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# Ostrowski-Type Inequalities for Functions Whose Derivative Modulus is Relatively Convex.

*Miguel Vivas Cortez*[1,*], *Carlos García* [2] *and Jorge E. Hernández H.*[3]

[1] Pontificia Universidad Católica del Ecuador (PUCE), Facultad de Ciencias Exactas y Naturales, Escuela de Ciencias Físicas y Matemática. Sede Quito, Ecuador.
[2] Universidad Centroccidental Lisandro Alvarado, Decanato de Ciencias y Tecnología, Departamento de Matemáticas, Barquisimeto, Venezuela.
[3] Universidad Centroccidental Lisandro Alvarado, Decanato de Ciencias Económicas y Empresariales, Departamento de Técnicas Cuantitativas, Barquisimeto, Venezuela.

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**Abstract:** We found some new Ostrowski-type inequalities for functions whose derivative module is relatively convex, also some others of the same type making use of relatively $s-$convex functions in the second sense. With these results we obtain generalizations of results found by M. Alomari et. al. using convex and $s-$convex in the second sense ([1,2]).

**Keywords:** Ostrowski inequality, relative convex functions, relative $s-$convex functions.

## 1 Introduction

The Ostrowski inequality is known in the classical literature since 1938 ([10]), when A. Ostrowski gave an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at the point $x \in [a,b]$ as follow: Let $f : I \subset [0,+\infty) \to \mathbb{R}$ a differentiable function in $I^\circ$, the interior of the interval $I$, such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ and $a < b$. If $|f'(x)| \leq M$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

Recently, many generalizations of the Ostrowski inequality for functions of bounded variation, Lipschitzian, monotone, absolutely continuous, convex functions, $s$-convex and $h$-convex functions, n-times differentiable mappings with error estimates with some special means together with some numerical quadrature are done. [1,2,3,4,6] .

In this work we give new Ostrowski type inequalities for functions whose derivative's module is relatively convex.

## 2 Preliminaries

This section is intended to give the fundamentals in which the present work is constructed. It is well known that the following definition was established by W.J. Jensen.

**Definition 1([1]).** *Let $I$ be an interval in $\mathbb{R}$. A function $f$ : $I \to \mathbb{R}$ is said to be convex, if for every $x,y \in I$ and every $t \in (0,1)$, the inequality*

$$f(tx+(1-t)y) \leq t f(x) + (1-t) f(y), \tag{1}$$

*holds.*

If the inequality in (1) holds in the opposite sense, then we say that $f$ is concave.

In 1961 W. Orlicz introduced the $s-$convexity concept in [9], and later, in 1978 W. Breckner introduced a second version of it in [5], and it is presented here.

**Definition 2([7]).** *Let $0 < s \leq 1$. A function $f : [0,+\infty) \to \mathbb{R}$ is s-convex in the first sense or $s_1$-convex if*

$$f(\alpha x+\beta y) \leq \alpha^s f(x) + \beta^s f(y),$$

*for every $x,y \in [0,+\infty)$ and $\alpha,\beta \in (0,1)$ and $\alpha^s+\beta^s = 1$. The function $f$ is s-convex in the second sense or $s_2$-convex*

* Corresponding author e-mail: mjvivas@puce.edu.ec

*if*

$$f(\alpha x+\beta y)\leq\alpha^{s}f(x)+\beta^{s}f(y),$$

*for every* $x,y\in[0,+\infty)$ *and* $\alpha,\beta\in(0,1)$ *and* $\alpha+\beta=1$.

If the inequalities in (2) hold in the opposite sense, then we say that $f$ is $s$-concave in the first and second sense, respectively.

The following results are of interest for the development of this work. The following theorem establishes an inequality of the Ostrowski type using a function whose derivative in modulus is convex, and can be found in [1].

**Theorem 1.** *(See [1]) let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable function in* $I^{\circ}$ *such that* $f'\in L([a,b])$ *where* $a,b\in I$ *with* $a<b$. *If* $|f'|$ *is convex in* $[a,b]$, *then the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\\&\leq\frac{b-a}{6}\left[\left(4\left(\frac{b-x}{b-a}\right)^{3}-3\left(\frac{b-x}{b-a}\right)^{2}+1\right)|f'(a)|\right.\\&\left.+\left(9\left(\frac{b-x}{b-a}\right)^{2}-4\left(\frac{b-x}{b-a}\right)^{3}-6\left(\frac{b-x}{b-a}\right)+2\right)|f'(b)|\right]\end{aligned}$$

*holds for each* $x\in[a,b]$. *The constant* $\frac{1}{6}$ *is the best possible in the sense it can not be replaced for any smaller.*

Also, with the use of a function whose derivative in modulus is $s-$convex in the second sense, M. Alomari et. al. in [2] established the next result which involves the Ostrowski inequality.

**Theorem 2.** *(See [2]) Let* $f: I\subset\mathbb{R}_{+}\to\mathbb{R}_{+}$ *a differentiable function in* $I^{\circ}$ *such that* $f'\in\mathscr{L}[a,b]$ *where* $a,b\in I$ *with* $a<b$. *If* $|f'|$ *is s-convex in the second sense in* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $|f'(x)|\leq M$, $x\in[a,b]$, *then the following inequality holds for each* $x\in[a,b]$.

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\leq\frac{M}{b-a}\left[\frac{(x-a)^{2}+(b-x)^{2}}{s+1}\right].$$

Other results, no less important, were established by M. Alomari in [3] , establishing the inequality of Ostrowski by using functions whose $q-$th powers of its derivative in module is s-convex in the second sense.

**Theorem 3.** *Let* $f: I\subseteq\mathbb{R}_{+}\to\mathbb{R}_{+}$ *a differentiable mapping in* $I^{\circ}$ *such that* $f'\in\mathscr{L}[a,b]$ *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^{q}$ *is s-convex in the second sense in* $[a,b]$ *for some fixed* $s\in(0,1]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $|f'(x)|\leq M$, $x\in[a,b]$, *then the following inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\\&\leq\frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-a)^{2}+(b-x)^{2}}{b-a}\right]\end{aligned}$$

*for each* $x\in[a,b]$.

**Theorem 4.** *Let* $f: I\subseteq\mathbb{R}_{+}\to\mathbb{R}_{+}$ *a differentiable mapping in* $I^{\circ}$ *such that* $f'\in\mathscr{L}[a,b]$ *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^{q}$ *is s-convex in the second sense in* $[a,b]$ *for some fixed* $s\in(0,1]$, $q\geq1$, *and* $|f'(x)|\leq M$, $x\in[a,b]$, *then the following inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\\&\leq M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-a)^{2}+(b-x)^{2}}{2(b-a)}\right]\end{aligned}$$

*holds for each* $x\in[a,b]$.

Among studies getting in the area of generalized convexity, those that make use of relatively convex functions with respect to a function stand out. M. A. Noor, K.I. Noor and M.U. Awan introduced the following definitions ([8]).

**Definition 3.** *Let* $K_g$ *a subset of* $H$. $K_g$ *is said to be relatively convex with respect to a function* $g: H\to H$ *if*

$$tg(v)+(1-t)u\in K_{g}$$

$\forall u,v\in H: u,g(v)\in K_g$, *and* $t\in[0,1]$.

**Definition 4.** *Let* $I$ *be an interval in* $\mathbb{R}$. *A function* $f: K_g\subseteq\mathbb{R}\to\mathbb{R}$ *is said to be relatively convex with respect to a function* $g:\mathbb{R}\to\mathbb{R}$ *if the inequality*

$$f(tg(x)+(1-t)y)\leq tf(g(x))+(1-t)f(y)$$

*holds for all* $g(x),y\in K_g$, $x,y\in\mathbb{R}$ *and* $t\in[0,1]$.

If the inequality in (4) holds in the opposite sense, then we say that $f$ is relatively concave.

An example of relatively convex set is showed in [8], taken from a study on the environmental impact of noise caused by a train. The authors define the function $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}x\ if\ x\in[0,50]\\0\quad otherwise\end{cases}$$

in such a way that the set $[0,50]\cup[125,130]$ is a relatively convex set.

Also M.Noor et. al. introduced the following definition ([8]).

**Definition 5.** *A function* $f: K_g\to[0,+\infty)$ *is said to be relatively s-convex in the second sense with respect to a function* $g: H\to H$, *where* $s\in(0,1]$, *if inequality*

$$f(tg(x)+(1-t)y)\leq t^{s}f(g(x))+(1-t)^{s}f(y)$$

*holds for each* $x,y\in[0,+\infty)$, $g(x),y\in K_g$ *and* $t\in[0,1]$.

If the inequality in (5) holds in the opposite sense, then we say that $f$ is relatively $s$-concave in the second sense.

## 3 Main Results

In this section, we present new Ostrowski types of inequality derivative modulus wich is relative convex and relative $s-$convex functions in the second sense.
The following Lemma is necessary.

**Lemma 1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ a differentiable function in $I^\circ$ where $a,b \in I$, $a < b$ and $g : \mathbb{R} \to \mathbb{R}$ is a function. If $f' \in \mathscr{L}[a,b]$, then the following equality holds*

$$f(g(x)) - \frac{1}{b-a}\int_a^b f(z)dz$$

$$= \frac{(g(x)-a)^2}{b-a}\int_0^1 t f'(tg(x)+(1-t)a)dt - \frac{(g(x)-b)^2}{b-a}\int_0^1 t f'(tg(x)+(1-t)b)dt$$

*for every $x \in g^{-1}(I)$.*

*Proof.* Integrating by parts we get:

$$\int_0^1 t f'(t(g(x)+(1-t)a)dt$$

$$= \frac{f(g(x))}{g(x)-a} - \frac{1}{g(x)-a}\int_0^1 f(t(g(x)+(1-t)a)dt,$$

so,

$$(g(x)-a)\int_0^1 t f'(t(g(x)+(1-t)a)dt = f(g(x)) - \int_0^1 f(tg(x)+(1-t)a)dt.$$

With the change of variable $z = tg(x)+(1-t)a$ it follows that

$$(g(x)-a)^2\int_0^1 t f'(t(g(x)+(1-t)a)dt = (g(x)-a)f(g(x)) - \int_a^{g(x)} f(z)dz. \qquad (2)$$

Similarly

$$(g(x)-b)^2\int_0^1 t f'(t(g(x)+(1-t)b)dt = (g(x)-b)f(g(x)) - \int_b^{g(x)} f(z)dz. \qquad (3)$$

subtracting (3) from (2) it follows the desired result.

**Theorem 5.** *Let $f : I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in \mathscr{L}[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is relatively convex with respect to a function $g : \mathbb{R} \to \mathbb{R}$ in $[a,b]$ and $|f'(x)| \leq M$, the inequality*

$$\left| f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{M}{b-a}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{2}\right]$$

*holds for all $x \in g^{-1}(I)$.*

*Proof.* Using Lemma 1 we have

$$\left| f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{(g(x)-a)^2}{b-a}\int_0^1 t\left|f'(tg(x)+(1-t)a)\right|dt + \frac{(g(x)-b)^2}{b-a}\int_0^1 t\left|f'(tg(x)+(1-t)b)\right|dt.$$

Now, since $|f'|$ is relatively convex y $|f'(x)| \leq M$ we get

$$\left| f(g(x)) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq M\frac{(g(x)-a)^2}{b-a}\int_0^1 tdt + M\frac{(g(x)-b)^2}{b-a}\int_0^1 tdt \leq \frac{M}{b-a}\left[\frac{(g(x)-a)^2-(g(x)-b)^2}{2}\right]$$

and the proof follows.

*Remark.* If in Theorem (5) we take $g(x) = x$, we obtain the classic Ostrowski inequality.

**Lemma 2.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function in $I^\circ$ where $a,b \in I$ with $a < b$ and $g : \mathbb{R} \to \mathbb{R}$ is a function. If $f' \in \mathscr{L}[a,b]$, then the following equality*

$$f(x) - \frac{1}{b-g(a)}\int_{g(a)}^b f(u)du = (g(a)-b)\int_0^1 p(t)f'(tg(a)+(1-t)b)dt$$

*holds for every $t \in [0,1]$ where*

$$p(t) = \begin{cases} t, & t \in [0, \frac{b-x}{b-g(a)}], \\ t-1, & t \in (\frac{b-x}{b-g(a)}, 1]. \end{cases}$$

*for every $x \in [a,b]$.*

*Proof.* Integrating by parts we get:

$$\int_0^{\frac{b-x}{b-g(a)}} t f'(t(g(a)+(1-t)b)dt = \frac{b-x}{b-g(a)}\frac{f(x)}{g(a)-b} - \int_0^{\frac{b-x}{b-g(a)}} \frac{f(t(g(a)+(1-t)b)}{g(a)-b}dt \qquad (4)$$

and

$$\int_{\frac{b-x}{b-g(a)}}^1 (t-1)f'(t(g(a)+(1-t)b)dt = \frac{x-g(a)}{b-g(a)}\frac{f(x)}{g(a)-b} - \int_{\frac{b-x}{b-g(a)}}^1 \frac{f(t(g(a)+(1-t)b)}{g(a)-b}dt \qquad (5)$$

Adding (4) and (5) we obtain

$$\int_0^{\frac{b-x}{b-g(a)}} t f'(t(g(a)+(1-t)b)dt$$

$$+\int_{\frac{b-x}{b-g(a)}}^1 (t-1)f'(t(g(a)+(1-t)b)dt$$

$$=\frac{f(x)}{g(a)-b}-\frac{1}{(g(a)-b)^2}\int_b^{g(a)} f(u)du,$$

equivalently,

$$f(x)-\frac{1}{b-g(a)}\int_{g(a)}^b f(u)du$$

$$=(g(a)-b)\int_0^1 p(t)f'(tg(a)+(1-t)b)dt.$$

The proof is complete.

**Theorem 6.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|$ is relatively convex with respect to a function $g:\mathbb{R}\to\mathbb{R}$ in $[a,b]$, then the following inequality holds*

$$\left|f(x)-\frac{1}{b-g(a)}\int_{g(a)}^b f(u)du\right|$$

$$\leq\frac{g(a)-b}{6}\left[\left(4\left(\frac{b-x}{b-g(a)}\right)^3-3\left(\frac{b-x}{b-g(a)}\right)^2+1\right)|f'(g(a))|\right.$$

$$\left.+\left(9\left(\frac{b-x}{b-g(a)}\right)^2-4\left(\frac{b-x}{b-g(a)}\right)^3-6\left(\frac{b-x}{b-g(a)}\right)+2\right)|f'(b)|\right]$$

*for each $x\in[a,b]$. The constant $\frac{1}{6}$ is the best possible in the sense that is can not be replaced by any smaller.*

*Proof.* Using Lemma (2), the triangular inequality and the fact that $|f'|$ is relatively convex with respect to a function $g:[a,b]\to\mathbb{R}$ we get that

$$\left|f(x)-\frac{1}{b-g(a)}\int_{g(a)}^b f(u)du\right|$$

$$\leq(g(a)-b)\int_0^{\frac{b-x}{b-g(a)}} t|f'(tg(a)+(1-t)b)|dt$$

$$+(g(a)-b)\int_{\frac{b-x}{b-g(a)}}^1 (1-t)|f'(tg(a)+(1-t)b)|dt$$

$$\leq(g(a)-b)\int_0^{\frac{b-x}{b-g(a)}} t(t|f'(g(a))|+(1-t)|f'(b)|)dt$$

$$+(g(a)-b)\int_{\frac{b-x}{b-g(a)}}^1 (1-t)(t|f'(g(a))|+(1-t)|f'(b)|)dt$$

$$=(g(a)-b)\int_0^{\frac{b-x}{b-g(a)}} (t^2|f'(g(a))|+t(1-t)|f'(b)|)dt$$

$$+(g(a)-b)\int_{\frac{b-x}{b-g(a)}}^1 ((1-t)t|f'(g(a))|+(1-t)^2\left|f'(b)\right|)dt.$$

Then, integrating over $t\in[0,1]$ we get

$$\left|f(x)-\frac{1}{b-g(a)}\int_{g(a)}^b f(u)du\right|$$

$$\leq\frac{(g(a)-b)}{6}\left[4\left(\frac{b-x}{b-g(a)}\right)^3-3\left(\frac{b-x}{b-g(a)}\right)^2+1\right)|f'(g(a))|$$

$$+9\left(\frac{b-x}{b-g(a)}\right)^2-4\left(\frac{b-x}{b-g(a)}\right)^3-6\left(\frac{b-x}{b-g(a)}\right)+2\Big)|f'(b)|\Big].$$

The proof is complete.

*Remark.* If in Theorem (6) we take $g(x)=x$, we obtain the result of Theorem (1).

The following results correspond to those functions whose derivatives in modulus are relatively $s-$convex in the second sense.

**Theorem 7.** *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function in $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|$ is relatively s-convex with respect to a function $g:\mathbb{R}\to\mathbb{R}$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq\frac{M}{b-a}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{s+1}\right]$$

*for each $x\in[a,b]$.*

*Proof.* By lemma (1), the triangle inequality and the fact that $|f'|$ is relatively $s$-convex with respect to a function $g:[a,b]\to\mathbb{R}$ and $|f'(x)|\leq M$, $x\in[a,b]$ we get:

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq\frac{(g(x)-a)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)a)|dt$$

$$+\frac{(g(x)-b)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)b)|dt$$

$$\leq\frac{(g(x)-a)^2}{b-a}\int_0^1 t(t^s|f'(g(x))|+(1-t)^s\left|f'(a)\right|)dt$$

$$+\frac{(g(x)-b)^2}{b-a}\int_0^1 t(t^s|f'(g(x))|+(1-t)^s\left|f'(b)\right|)dt$$

$$=M\frac{(g(x)-a)^2}{b-a}\int_0^1 t(t^s+(1-t)^s)dt$$

$$+M\frac{(g(x)-b)^2}{b-a}\int_0^1 t(t^s+(1-t)^s)dt,$$

then, integrating over $t\in[0,1]$ we get

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq M\left(\frac{1}{s+2}+\frac{1}{(s+1)(s+2)}\right)\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{b-a}\right]$$
$$=\frac{M}{b-a}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{s+1}\right]$$

The proof is complete.

*Remark.* If in Theorem 7 we let $g(x)=x$, we obtain the result of Theorem 2.

**Theorem 8.** *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function in $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is relatively s-convex in the second sense with respect to a function $g:\mathbb{R}\to\mathbb{R}$ for some fixed $s\in(0,1]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right]$$

*for each $x\in[a,b]$.*

*Proof.* Let $q>1$ . From lemma 1 and using the Hölder inequality, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(g(x)-a)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)a)|dt$$
$$+\frac{(g(x)-b)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)b)|dt$$
$$\leq\frac{(g(x)-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tg(x)+(1-t)a)|\,dt\right)^{1/q}$$
$$+\frac{(g(x)-b)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tg(x)+(1-t)b)|^q\right)^{1/q},$$

where $1/p+1/q=1$.

Since $|f'|^q$ is relatively s-convex in the second sense with respect to a function $g$ and $|f'(x)|\leq M$, then

$$\int_0^1|f'(tg(x)+(1-t)a)|^q dt$$
$$\leq\int_0^1\left(t^s|f'(g(x))|^q+(1-t)^s|f'(a)|^q\right)dt$$
$$\leq\frac{2M^q}{s+1}$$

and

$$\int_0^1|f'(tg(x)+(1-t)b)|^q dt$$
$$\leq\int_0^1\left(t^s|f'(g(x))|^q+(1-t)^s|f'(b)|^q\right)dt$$
$$\leq\frac{2M^q}{s+1}.$$

Therefore, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right].$$

The proof is complete.

*Remark.* If in Theorem (8) we let $g(x)=x$, we obtain the result of Theorem (3).

**Theorem 9.** *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function in $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is relatively s-convex with respect to a function $g:\mathbb{R}\to\mathbb{R}$ for some fixed $s\in(0,1]$, $q\geq1$, and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{2(b-a)}\right]$$

*for each $x\in[a,b]$.*

*Proof.* Let $q>1$ from Lemma 1, and using the power mean inequality, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(g(x)-a)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)a)|dt$$
$$+\frac{(g(x)-b)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)b)|dt$$
$$\leq\frac{(g(x)-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t|f'(tg(x)+(1-t)a)|^q\right)^{1/q}$$
$$+\frac{(g(x)-b)^2}{b-a}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t|f'(tg(x)+(1-t)b)|^q\right)^{1/q}$$

Since $|f'|^q$ is relatively s-convex in the second sense with respect to a function $g$ and $|f'(x)|\leq M$, then we have

$$\int_0^1 t|f'(tg(x)+(1-t)a)|^q dt$$
$$\leq\int_0^1\left(t^{s+1}|f'(g(x))|^q+t(1-t)^s|f'(a)|^q\right)dt$$
$$\leq\frac{M^q}{s+1}$$

and

$$\int_0^1 t|f'(tg(x)+(1-t)b)|^q dt$$
$$\leq\int_0^1\left(t^{s+1}|f'(g(x))|^q+t(1-t)^s|f'(b)|^q\right)dt$$
$$\leq\frac{M^q}{s+1}.$$

Therefore, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq M\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right]$$
$$=M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(b-a)}\right].$$

The proof is complete.

*Remark.* If in Theorem (9) we take $g(x)=x$, we obtain the result of Theorem (4).

The next result is known as the inequality of Hermite-Hadamard for functions relatively $s$-convex in the second sense.

**Theorem 10.** *Let $f:K_g\subset\mathbb{R}\to\mathbb{R}$ be a relatevily $s$-convex in the second sense, then we have*

$$2^{s-1}f\left(\frac{a+g(b)}{2}\right)\leq\frac{1}{g(b)-a}\int_a^{g(b)}f(u)du\leq\frac{f(a)+f(g(b))}{s+1}.$$

*If $f$ is relatively $s$-concave in the second sense, then the inequalities holds in the opposite sense.*

The proof of that theorem can be found in [8].

**Theorem 11.** *Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function in $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is relatively $s$-concave with respect to a function $g:\mathbb{R}\to\mathbb{R}$ for some fixed $s\in(0,1]$, $q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{2^{(s-1)/q}}{(1+p)^{1/q}(b-a)}\left[(g(x)-a)^2\left|f'\left(\frac{a+g(x)}{2}\right)\right|\right.$$
$$\left.+\ (g(x)-b)^2\left|f'\left(\frac{b+g(x)}{2}\right)\right|\right]$$

*for each $x\in[a,b]$*

*Proof.* Suppose that $q>1$ from Lemma 1, and using the Hölder inequality, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(g(x)-a)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)a)|dt$$
$$+\frac{(g(x)-b)^2}{b-a}\int_0^1 t|f'(tg(x)+(1-t)b)|dt$$
$$\leq\frac{(g(x)-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tg(x)+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(g(x)-b)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tg(x)+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

Since $|f'|^q$ is relatively $s$-concave in the second sense with respect to a function $g$, using the theorem (10), we have

$$\int_0^1|f'(tg(x)+(1-t)a)|^q dt\leq 2^{s-1}\left|f'\left(\frac{g(x)+a}{2}\right)\right|^q$$

and

$$\int_0^1|f'(tg(x)+(1-t)b)|^q dt\leq 2^{s-1}\left|f'\left(\frac{g(x)+b}{2}\right)\right|^q.$$

Therefore, we have

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{2^{(s-1)/q}}{(1+p)^{1/q}(b-a)}\left[(g(x)-a)^2\left|f'\left(\frac{a+g(x)}{2}\right)\right|\right.$$
$$\left.+(g(x)-b)^2\left|f'\left(\frac{b+g(x)}{2}\right)\right|\right].$$

The proof is complete.

## 4 Some Consequences

**Corollary 1.** *Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ a differentiable function in $I^\circ$ such that $f'\in\mathscr{L}[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is relatively convex with respect to a function $g:\mathbb{R}\to\mathbb{R}$ , $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\left|f(g(x))-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{M}{(b-a)}\left[\frac{(g(x)-a)^2+(g(x)-b)^2}{(1+p)^{1/p}}\right]$$

*for each $x\in[a,b]$.*

*Proof.* If in Theorem 8 we let $s=1$, we obtain the desired result.

**Corollary 2.** *If in Theorem 9 we choose the function $g(x)=\frac{a+b}{2}$, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M(b-a)}{4}\left(\frac{2}{s+1}\right)^{1/q}$$

*$q\geq 1$, where $s\in(0,1]$ and $|f'|^q$ is relatively $s$-convex in the second sense with respect to a function $g:\mathbb{R}\to\mathbb{R}$.*

**Corollary 3.** *If in (11) we choose $s=1$ and $g(x)=\frac{a+b}{2}$. then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{b-a}{4(1+p)^{1/q}}\left[\left|f'\left(\frac{3a+d}{2}\right)\right|+\left|f'\left(\frac{3b+a}{2}\right)\right|\right],$$

*where $|f'|^q$ is relatively $s$-concave in the second sense with respect to a function $g:\mathbb{R}\to\mathbb{R}$.*

## 5 Conclusions

In the development of this work we have established an Ostrowski-type inequality for functions whose derivative module is relatively convex, also we give a new version of a result obtained by M. Alomari and M. Darus. Using the relative $s-$convexity with respect to a given function we obtain some others results of the same type. With these results we obtain generalizations of results found by M. Alomari et. al. using convex and $s-$convex in the second sense ([1,2]). We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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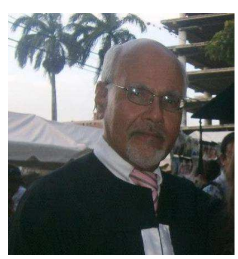
**Miguel J. Vivas C.** earned his Ph.D. degree from Universidad Central de Venezuela, Caracas, Distrito Capital (2014) in the field Pure Mathematics (Nonlinear Analysis), and earned his Master Degree in Pure Mathematics in the area of Differential Equations (Ecological Models). He has vast experience of teaching and research at university levels. It covers many areas of Mathematical such as Inequalities, Bounded Variation Functions and Ordinary Differential Equations. He has written and published several research articles in reputed international journals of mathematical and textbooks. He was Titular Professor in Decanato de Ciencias y Tecnología of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela, and invited Professor in Facultad de Ciencias Naturales y Matemáticas from Escuela Superior Politécnica del Litoral (ESPOL), Guayaquil, Ecuador, actually is Principal Professor and Researcher in Pontificia Universidad Católica del Ecuador. Sede Quito, Ecuador.

**Carlos Eduardo García.** obtained his Master of the University Centroccidental Lisandro Alvarado, State Lara of Barquisimeto, in the field of the mathematical pure (ergodic theory ). A student of PhD from the Universidad Centroccidental Lisandro Alvarado. Currently is Professor in the Decanato de Ciencias y Tecnologa de la University Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, State Lara, Venezuela. His field of interest is the ergodic theory and the nonlinear analysis (convexity generalized).



**Jorge E. Hernández H.** earned his M.Sc. degree from Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Estado Lara (2001) in the field Pure Mathematics (Harmonic Analysis). He has vast experience of teaching at university levels. It covers many areas of Mathematical such as Mathematics applied to Economy, Functional Analysis, Harmonic Analysis (Wavelets). He is currently Associated Professor in Decanato de Ciencias Económicas y Empresariales of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela.