Inequalities for Power Series

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Abstract: The aim of this paper is to give several inequalities for power series starting from a generalization of Young’s inequality for sequences of complex numbers. Then some inequalities deduced from some variants of the arithmetic-geometric mean inequality will be given. Thus by Theorem 1, Theorem 2 and Theorem 3 several refinements of Young’s inequality for functions defined by power series with real coefficients are given and by Theorem 4 a generalization of a sharp Hölder’s inequality for functions defined by power series with real coefficients is presented. Then a generalization of Young’s inequality for \(m\) pair of complex numbers in the case of the functions defined by power series is given in Remark 1, and a variant of Muirhead’s inequality for functions defined by power series with real coefficients is given in Proposition 3. There are a lot of examples related to some fundamental complex functions such as the exponential, logarithm, trigonometric and hyperbolic functions and also there are applications for some special functions such as polylogarithm, hypergeometric and Bessel functions for the first kind. Finally, we present an application related to the average information.

Keywords: Power series, Young’s inequality, Muirhead’s inequality, arithmetic-geometric mean inequality

1 Introduction

Power series is a special type of series of a function. The applications to power series can be found in mathematics, computer science, physics and in information theory. We will study the power series related to inequalities. Using a refinement of the Cauchy-Bunyakovsky-Schwarz inequality, Cerone and Dragomir in [12], established some inequalities concerning functions defined by convergent power series with real or nonnegative coefficients. The technique to find other inequalities of functions using power series was given by Ibrahim and Dragomir in [3], Mortici in [11] and Ibrahim, Dragomir and Darus in [4]. This method is important because can be improved and extended some of the known inequalities, which have applications in many fields.

We consider an analytic function defined by the power series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

with real coefficients and convergent on the disk \(D(0,R), R > 0\). As in [4] the weighted version of Hölder’s inequality can be stated as below:

\[ |f(xy)| = \left| \sum_{n=0}^{\infty} a_n x^n y^n \right| \leq \left( \sum_{n=0}^{\infty} |a_n| |x|^n |y|^n \right)^{\frac{1}{p}} \left( \sum_{n=0}^{\infty} |a_n| x^n y^n \right)^{\frac{1}{q}} \]

\[ = f^\frac{1}{p}_A (|x|^{p}) f^\frac{1}{q}_A (|y|^{q}) \]

for any \(x, y \in \mathbb{C}\) with \(|x|^{p}, |y|^{q} \in D(0,R)\) and \(f_A(z)\) is a power series defined by \(\sum_{n=0}^{\infty} |a_n| z^n\). The power series \(f_A(z)\) have the same radius of convergence as the original power series \(f(z)\).

In the case when all coefficients of the series \(f(z)\) are positive we have \(f(z) = f_A(z)\).

Next, we present several results related to inequalities, that will be useful in our study.

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We consider the following inequality, which represents an improvement of Young’s inequality:

**Lemma 1. ([8])** For \(0 < a, b \leq 1\) and \(\lambda \in (0, 1)\) we have
\[
(r(\sqrt{a} - \sqrt{b})^2 + A(\lambda)ab \log^2 \left( \frac{a}{b} \right)) \leq \lambda a + (1 - \lambda)b - a^{\frac{1}{\lambda}}b^{1 - \lambda}
\]
where \(r = \min\{\lambda, 1 - \lambda\}, A(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{\lambda}{4}\), and \(B(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{\lambda}{4}\).

If we take here \(\lambda = \frac{1}{2}\) and replace \(a^{\frac{1}{\lambda}}\) by \(a\) and \(b^{1 - \lambda}\) by \(b\) then \(1 - \lambda = \frac{1}{2}\) and we obtain:
\[
ab + r(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + A\left(\frac{1}{p}\right)a^p b^q \log^2 \left( \frac{a^p}{b^q} \right) \leq \frac{a^p}{p} + \frac{b^q}{q}
\]
\[
\leq ab + (1 - r)(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + B\left(\frac{1}{p}\right)a^p b^q \log^2 \left( \frac{a^p}{b^q} \right). \quad (1)
\]

We also need the inequality from below which is given in \([5]\), Lemma 2.

**Lemma 2.** For \(a_{ij} \geq 0\), \(p_j > 0\), \(i \in \{1, 2, \ldots, n\}\) and \(j \in \{1, 2, \ldots, m\}\) such that \(\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \geq 1\) we have
\[
\sum_{i=1}^n a_{i1} a_{i2} \ldots a_{im} \leq \left( \sum_{i=1}^n a_{i1}^{p_1} \right)^{\frac{1}{p_1}} \left( \sum_{i=1}^n a_{i2}^{p_2} \right)^{\frac{1}{p_2}} \ldots \left( \sum_{i=1}^n a_{im}^{p_m} \right)^{\frac{1}{p_m}}.
\]

Next inequality is given in \([2]\), Proposition 5.1 and will be used in Theorem 4.

**Proposition 1. ([2])** Let \(a_1, \ldots, a_n \geq 0\) and \(p_1, \ldots, p_n \geq 0\) with \(\sum_{j=1}^n p_j = 1\) we have
\[
\sum_{i=1}^n p_i a_i - a_{P_1}^{p_1} \ldots a_{P_n}^{p_n} \geq n \lambda \left( \frac{1}{n} \sum_{i=1}^n a_i - a_{P_1} \ldots a_{P_n} \right),
\]
with equality if and only if \(a_1 = \ldots = a_n\), where \(\lambda = \min\{p_1, \ldots, p_n\}\).

Using the above results in this paper we give by Theorem 1, Theorem 2 and Theorem 3 several refinements of Young’s inequality presented in \([4]\) for functions defined by power series with real coefficients and by Theorem 4 a generalization of a sharp Hölder’s inequality for functions defined by power series with real coefficients is presented. Then motivated by some results from \([6, 7]\), a generalization of Young’s inequality for \(m\) pair of complex numbers in the case of the functions defined by power series is given in Remark 1, and a variant of Muirhead’s inequality for functions defined by power series with real coefficients is given in Proposition 3.

These results are important due to their applications for special functions such as polylogarithm, hypergeometric and Bessel and modified Bessel functions for the first kind. Moreover, in information sciences, many applications of Hölder’s inequality have also been studied by many authors as \([22]\). In section 3 an application related to the average information is presented.

## 2 Main results

The three results were obtained using a refinement of Young’s inequality given in \([8]\) for two positive numbers \(a\) and \(b\) in \((0, 1)\) for power series with real coefficients, and the same method as in \([4]\), Theorem 1, 2 and 3.

**Theorem 1.** Let \(f(z) = \sum_{n=0}^m p_n z^n\); \(g(z) = \sum_{n=0}^m q_n z^n\) be the power series with real coefficients and convergent on the open disk \(D(0, R), 0 < R < 1\). If \(p, q\) are real numbers with \(p > 1\), \(\frac{1}{p} + \frac{1}{q} = 1\) and \(a, b \in \mathbb{C}, a, b \neq 0\), \(|a| < 1, |b| < 1\) so that \(|ab|, |a|^p, |a|^q, |b|^p, |b|^q, |a|^p|b|^q, |a|^q|b|^p\) are real numbers and \(|a|^q|b|^p \in D(0, R)\), then we have
\[
|f(ab)|g(\|ab\|) + rM_1 + A\left(\frac{1}{p}\right)T_1 \leq \frac{1}{p} f_A(|a|^q)g_A(|ab|) + rM_1 + A\left(\frac{1}{p}\right)T_1 \leq \frac{1}{p} f_A(|a|^p)g_A(|b|^p) + \frac{1}{q} f_A(|b|^q)g_A(|a|^q) \quad (2)
\]
\[
\leq f_A(|a||b|)g_A(|ab|) + (1 - r)M_2 + B\left(\frac{1}{p}\right)T_1,
\]
and
\[
|f(ab)|g_A(|a| |b|^{q-1})| + rM_2 + A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2 \leq f_A(|a||b|^{q-1})g_A(|a| |b|^{q-1}) + rM_2 + A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2 \leq \frac{1}{p} f_A(|a|^p)g_A(|b|^q) + \frac{1}{q} f_A(|b|^p)g_A(|a|^q) \leq f_A(|a|^q|b|^p)g_A(|a|^q) + (1 - r)M_2 + B\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2,
\]
if in this case \(|a| < |b|, \quad |a||b|^{p-1}, |a||b|^{q-1}, |a|^p, |a|^q, |b|^p, |b|^q, |a|^q|b|^p, |a|^q|b|^q \in D(0, R)\), where
\[
M_1 = f_A(|a|^p)g_A(|b|^p) + f_A(|b|^q)g_A(|a|^q) - 2f_A(|a|^q|b|^p)g_A(|a|^q) - 2f_A(|a|^q|b|^q)g_A(|a|^q),
\]
Denoting by $P_1$ the quantity
\[
\sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| \log^2 \left( \frac{|a|^{p-k} |b|^{q-j}}{|b|^{q-k} |a|^{p-j}} \right) \leq \left( |a|^{p} |b|^{q} \right)^k \left( |a|^{q} |b|^{p} \right)^k
\]
by computation we have,
\[
P_1 = \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| |(j - k)| \log |a| - (j - k) \log |b| = \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| |(j - k)| \log |a| - (j - k) \log |b|
\]
for any $j, k \in \{0, 1, 2, \ldots, n\}$. All the series whose partial sums which appear here in inequality (4) are convergent on the disk $D(0, R)$ therefore we can take the limit when $n$ tends to $\infty$ in (4) and obtain the inequality (2) taking into account that because $T_1$ is the limit when $n$ tends of $\infty$ of $P_1$.

In the second case, if we replace in (1) $a$ by $|a^j| |b^k|$ and $b$ by $|a^k| |b^j|$ then we have
\[
\sum_{j=0}^{n} |p_j| |a^j| |b^k| \sum_{k=0}^{n} |q_k| |a^k| |b^j| + r \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| |a^j| |b^k|^{p-k} + \frac{1}{p} \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| \log^2 \left( \frac{|a|^{p-k} |b|^{q-j}}{|b|^{q-k} |a|^{p-j}} \right) \leq \left( |a|^{p} |b|^{q} \right)^k \left( |a|^{q} |b|^{p} \right)^k
\]
for any $j, k \in \{0, 1, 2, \ldots, n\}$. If we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over $j$ and $k$ from 0 to $n$, we obtain
\[
\sum_{j=0}^{n} |p_j| |a^j| |b^k| \sum_{k=0}^{n} |q_k| |a^k| |b^j| + r \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| |a^j| |b^k|^{p-k} + \frac{1}{p} \sum_{j=0}^{n} |p_j| \sum_{k=0}^{n} |q_k| \log^2 \left( \frac{|a|^{p-k} |b|^{q-j}}{|b|^{q-k} |a|^{p-j}} \right) \leq \left( |a|^{p} |b|^{q} \right)^k \left( |a|^{q} |b|^{p} \right)^k
\]
\[-2|a|^j\frac{1}{q_j}b^j + |a|^j b^j + kn + A(1)\log^2 \left( \frac{|a|^{j+q_k}}{|b|^{j+q_k}} \right) \leq \frac{1}{p} |a|^j b^{q_k} + \frac{1}{q} |q_k| |b|^{q_k} \leq (6)\]

\[
\leq |a| |b|^j |b|^{q_k} + (1 - r)|a|^j |b|^{q_k} + |a|^q |b|^{j+q_k} - 2|a|^j b^j + B(1)|p| \log^2 \left( \frac{|a|^{j+q_k}}{|b|^{j+q_k}} \right) |a|^j |b|^{q_k} \]

for any \( j, k \in \{0, 1, 2, \ldots, n\} \).

Now we multiply (6) by \( |p_j||q_k| \geq 0 \), \( j, k \in \{0, 1, 2, \ldots, n\} \) and summing over \( j \) and \( k \) from 0 to \( n \), we have

\[
\sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |a||b|^{q_k} - 2|a|^j b^j + B \left( \frac{1}{p} \right) \log^2 \left( \frac{|a|^{j+q_k}}{|b|^{j+q_k}} \right) |a|^j |b|^{q_k} \leq (7)\]

In this case

\[
P_2 = \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |p_j| |q_k| |j| |p_j| |b|^k q^2 - 2pqjk |a|^j |b|^{q_k} = \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |j| |p_j| |b|^k q^2 - 2pqjk |a|^j |b|^{q_k}.

Taking into account that all the series whose partial sums are involved in previous inequality are convergent on the disk \( D(0, R) \), and letting \( n \to \infty \) in the inequality (7), we notice that the desired inequality (3) takes place, because \( T_2 \) is the limit when \( n \) tends of \( \infty \) of \( P_2 \). \( \Box \)

**Theorem 2.** Let \( f(z) = \sum_{n=0}^{\infty} p_n z^n \), \( g(z) = \sum_{n=0}^{\infty} q_n z^n \) be the power series with real coefficients and convergent on the open disk \( D(0, R) \), \( 0 < R < 1 \). If \( p, q \) are real numbers with \( p > 1, \frac{1}{q} + \frac{1}{q} = 1 \) and \( a, b \in C \), \( a, b \neq 0 \), \( |a| < 1, |b| < 1 \) such that \( |a||b|, |a|^q, |b|^p, |b|^2, |a|^k |b|^q, |a|^q |b|^j \in D(0, R) \), then we have

\[
|f(ab)g(|a|^2 |b|^2)| + rM_3 + A(1)pT_3 \leq \left( \frac{1}{p} \right) f_a(|a|^2 |b|^2) + rM_3 + A(1)pT_3 \leq \frac{1}{p} f_a(|a|^2 |b|^2)g_A(|a|^2 |b|^2) \leq (8)\]

\[
\leq f_a(|a||b|)g_A(|a|^2 |b|^2) + (1 - r)M_3 + B(1)pT_3, \]

where

\[
M_3 = f_a(|a|^2 |b|^2)g_A(|a|^2 |b|^2) - 2f_a(|a|^2 |b|^2)g_A(|a||b|),
\]

\[
T_3 = 4\log \left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_1(|a|^2 |b|^2) + \log \left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_2(|a|^2 |b|^2) + 4\log \left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_3(|a|^2 |b|^2).
\]

**Proof.**

Now, we replace \( a \) by \( |a|^k |b|^j \), and \( b \) by \( |a|^j |b|^k \) in inequality (1), we multiply by \( |p_j||q_k| \geq 0 \) and then summing over \( j \) and \( k \) from 0 to \( n \) we get

\[
\sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |a|^k |b|^j |a|^j |b|^k \leq \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |q_j| |j| |p_j| |b|^k q^2 - 2pqjk |a|^j |b|^{q_k} = \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |j| |p_j| |b|^k q^2 - 2pqjk |a|^j |b|^{q_k}.
\]

\[
\leq \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |j| |p_j| |b|^k q^2 - 2pqjk |a|^j |b|^{q_k} \leq (9)\]

\[
+ (1 - r) \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |a|^k |b|^j |a|^j |b|^k - 2|a|^k |b|^j |a|^j |b|^k + B \left( \frac{1}{p} \right) \sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| |a|^k |b|^j |a|^j |b|^k
\]

\[
\left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_1(|a|^2 |b|^2) + \left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_2(|a|^2 |b|^2) + 4\log \left( \frac{|a|^2 |b|^2} {4} \right) f_a(|a|^2 |b|^2) S_3(|a|^2 |b|^2) \]

where \( P_3 \) is the quantity

\[
\sum_{j=0}^{n} \sum_{k=0}^{n} \left| p_j \right| \left| q_k \right| \log^2 \left( \frac{|a|^2 |b|^j |a|^j |b|^k} {4} \right) |a|^2 |b|^j |a|^j |b|^k.
\]
By computation, we find,

\[ P_3 = \sum_{j=0}^{n} \sum_{k=0}^{n} |p_j| |q_k| \log^2 \left( \left| a \right|^{\frac{2j-k}{p}} \left| b \right|^{-j} \right) |a|^{2k}|b|^{|n|} |a|^{|q|} |b|^{2k} = \]

\[ = \sum_{j=0}^{n} \sum_{k=0}^{n} |p_j| |q_k| (2k \log |a| - jk \log |a| + jk \log |b|^2 - 2a^{2k}b^{|n|} |a|^{|q|} |b|^{2k} = \]

\[ = \sum_{j=0}^{n} \sum_{k=0}^{n} |p_j| |q_k| (4k \log^2 |a|/\log |b|^2 + j^2 \log^2 |a|^{|q|} + 4jk \log |a| |\log |b|^2 |a|^{2k}b^{|n|} |a|^{|q|} |b|^{2k}. \]

Since all the series whose partial sums are involved in the inequality (9) are convergent on the disk \( D(0,R) \), letting \( n \) tend to \( \infty \) in (9), we deduce the desired inequality, because \( T_3 \) is the limit when \( n \) tends to \( \infty \) of \( P_3 \). □

**Theorem 3.** Let \( f(z) \) and \( g(z) \) be as in Theorem 1. If \( |a|^2 |b|^{|p|} |a|^{|q|} |b|^{2} \) \( D(0,R) \) then one has the following inequality

\[ \left| f(a^{\frac{1}{p}}b)g(a^{\frac{1}{q}}b) \right| + rM_4 + A(\frac{1}{p})T_4 \leq \]

\[ \leq f_3(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) + rM_4 + A(\frac{1}{p})T_4 \leq \]

\[ \frac{-1}{p} f_3(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) + \frac{1}{q} f_3(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) \leq \]

\[ \leq f_A(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) + (1-r)M_4 + B(\frac{1}{p})T_4, \]

where

\[ M_4 = f_3(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) + f_3(a^{\frac{1}{q}}b)g_3(a^{\frac{1}{q}}b) - \]

\[ -2f_3(a|\frac{1}{q}|b^{\frac{1}{q}})g_3(a|\frac{1}{q}|b^{\frac{1}{q}}), \]

\[ T_4 = \log^2 \left( \left| a^{\frac{1}{q}}b^{\frac{1}{q}} \right|^2 \right) g_3(a^{\frac{1}{q}}b^{\frac{1}{q}})S_1(a^{\frac{1}{q}}b^{\frac{1}{q}}) + \]

\[ + \log^2 \left( \left| a^{\frac{1}{q}}b^{\frac{1}{q}} \right|^2 \right) f_3(a^{\frac{1}{q}}b^{\frac{1}{q}})S_2(a^{\frac{1}{q}}b^{\frac{1}{q}}) + \]

\[ + 2S_3(a^{\frac{1}{q}}b^{\frac{1}{q}})S_4(a^{\frac{1}{q}}b^{\frac{1}{q}}) \log \left( \left| a^{\frac{1}{q}}b^{\frac{1}{q}} \right|^2 \right). \]

**Proof.** Using again the inequality (1) with \( |a|^{\frac{1}{p}} |b|^k \) instead of \( a \) and \( |a|^{\frac{1}{q}} |b|^l \) instead of \( b \) we obtain for any \( j,k \in \{0,1,2,...,n\} \) the following inequality

\[ (|a|^{\frac{1}{p}} |b|^l) (|b|^{\frac{1}{q}} |a|^l)^k + r(|a|^2 |b|^k + |a|^{2k} |b|^q - \]

\[ -2|a|^l |b|^k |a|^l |b|^q + \]

\[ + A(\frac{1}{p}) \log^2 \left( \left| a^{\frac{1}{p}} |b|^k \right|^2 \right) \left( |a|^2 |b|^q \right) \left( |a|^{2k} |b|^p \right)^q \leq \]

\[ \leq \frac{1}{p} |a|^{2j} |b|^p + \frac{1}{q} |a|^{2k} |b|^q \leq \]

\[ \leq (|a|^{\frac{1}{p}} |b|^l) (|b|^{\frac{1}{q}} |a|^l)^k + (1-r) (|a|^2 |b|^k + |a|^{2k} |b|^q - \]

\[ -2|a|^l |b|^k |a|^l |b|^q + \]

\[ + B(\frac{1}{p}) \log^2 \left( \left| a^{\frac{1}{p}} |b|^k \right|^2 \right) \left( |a|^2 |b|^q \right) \left( |a|^{2k} |b|^p \right)^q. \]

By the same method as in Theorem 1 we find the desired inequality. □

**Remark 1.** Let \( r_1, r_2, ..., r_m \neq 0 \) be real numbers such that \( \frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_m} = 1 \) and \( f(z) = \sum_{n=0}^{\infty} p_n z^n \), \( g(z) = \sum_{n=0}^{\infty} q_n z^n \) be the power series with real coefficients and convergent on the open disk \( D(0,R), 0 < R \). If \( a_1, a_2, ..., a_m, b_1, b_2, ..., b_m \in \mathbb{C} \), such that \( a_1 a_2 ... a_m, b_1 b_2 ... b_m, |a_1|^r, |b_1|^r \in D(0,R), i \in \{1,2,...,m\} \) then we have

\[ |f(a_1 a_2 ... a_m)g(b_1 b_2 ... b_m)| \leq \]

\[ \leq f_A(|a_1|^r |a_2|^r ... |a_m|^r)g_b(|b_1|^r |b_2|^r ... |b_m|^r) \leq \]

\[ \leq \frac{1}{r_1} f_A(|a_1|^r)g_A(|b_1|^r) + \frac{1}{r_2} f_A(|a_2|^r)g_A(|b_2|^r) + ... + \]

\[ + \frac{1}{r_m} f_A(|a_m|^r)g_A(|b_m|^r). \]

**Proof.** We use the well-known inequality

\[ \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m \geq \alpha_1 x_1^2 x_2^2 ... x_m^2 \]

which takes place for any \( x_1, x_2, ..., x_m > 0 \) and \( \alpha_1, \alpha_2, ..., \alpha_m \) real numbers such that \( \alpha_1 + \alpha_2 + ... + \alpha_m = 1 \) and replacing \( \alpha_i \) by \( \frac{r_i}{m} \) and \( x_i^{\frac{1}{r_i}} \) by \( x_i \) we obtain

\[ \frac{1}{r_1} x_1^r + \frac{1}{r_2} x_2^r + ... + \frac{1}{r_m} x_m^r \geq x_1 x_2 ... x_m. \]

Taking above \( x_1 = |a_1|^r |b_1|^k, x_2 = |a_2|^r |b_2|^k, ..., x_m = |a_m|^r |b_m|^k \) for \( j,k \in \{0,1,2,...,n\} \) and using the same method like in Theorem 1 we find the desired inequality. □

**Proposition 2.** Let \( a_j \) be complex numbers and \( p_j > 0, j \in \{1,2,...,m\} \) such that \( \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} \geq 1 \). If \( f(z) = \sum_{n=0}^{\infty} p_n z^n \) is the power series with real coefficients and convergent on the open disk \( D(0,R), 0 < R \) and \( a_1 a_2 ... a_m, |a_1|^p_1 |a_2|^p_2 ... |a_m|^p_m \in D(0,R) \), and \( p_j \geq 1 \) for all \( i \in \mathbb{N} \) then the following inequality holds:

\[ |f(a_1 a_2 ... a_m)| \leq f_A(|a_1|^p_1 |a_2|^p_2 ... |a_m|^p_m) \]

\[ \leq \]
\[ \leq \frac{1}{p_1} (|a_1|^{p_1}) \frac{1}{p_2} (|a_2|^{p_2}) \cdots \frac{1}{p_m} (|a_m|^{p_m}). \]

**Proof.** If we consider \( a_{ij} = |p_j|^{\frac{1}{p_j}} |a_i|^{|1|} \) with \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m\} \) in Lemma 2, see [5] page 743, the inequality

\[ \sum_{i=1}^{n} a_1 a_2 \cdots a_m \leq \left( \sum_{i=1}^{n} |a_i|^{p_1} \right)^{\frac{1}{p_1}} \left( \sum_{i=1}^{n} |a_i|^{p_2} \right)^{\frac{1}{p_2}} \cdots \left( \sum_{i=1}^{n} |a_i|^{p_m} \right)^{\frac{1}{p_m}} \]

becomes:

\[ \sum_{i=1}^{n} |p_i|^{\frac{1}{p_1}} |a_1|^{|1|} |a_2|^{|2|} \cdots |a_m|^{|m|} \leq \left( \sum_{i=1}^{n} |p_i||a_i|^{p_1} \right)^{\frac{1}{p_1}} \left( \sum_{i=1}^{n} |p_i||a_i|^{p_2} \right)^{\frac{1}{p_2}} \cdots \left( \sum_{i=1}^{n} |p_i||a_i|^{p_m} \right)^{\frac{1}{p_m}} \]

or

\[ \sum_{i=1}^{n} |p_i||a_1|^{|1|} |a_2|^{|2|} \cdots |a_m|^{|m|} \leq \left( \sum_{i=1}^{n} |p_i||a_i|^{p_1} \right)^{\frac{1}{p_1}} \left( \sum_{i=1}^{n} |p_i||a_i|^{p_2} \right)^{\frac{1}{p_2}} \cdots \left( \sum_{i=1}^{n} |p_i||a_i|^{p_m} \right)^{\frac{1}{p_m}} \]

Taking into account that \( a_1 a_2 \cdots a_m, |a_1|, |a_2|, \ldots, |a_m| \in D(0, R) \), when \( n \) tends to \( \infty \) we get inequality (11). \( \square \)

Using a refinement of the weighted arithmetic-geometric mean inequality for \( n \) real numbers, see [2], we find the following:

**Theorem 4.** Let \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0 \), \( p_1, p_2, \ldots, p_n > 0 \) with \( \sum_{j=1}^{n} p_j = 1 \) and \( \lambda = \min\{p_1, \ldots, p_n\} \). If we assume that the multiplicity attaining \( \lambda \) is 1, then we have the following inequality:

\[ \sum_{i=1}^{n} p_i f(a_i) g(b_i) = \left[ f(a_1 a_2^2 \cdots a_n^n) g(b_1 b_2^2 \cdots b_n^n) \right] \geq \sum_{i=1}^{n} p_i f(a_i) g(b_i) - \left[ f(a_1 a_2^2 \cdots a_n^n) g(b_1 b_2^2 \cdots b_n^n) \right] \]

\[ \geq n \lambda \left( \frac{1}{n} \sum_{i=1}^{n} f(a_i) g(b_i) - \left[ f(a_1 a_2^2 \cdots a_n^n) g(b_1 b_2^2 \cdots b_n^n) \right] \right), \]

where \( f, g, f_a, g_a \) are as in Theorem 1 and \( a_1 a_2^2 \cdots a_n^n, b_1 b_2^2 \cdots b_n^n \), \( a_1, b_1, \ldots, a_n, b_n \) are elements of \( D(0, R) \).

**Proof.** We replace \( a_i > 0 \) by \( a_i^{\lambda} b_i^{\lambda} \) for \( j, k \in \{1, 2, \ldots, m\}, i \in \{1, \ldots, n\} \) in inequality from below and write again this inequality

\[ \sum_{i=1}^{n} p_i a_i - a_1 a_2 \cdots a_n \geq n \lambda \left( \frac{1}{n} \sum_{i=1}^{n} a_i - a_1 a_2^2 \cdots a_n^n \right). \]

From Proposition 5.1 (Proposition 1), see [2] obtaining:

\[ \sum_{i=1}^{n} p_i a_i b_i = a_1 b_1 a_2 b_2 \cdots a_n b_n \geq \]

\[ \geq n \lambda \left( a_1 b_1 + \cdots + a_n b_n \right) - \left( a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \right) \]

which by multiplication by \( |p_j||a_i^{\lambda} b_i^{\lambda} g(b_i^n) \) and summing over \( j \) and \( k \) will give the desired inequality from conclusion when \( m \) tend to infinity. \( \square \)

For finite sequences of real numbers we use the majorization relation from [6]. Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) two finite sequences of real numbers. We say that the sequence \( a \) majorizes the sequence \( b \) and we write

\( a \gg b \) or \( b \ll a \),

if after rearranging of the sequence \( a \) and \( b \) satisfy the following three conditions:

\[ a_1 \geq a_2 \geq \cdots \geq a_n \quad \text{and} \quad b_1 \geq b_2 \geq \cdots \geq b_n \]

\[ a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k, \quad \text{for each} \ k, 1 \leq k \leq n-1; \]

\[ a_1 + a_2 + \cdots + a_n = a_1 + b_2 + \cdots + b_n. \]

As in [6], Definition 2, let \( F(x_1, x_2, \ldots, x_n) \) be a function in \( n \) nonnegative real variables. Define

\[ \sum_{1}^{n} F(x_1, x_2, \ldots, x_n) \]

as the sum of \( n! \) summands, obtained from the expression \( F(x_1, x_2, \ldots, x_n) \) as all the possible permutations of the sequence \( x = (x_i)_{i=1}^{n} \).

Particularly, if for some sequence of nonnegative exponents \( a = (a_i)_{i=1}^{n} \) the function \( F \) is of the form \( P(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \), then instead of

\[ \sum_{1}^{n} F(x_1, x_2, \ldots, x_n) \]

we shall write also

\[ T[a_1, a_2, \ldots, a_n](x_1, x_2, \ldots, x_n) \]

or just \( T[a_1, a_2, \ldots, a_n] \) if it is clear which is the sequence \( x \) used here.

Using the technique given in [3] for Muirhead’s theorem, we find the following inequality:

**Proposition 3.** If \( a \ll b \) and \( y_i, z_i \in \mathbb{C}, \quad y_i, z_i \neq 0, \quad i \in \{1, \ldots, n\} \)

then

\[ \sum_{1}^{n} f_a(|y_1|^{a_1} |y_2|^{a_2} \cdots |y_n|^{a_n}) g_a(|z_1|^{a_1} |z_2|^{a_2} \cdots |z_n|^{a_n}) \leq \]

\[ \frac{1}{n} \sum_{i=1}^{n} a_i - a_1 a_2^2 \cdots a_n^n \]
\[ \leq \sum f_A(|y_1|^{b_1}|y_2|^{b_2}|y_n|^{b_n}) g_A(|z_1|^{b_1}|z_2|^{b_2}|z_n|^{b_n}), \]

where \( f, g, f_A \) and \( g_A \) are as in Theorem 1, \( |y_\sigma(1)|^{b_1}|y_\sigma(2)|^{b_2}|...|y_\sigma(n)|^{b_n} \), \( |z_\sigma(1)|^{b_1}|z_\sigma(2)|^{b_2}|...|z_\sigma(n)|^{b_n} \) \( \in D(0,R) \) for any \( \sigma, \sigma \), being an arbitrary permutation of the numbers \( \{1,2,...,n\} \).

**Proof.** We consider in Muirhead’s inequality instead of \( x_i, |y_\sigma(i)|^{b_k}, i \in \{1,2,...,n\} \) we multiply by \( |q_i| |q_k| \), and summing over \( j, k \in \{0,...,m\} \) we get the desired inequality when \( m \) tends to infinity. \( \square \).

### 3 Applications related to the average information

1. Next, we present an application related to the average information for two messages.

   For \( n = 2 \) in Theorem 4, with \( f(z) = \sum_{n \geq 0} a_n c^n, g(z) = \sum_{n \geq 0} b_n c^n, a_n, b_n \geq 0 \), for all \( n = 1,2,... \), we deduce the following inequality:

   \[ \lambda f(a)g(c) + (1 - \lambda) f(b)g(d) - f(a^\lambda b^{1-\lambda})g(c^\lambda d^{1-\lambda}) \geq \]

   \[ \geq \min\{\lambda,1-\lambda\} [f(a)g(c) + f(b)g(d) - 2f(\sqrt{ab})g(\sqrt{cd})]. \quad (12) \]

   If we take \( g(x) = 1 \) in inequality (12), we obtain the inequality:

   \[ \lambda f(a) + (1 - \lambda) f(b) - f(a^\lambda b^{1-\lambda}) \geq \]

   \[ \geq \min\{\lambda,1-\lambda\} [f(a) + f(b) - 2f(\sqrt{ab})]. \quad (13) \]

   But, we have

   \[ f(a) + f(b) = \sum_{n \geq 0} a_n a^n + \sum_{n \geq 0} a_n b^n = \sum_{n \geq 0} a_n (a^n + b^n) \geq \]

   \[ \geq 2 \sum_{n \geq 0} (\sqrt{ab})^n = 2f(\sqrt{ab}), \]

   so we find the inequality

   \[ f(a) + f(b) = \sum_{n \geq 0} a_n a^n + \sum_{n \geq 0} a_n b^n = \sum_{n \geq 0} a_n (a^n + b^n) \geq \]

   \[ \geq 2 \sum_{n \geq 0} (\sqrt{ab})^n = 2f(\sqrt{ab}). \]

   Therefore \( \lambda f(a) + (1 - \lambda) f(b) - f(a^\lambda b^{1-\lambda}) \geq 0 \) for all \( \lambda \in [0,1] \) and \( 0 < a, b < 1 \), i.e. \( f \) is an GA-convex function.

   From Information Theory and Coding, let messages be \( m_1 \) and \( m_2 \) and they have probabilities of occurrence as \( p \) and \( 1 - p \). Suppose that a sequence of \( n \) messages is transmitted. If \( n \) is sufficiently large, then we say that \( np \) messages of \( m_1 \) are transmitted and \( n(1-p) \) messages of \( m_2 \) are transmitted.

   The information due to message \( m_1 \) will be \( I_1 = \log_2 \left( \frac{1}{p} \right) \) and the information due to message \( m_2 \) will be \( I_2 = \log_2 \left( \frac{1}{1-p} \right) \). Then the total information carried due to the sequence of \( n \) message will be

   \[ I = np I_1 + n(1-p) I_2 = \]

   \[ = n[p \log_2 \left( \frac{1}{p} \right) + (1-p) \log_2 \left( \frac{1}{1-p} \right)]. \]

   Average information is the ratio between \( I \) and \( n \), so is represented by Shannon entropy \( H \) given by

   \[ H = p \log_2 \left( \frac{1}{p} \right) + (1-p) \log_2 \left( \frac{1}{1-p} \right). \]

   This function denoted by \( \Omega(\cdot) \) is also called as Horseshoe function, where

   \[ \Omega(p) = p \log_2 \left( \frac{1}{p} \right) + (1-p) \log_2 \left( \frac{1}{1-p} \right). \]

   In inequality (13), we consider \( f(x) = \ln \left( \frac{1}{1-x} \right) = \ln 2 - \log_2 \left( \frac{1}{1-x} \right), \lambda = p, a = 1-p \) and \( b = p \).

   Therefore we obtain the following inequality

   \[ H = \ln 2 \cdot \Omega(p) \geq \]

   \[ \geq \ln \left( \frac{1}{1 - (1-p)^p p^{1-p}} \right) + r \ln \left( \frac{1 - \sqrt{p(1-p)}}{p(1-p)} \right) \geq \]

   \[ \geq \ln \left( \frac{1}{1 - (1-p)^p p^{1-p}} \right); \quad (14) \]

   where \( r = \min\{p,1-p\} \).

2. Using a similar method as in [19, 20] we present an application of such inequalities in Information Theory. For that we consider the inequality

   \[ AB + r(A^p + B^q - 2A^{\frac{p}{2}} B^{\frac{q}{2}}) \leq \]

   \[ \frac{1}{p} A^p + \frac{1}{q} B^q \leq AB + (1-r)(A^p + B^q - 2A^{\frac{p}{2}} B^{\frac{q}{2}}), \]

   where \( r \) is as in [8], inequality (1.5). In this inequality we put, as in the proof of the classical H"{o}lder’s inequality,

   \[ A = \frac{a_i}{b_i} \] \( \text{and} \) \( B = \frac{b_i}{a_i} \) \( \text{where} \)

   \[ a_i, b_i > 0, i \in \{1,2,...,n\} \] \( \text{here} \) \( p > 1 \) and summing over \( i \) from 1 to \( n \) we get

   \[ \sum_{i=1}^{n} a_i b_i \]

   \[ \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} + \]

   \[ + 2r \left[ 1 - \frac{\left( \sum_{i=1}^{n} \frac{a_i}{b_i} \frac{b_i}{a_i} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{2}}}{\left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}} \right] \leq 1 \]

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and the exponential "useful" mean lengths of codewords weighted with the function of power probabilities and utilities is defined as

$$\frac{\alpha}{1 - \alpha} \sum_{k=1}^{n} p_k^\beta \left( \frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) ^{1 - \frac{1}{\alpha}} D^{-(1-\alpha)\beta_i}.$$ 

The last inequality is a generalization of Shannons inequality.

**Theorem 5.** Let $\alpha > 0$, $\beta > 0$, $\alpha, \beta \neq 1$, $p_k > 0$, $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} p_k^\beta = 1$, let $D(D \geq 2)$ is the size of the code alphabet. If $h_k, k = 1, 2, 3, \ldots, n$ are the average length of the codewords satisfying $\sum_{k=1}^{n} D^{-h_k} \leq 1$ then for every uniquely decipherable code, the "useful" $\alpha$-average length of codewords satisfies

$$\frac{\alpha}{1 - \alpha} \log_D \left( \sum_{k=1}^{n} p_k^\beta \left( \frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) ^{1 - \frac{1}{\alpha}} D^{-(1-\alpha)\beta_i} \right) \geq \log_2 \sum_{k=1}^{n} \left( \frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) \left( \frac{p_k^\beta u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) \left( \frac{D^{-h_k}}{\sum_{i=1}^{n} D^{-h_i}} \right) D^{-h_k},$$

where

$$M(p_k, u_k; \alpha) = \sum_{k=1}^{n} \left( \frac{p_k^\beta u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) D^{-h_k}.$$ 

**Proof.** Using the substitution $r = \frac{\alpha - 1}{\alpha}, s = 1 - \alpha < 0$, $h_k = p_k^\beta \left( \frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) ^{1 - \frac{1}{\alpha}} D^{-l_k}$ and $u_k = p_k^\beta \left( \frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i} \right) ^{1 - \frac{1}{\alpha}}$ in (15) and after suitable calculus we obtain inequality (16) when $\alpha > 1$.

### 4 Conclusions

This paper has proposed several inequalities concerning functions defined by convergent power series with real or nonnegative coefficients. This method is useful, because many difficult inequalities can be easily solved and often they can be extended.
As in [4], there exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that \( L_n(z) \), \( {}_2F_1(a,b;c;z) \), \( J_n(z) \) and \( I_n(z) \) are power series with real coefficients and convergent on the open disk \( D(0,1) \). Therefore, like in [4], we can think to rewrite the inequalities given before under conditions from our theorems.

In addition, as in [4], because the functions \( \exp(z), z \in \mathbb{C}, 1/z \in D(0,1) \), \( \ln(1/z), z \in D(0,1) \), \( \sinh(z), z \in \mathbb{C} \) are power series with real coefficients and convergent on the open disk \( D(0,1) \) we can think to rewrite the inequalities given before under conditions from our theorems.

Also many inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions can be found in the literature, see [13,14,15,16,17,18] and references therein, and on the other hand it is wellknown that the power series and special functions have important applications in engineering sciences and applied mathematics as parts of Information Sciences, therefore new questions will arise from new applications.

Moreover, in Information Theory appear many inequalities and concepts such as Singh’s inequality ([21]) and Shannon entropy which can be obtained from our theorems. There exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that \( L_n(z) \), \( {}_2F_1(a,b;c;z) \), \( J_n(z) \) and \( I_n(z) \) are power series with real coefficients and convergent on the open disk \( D(0,1) \). Therefore, like in [4], we can think to rewrite the inequalities given before under conditions from our theorems.

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References

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