p^m + 2)(p^m + 1) = p^{2m} + 3p^m + 2.$$

Also
$$|E(\Gamma(G))| = 2^{-1}(p^{2m} + 3p^m + 2).$$

Since $\Gamma(G)$ is a complete graph, therefore every independent set includes only one vertex.
Thus the independence number of graph $\Gamma(G)$ equals 1. Therefore $\alpha(\Gamma(G)) = 1$.
On the other hand, by Lemma 2.3:

$$\alpha(\Gamma(G)) + \beta(\Gamma(G)) = |V(\Gamma(G))|.$$

So we have:

$$\beta(\Gamma(G)) = p^m + 1.\square$$

Corollary 3.3. Suppose that $G \cong SL_2(q)$, where $q \neq 2$ and q is a prime power, then the graph $\Gamma(G)$ is$(|V(\Gamma(G))| - 1)$-edge-connected and $\Gamma(G)$ has no cutset consisting of pairwise adjacent vertices.
Proof. By Theorem 3.2, the graph $\Gamma(G)$ is a complete graph. Thus the graph $\Gamma(G)$ is $|V(\Gamma(G))|$-critical, therefore by Lemma 2.5 $\Gamma(G)$ is $(|V(\Gamma(G))| - 1)$-edge-connected and according to Lemma 2.6 $\Gamma(G)$ has no cutset consisting of pairwise adjacent vertices. $\square$
Proposition 3.4. Let $G \cong SL_2(q)$:

i)If $q = 2$, then $\chi(\Gamma(G)) = 1$, $\omega(\Gamma(G)) = 1$.
ii)If $q = 2^m, m \geq 2$, then $\chi(\Gamma(G)) = \omega(\Gamma(G)) = 2^m$.
iii)If $q = p^m$, where p is an odd prime number, $m \geq 1$, then $\chi(\Gamma(G)) = \omega(\Gamma(G)) = p^m + 2$.

In (ii) and (iii), the girth of graph equals 3 and it is a Hamiltonian-connected graph with $diam(\Gamma(G)) = 1$.
Proof. For the first case, by Theorem 3.2 the minimum number of colors needed to color the graph $\Gamma(G)$ in which no two adjacent vertex have the same color equals 1. Therefore $\chi(\Gamma(G)) = 1$ and since the graph $\Gamma(G)$ is not connected, then $\omega(\Gamma(G)) = 1$.
Since the graph $\Gamma(G)$ is a complete graph for cases (ii) and (iii) by Theorem 3.2, and $\omega(\Gamma(G))$ is the maximum size of a set of pairwise adjacent vertices in $\Gamma(G)$, then $\omega(\Gamma(G)) = |V(\Gamma(G))|$.
As $\chi(\Gamma)$ is the minimum number of colors needed to color a graph $\Gamma(G)$ such that each two adjacent vertices have different colors, thus
$\chi(\Gamma(G)) = |V(\Gamma(G))| = \omega(\Gamma(G))$.
Now according to Theorem 3.1 we have:
$\chi(\Gamma(G)) = \omega(\Gamma(G)) = 2^m$ for case (ii),
and $\chi(\Gamma(G)) = \omega(\Gamma(G)) = p^m + 2$ for case (iii).
In both cases (ii) and (iii) the graph is connected because it is a complete graph and also it includes at least one

cycle which visits every edge once, thus there is a Hamiltonian cycle in the graph $\Gamma(G)$ in both cases, therefore $\Gamma(G)$ is a Hamiltonian graph. The number of vertices of $\Gamma(G)$ is at least 4 in both cases. Thus it includes a triangle, hence it is the shortest cycle that exists in $\Gamma(G)$. Therefore in both cases the girth of graph equals 3.

Since for every pair of vertices u and v in $\Gamma(G)$ we have: $d(u,v)=1$, then $diam(\Gamma(G))=1$. □

Definition. [2] Let Γ be a graph and also $|V(\Gamma)|=n$ and u be a complex number. For each natural number r, let $m_r(\Gamma)$ denotes the number of distinct color-partitions of $V(\Gamma)$ into r color-classes, and define $u_{(r)}$ to be the complex number $u(u-1)\cdots(u-r+1)$.

The chromatic polynomial of Γ is the polynomial

$$C(\Gamma;u)=\sum_{r=1}^{|V(\Gamma)|} m_r(\Gamma)u_{(r)}$$

Proposition 3.5. Let $G\cong SL_2(q)$:

i)If $q=2$, then $C(\Gamma(G);u)=u^2$.

ii)If $q=2^m$, $m\geq 2$, then the chromatic polynomial of graph $\Gamma(G)$ is of the form:

$$C(\Gamma(G);u)=u(u-1)\cdots(u-2^m+1).$$

iii)If $q=p^m$, where p is an odd prime number, $m\geq 1$, then the chromatic polynomial of graph $\Gamma(G)$ is of the form:

$$C(\Gamma(G);u)=u(u-1)\cdots(u-p^m-1).$$

Proof. In the first case $G\cong S_3$, therefore $m_1(\Gamma(G))=1$, $m_2(\Gamma(G))=2$ and by definition of $C(\Gamma(G);u)$ we have:

$$C(\Gamma(G);u)=u^2.$$

Now according to Theorem 3.2, $\Gamma(G)$ is a complete graph in cases (ii) and (iii), thus each vertex of the graph $\Gamma(G)$ is adjacent by the others and its chromatic polynomial is as following:

$$m_1(\Gamma(G))=m_2(\Gamma(G))=\cdots=m_{|V(\Gamma(G))|-1}(\Gamma(G))=0,$$

$$m_{|V(\Gamma(G))|}(\Gamma(G))=1$$

$$\begin{aligned}C(\Gamma(G);u)&=\sum_{r=1}^{|V(\Gamma(G))|} m_r(\Gamma(G))u_{(r)}\\&=m_{|V(\Gamma(G))|}(\Gamma(G))u_{|V(\Gamma(G))|}=u_{(|V(\Gamma(G))|)}.\end{aligned}$$

Now, according to Theorem 3.1 we have the following relation for the second part of proposition

$$C(\Gamma(G);u)=u(u-1)\cdots(u-2^m+1).$$

For the third part:

$$C(\Gamma(G);u)=u(u-1)\cdots(u-p^m-1).□$$

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