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Some Optimal Quadrature Formulas and Error Bounds

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Abstract: The corrected quadrature rules are considered and the estimations of error involving the second derivative are given. The numerical examples which provides that the approximation in corrected rule of a optimal quadrature formula in sense Nikolski is better than in the original rule are considered.

Keywords: Optimal quadrature formula; corrected quadrature formula; Peano's Theorem; remainder term; optimal in sense Nikolski.

1. Introduction

Let $\mathcal{F}[a,b]$ be a linear space of real valued functions, defined and integrable on a finite interval $[a,b] \subset \mathbf{R}$. It is called a *quadrature formula* or *formula of numerical integration*, the following formula

$$\int_a^b f(x)dx = \sum_{k=0}^m A_k f(a_k) + \mathcal{R}[f],\ f \in \mathcal{F}[a,b], \tag{1}$$

where $a_k \in [a,b]$, respectively $A_k,\ k = \overline{0,m}$ are called the nodes, respectively the coefficients of the quadrature formula, and $\mathcal{R}[f]$ is the remainder term. A quadrature formula has *degree of exactness equal n*, if

$$\mathcal{R}[e_0] = 0\,,\ \mathcal{R}[e_1] = 0\,,\ \cdots,\ \mathcal{R}[e_n] = 0,$$

where $e_j(t) = t^j$. Moreover, if $\mathcal{R}[e_{n+1}] \neq 0$, then the quadrature formula has *degree of exactness effectively equal n*.

The quadrature formula (1) is called *optimal in sense Nikolski* in the space $\mathcal{F}$, if

$$\mathcal{E}_{m,n}(\mathcal{F},A,X) = \sup_{f\in\mathcal{F}} |\mathcal{R}[f]|$$

attains the minimum value with regard to A and X, where $A = \{A_k\}_{k=0}^m$ are the coefficients and $X = \{a_k\}_{k=0}^m$ are the nodes of quadrature formula.

Denote by

$$W_p^n[a,b] := \left\{ f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous },\ \left\| f^{(n)} \right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p\,dx \right\}^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|\,.$$

We consider the quadrature formula (1) has degree of exactness equal $n-1$. If $f \in W_p^n[a,b]$, by using Peano′s theorem, the remainder term can be written

$$\mathcal{R}[f] = \int_a^b K(t) f^{(n)}(t)dt, \text{ where } K(t) = \mathcal{R}\left[\frac{(x-t)_+^{n-1}}{(n-1)!}\right].$$

For the remainder term we have the evaluation

$$|\mathcal{R}[f]| \le \left[\int_a^b \left|f^{(n)}(t)\right|^p dt\right]^{\frac{1}{p}} \left[\int_a^b |K(t)|^q\,dt\right]^{\frac{1}{q}}, \tag{2}$$

$\frac{1}{p} + \frac{1}{q} = 1$, with remark that in the cases $p = 1$ and $p = \infty$ this evaluation is

$$|\mathcal{R}[f]| \le \int_a^b \left|f^{(n)}(t)\right| dt \sup_{t\in[a,b]} |K(t)|\,, \tag{3}$$

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$$|\mathcal{R}[f]| \le \sup_{t\in[a,b]} \left|f^{(n)}(t)\right| \int_a^b |K(t)|\, dt. \tag{4}$$

Therefore, the quadrature formula (1) is optimal in the sense Nikolski in the space $W_p^n[a,b]$, if

$$\mathcal{F}(A,X) = \begin{cases} \displaystyle\int_a^b |K(t)|^q\, dt, \ \frac{1}{p}+\frac{1}{q}=1,\ 1<p\le\infty, \\ \displaystyle\sup_{t\in[a,b]} |K(t)|,\ p=1, \end{cases}$$

attains the minimum value with regard to A and X, where $A=\{A_k\}_{k=0}^m$ are the coefficients and $X=\{a_k\}_{k=0}^m$ are the nodes of quadrature formula.

The problem to construct the optimal quadrature formulas was studied by many authors. The first results was obtained by A. Sard, L.S. Meyers and S.M. Nikolski. In the last years a number of authors have obtained in many different ways the optimal quadrature formulas ([1], [5], [6], [10], [13], [14]).

In 2008, N. Ujević and L. Mijić constructed a class of quadrature formulas of close type with 3 nodes.

The main result obtained by N. Ujević and L. Mijić is formulated bellow.

Theorem 1*[15] Let $I\subset \mathbf{R}$ be an open interval such that $[0,1]\subset I$ and let $f: I\to\mathbf{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have*

$$\left|\int_0^1 f(t)dt - \frac{\sqrt{2}}{8}f(0) - \left(1-\frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{8} f(1)\right| \le \frac{2-\sqrt{2}}{48}\|f''\|_\infty. \tag{5}$$

The structure of this paper is as follows: in Section 2 we construct a corrected rule of the quadrature formula which has degree of exactness equal 1. The estimations of the error in term of a variety on norms involving the second derivative are given. In Section 3 we construct some optimal quadrature formulas in sense Nikolski. Finally, in the Section 4 we consider the 4-points optimal quadrature formula and estimations of the error in corrected rule are given. These results are obtained from some inequalities point of view. Using a numerical example we will show that the estimations of the remainder term in corrected rule are better than in original quadrature formula.

2. The corrected quadrature formulas

It is called the corrected quadrature rule the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points. These formulae have a higher degree of exactness than the original rule. The estimate of the error in corrected rule is better then in the original rule, in generally. In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [2], [3], [4], [7], [8], [16]).

Let us consider the quadrature formula (1) which has degree of exactness equal 1. The remainder term of this quadrature formula has the following representation

$$\mathcal{R}[f] = \int_a^b K(t) f''(t)dt, \ \text{where}$$

$$K(t) = \mathcal{R}[(x-t)_+] = \frac{(b-t)^2}{2} - \sum_{k=0}^m A_k(a_k-t)_+.$$

Let

$$\int_a^b f(x)dx = \sum_{k=0}^m A_k f(a_k) + A\left[f'(b)-f'(a)\right] + \tilde{\mathcal{R}}[f],$$

where $A = \frac{1}{b-a}\int_a^b K(t)dt$, be the corrected quadrature formula of the rule (1).

Since the remainder term has degree of exactness 1 we can write

$$\tilde{\mathcal{R}}[f] = \int_a^b \tilde{K}(t) f''(t)dt, \ \text{where}$$

$$\tilde{K}(t) = \tilde{R}[(x-t)_+] = K(t) - A.$$

From the above relation we have $\int_a^b \tilde{K}(t)dt = 0.$

Theorem 2 *Let $f:[a,b]\to\mathbf{R}$ be an absolutely continuous function such that $f''\in L[a,b]$ and there exist real number $m[f], M[f]$ such that $m[f]\le f''(t)\le M[f]$, $t\in[a,b]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \frac{M[f]-m[f]}{2}\cdot\|\tilde{K}\|_1.$$

Proof. Since $\int_a^b \tilde{K}(t)dt = 0$, the remainder term

$$\tilde{\mathcal{R}}[f] = \int_a^b \tilde{K}(t) f''(t)dt, \tag{6}$$

can be written in the following way

$$\tilde{\mathcal{R}}[f] = \int_a^b \tilde{K}(t)\left(f''(t) - \frac{M[f]+m[f]}{2}\right)dt.$$

Therefore

$$\left|\tilde{\mathcal{R}}[f]\right| \le \left\|f'' - \frac{M[f]+m[f]}{2}\right\|_\infty \cdot \|\tilde{K}\|_1 \le \frac{M[f]-m[f]}{2}\cdot\|\tilde{K}\|_1.$$

Theorem 3 *Let $f:[a,b]\to\mathbf{R}$ be an absolutely continuous function such that $f''\in L[a,b]$. If there exist a real number $m[f]$ such that $m[f]\le f''(t)$, $t\in[a,b]$, then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \left\|\tilde{K}\right\|_\infty\cdot\left(\frac{f'(b)-f'(a)}{b-a} - m[f]\right)(b-a).$$

Proof. We have

$$\left|\tilde{\mathcal{R}}[f]\right| = \left|\int_a^b \tilde{K}(t)\left(f''(t) - m[f]\right)dt\right|$$
$$\leq \sup_{t\in[a,b]} |\tilde{K}(t)| \cdot \int_a^b \left(f''(t) - m[f]\right)dt$$
$$= \left\|\tilde{K}\right\|_\infty \cdot \left(\frac{f'(b) - f'(a)}{b-a} - m[f]\right)(b-a).$$

Theorem 4 *Let $f : [a,b] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L[a,b]$. If there exist a real number $M[f]$ such that $f''(t) \leq M[f]$, $t \in [a,b]$, then*

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \left\|\tilde{K}\right\|_\infty \cdot \left(M[f] - \frac{f'(b) - f'(a)}{b-a}\right)(b-a).$$

Let $f, g : [a,b] \to \mathbf{R}$ be integrable functions on $[a,b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \qquad (7)$$

is well known in the literature as the Čebyšev functional. It was proved that $T(f,f) \geq 0$ and the inequality

$$|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$$

holds. Denote by

$$\sigma(f;a,b) = \sqrt{(b-a)T(f,f)}.$$

Theorem 5 *Let $f : [a,b] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L_2[a,b]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \sigma(K;a,b) \cdot \sigma(f'';a,b). \qquad (8)$$

The inequality (8) is sharp in the sense that the constant $\sigma(K;a,b)$ cannot be replaced by a smaller ones.

Proof. The remainder term of the corrected quadrature formula can be written in such way

$$\tilde{\mathcal{R}}[f] = \int_a^b \tilde{K}(t)f''(t)dt$$
$$= \int_a^b \left[K(t) - \frac{1}{b-a}\int_a^b K(t)dt\right] f''(t)dt$$
$$= \int_a^b K(t)f''(t)dt - \frac{1}{b-a}\int_a^b K(t)dt \cdot \int_a^b f''(t)dt$$
$$= (b-a)T(K,f'').$$

From the above relation we obtain

$$\left|\tilde{\mathcal{R}}[f]\right| = |(b-a)T(K,f'')|$$
$$\leq \sqrt{(b-a)T(K,K)}\sqrt{(b-a)T(f'',f'')}$$
$$= \sigma(K;a,b) \cdot \sigma(f'';a,b).$$

To prove that the constant $\sigma(K;a,b)$ cannot be replaced by a smaller ones we define the function $F \in C^2[a,b]$ such that $F''(x) = K(x)$, $x \in [a,b]$. For the function F the right-hand side of (8) is equal with $(b-a)T(K,K)$ and the left-hand side becomes

$$\left|\tilde{\mathcal{R}}[F]\right| = \int_a^b \tilde{K}(t)K(t)dt$$
$$= \int_a^b \left(K(t) - \frac{1}{b-a}\int_a^b K(t)dt\right)K(t)dt$$
$$= \int_a^b K(t)^2dt - \frac{1}{b-a}\int_a^b K(t)dt \int_a^b K(t)dt$$
$$= (b-a)T(K,K).$$

3. The optimal quadrature formulas

In [12] were constructed the quadrature formulas of open type, optimal in sense Nikolski in the space $W_p^2[a,b]$, where $p = 1, \infty, 2$. The main purpose of this section is to derive some quadrature formulas of close type which are optimal in sense Nikolski in $W_\infty^2[a,b]$.

Let

$$\int_0^1 f(x)dx = \sum_{i=0}^{m-1} A_i f(a_i) + \mathcal{R}[f], \qquad (9)$$

be a quadrature formula with degree of exactness equal 1, where the nodes verifies $0 = a_0 < a_1 < \cdots < a_{m-1} < 1$.

We will calculate the coefficients A_i, $i = \overline{0, m-1}$ and the nodes a_i, $i = 0, m-1$ such that the quadrature formula (9) to be optimal, considering that the remainder term is evaluate in sense of (4). Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}[e_i] = 0$, $e_i(x) = x^i$, $i = 0,1$, namely $\sum_{i=0}^{m-1} A_i = 1$, $\sum_{i=0}^{m-1} A_i a_i = \frac{1}{2}$. Using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}[f] = \int_0^1 K(t)f''(t)dt, \text{ where} \qquad (10)$$

$$K(t) = \mathcal{R}[(x-t)_+] = \frac{1}{2}(1-t)^2 - \sum_{i=0}^{m-1} A_i(a_i - t)_+. \qquad (11)$$

Theorem 6 *For $f \in W_\infty^2[0,1]$ the quadrature formula of the form (9), optimal to the error has the following nodes and coefficients*

$$A_0 = \frac{1}{\sqrt{2}}h,\ A_1 = \frac{6-\sqrt{2}}{2}h,\ A_k = 2h,\ k = \overline{2, m-2},$$

$$A_{m-1} = \frac{2+\sqrt{3}}{2}h,$$

$$a_i = 2ih,\ i = \overline{0, m-1},\ \textit{where}\ h = \frac{2}{4(m-1)+\sqrt{3}}.$$

The remainder term has the following evaluation

$$|\mathcal{R}[f]| \leq \frac{h^3}{48}\left(3\sqrt{3}-32\sqrt{2}+40+12m\right)\|f''\|_\infty.$$

Proof. The remainder term can be evaluate in the following way

$$|\mathcal{R}[f]| \leq \|f''\|_\infty \cdot \int_0^1 |K(t)|dt.$$

The quadrature formula is optimal with regard to the error if $\int_0^1 |K(t)|dt \to$ minimum.

If we consider the substitution $1-t=u$, the function K can be written

$$K(t)=K(1-u)=\tilde{K}(u)=\frac{1}{2}u^2-\sum_{i=0}^{m-1} A_i\,[u-(1-a_i)]_+.$$

Denote by $1-a_i=u_{m-i}$, $i=\overline{0,m-1}$. Then the nodes u_k, $k=\overline{1,m}$ verifies

$$0<u_1<u_2<\cdots<u_m=1, \text{ and}$$

$$\begin{aligned}\tilde{K}(u) &= \frac{1}{2}u^2-\sum_{i=0}^{m-1}A_i(u-u_{m-i})_+\\ &= \frac{1}{2}u^2-\sum_{i=1}^{m}A_{m-i}(u-u_i)_+.\end{aligned}$$

If we denote $A_{m-i}=\lambda_i, i=\overline{1,m}$, then

$$\tilde{K}(u)=\frac{1}{2}u^2-\sum_{i=1}^{m}\lambda_i(u-u_i)_+.$$

Since the quadrature formula has degree of exactness 1 it follows

$$\sum_{i=0}^{m-1}A_i=1,\ \sum_{i=0}^{m-1}A_ia_i=\frac{1}{2}, \text{ namely}$$

$$\sum_{i=1}^{m}\lambda_i=1,\ \sum_{i=1}^{m}\lambda_iu_i=\frac{1}{2}, \tag{12}$$

and the function $\tilde{K}$ can be written

$$\tilde{K}(u)=\begin{cases}\frac{1}{2}u^2,\ 0\leq u\leq u_1,\\ \frac{1}{2}u^2-\left(\sum_{i=1}^{k}\lambda_i\right)u+\sum_{i=1}^{k}\lambda_iu_i,\\ u_k\leq u\leq u_{k+1},\ k=\overline{1,m-2},\\ \frac{1}{2}u^2-(1-\lambda_m)u+\frac{1}{2}-\lambda_m,\ u_{m-1}\leq u\leq 1.\end{cases}$$

If we denote $\sum_{i=1}^{k}\lambda_i=\alpha_k,\ -\sum_{i=1}^{k}\lambda_iu_i=\beta_k,\ k=\overline{1,m-2},$ it follows

$$\tilde{K}(u)=\begin{cases}\frac{1}{2}u^2,\ 0\leq u\leq u_1,\\ \frac{1}{2}u^2-\alpha_ku-\beta_k,\ u_k\leq u\leq u_{k+1},\ k=\overline{1,m-2},\\ \frac{1}{2}(1-u)^2-\lambda_m(1-u),\ u_{m-1}\leq u\leq 1.\end{cases}$$

Let us consider $[u_k,u_{k+1}]=[a-h,a+h]$. The parameters of the optimal formula can be obtained by identifying the function $\tilde{K}\Big|_{[u_k,u_{k+1}]}$ with the Chebyshev orthogonal polynomial of the second kind of degree 2, on the interval $[u_k,u_{k+1}]$, with the coefficient of u^2 equal with $1/2$, namely

$$\tilde{K}(u)=\frac{1}{2}u^2-\alpha_ku-\beta_k=\frac{1}{2}h^2\tilde{U}_2\left(\frac{u-a}{h}\right),$$

where $\tilde{U}_2(x)=x^2-\frac{1}{4}$ is the Chebyshev polynomial of the second kind of degree 2, on the interval $[-1,1]$.

By identifying the coefficients we obtain

$$\alpha_k=a=\frac{u_k+u_{k+1}}{2},\ \beta_k=\frac{h^2}{8}-\frac{a^2}{2}=\frac{h^2}{8}-\frac{(u_k+u_{k+1})^2}{8}.$$

If we denote by $[u_{k+1},u_{k+2}]=[b-h_1,b+h_1]$, we have $\tilde{K}(u_{k+1}-0)=\tilde{K}(a+h)=\frac{1}{2}h^2\tilde{U}_2(1)=\frac{3h^2}{8}$ and $\tilde{K}(u_{k+1}+0)=\tilde{K}(b-h_1)=\frac{1}{2}h_1^2\tilde{U}_2(-1)=\frac{3h_1^2}{8}$. Since $\tilde{K}\in C[0,1]$it follows that $u_1,\ldots,u_m$ are equidistant nodes and $u_{k+1}-u_k=2h$, $k=\overline{1,m-1}$. From the condition $\frac{u_1^2}{2}=\tilde{K}(u_1-0)=\tilde{K}(u_1+0)=\frac{3h^2}{8}$ we obtain $u_1=\frac{\sqrt{3}h}{2}$, and $u_k=\frac{4(k-1)+\sqrt{3}}{2}h, k=\overline{2,m}.$ To obtain the parameter h can be used the relation $u_1+(u_2-u_1)+\cdots+(u_m-u_{m-1})=1$ and we have $h=\frac{2}{4(m-1)+\sqrt{3}}$. Therefore, the nodes of the optimal quadrature formula are $a_i=2ih$, $i=\overline{0,m-1}$.

The quadrature formula is optimal with regard to the error if

$$I=\int_0^1\left|\tilde{K}(u)\right|du\to\text{minimum, where}$$

$$I=\sum_{k=0}^{m-1}I_k,\ I_0=\int_0^{u_1}\frac{u^2}{2}du,$$

$$I_k=\int_{u_k}^{u_{k+1}}\left|\tilde{K}(u)\right|du,\ k=\overline{1,m-1}.$$

We have

$$I_0=\int_0^{u_1}\frac{u^2}{2}du=\frac{u_1^3}{6},$$

$$\begin{aligned} I_k &= \int_{u_k}^{u_{k+1}} \left| \frac{1}{2}u^2 - \alpha_k u - \beta_k \right| du \\ &= \int_{a-h}^{a+h} \left| \frac{1}{2}u^2 - \alpha_k u - \beta_k \right| du \\ &= \frac{h^3}{2} \int_{-1}^{1} \left| t^2 - \frac{1}{4} \right| dt = \frac{h^3}{4}, \;\; k = \overline{1, m-2}, \end{aligned}$$

$$\begin{aligned} I_{m-1} &= \int_{u_{m-1}}^{1} \left| \frac{(1-u)^2}{2} - \lambda_m (1-u) \right| du \\ &= \int_{u_{m-1}}^{1-2\lambda_m} \left[\frac{(1-u)^2}{2} - \lambda_m (1-u) \right] dt \\ &- \int_{1-2\lambda_m}^{1} \left[\frac{(1-u)^2}{2} - \lambda_m (1-u) \right] dt \\ &= \frac{4}{3}\lambda_m^3 - \frac{1}{2}(1-u_{m-1})^2 \lambda_m + \frac{1}{6}(1-u_{m-1})^3. \end{aligned}$$

From the condition that I_{m-1} to attains minimum value we obtain

$$\lambda_m = \frac{1}{2\sqrt{2}}(1-u_{m-1}) = \frac{1}{\sqrt{2}}h.$$

Using the relation $\sum_{i=1}^{k} \lambda_i = \alpha_k, \; k = \overline{1, m-2}$ it follows $\lambda_k = \alpha_k - \alpha_{k-1} = 2h, \; k = \overline{2, m-2}$. Also, we have $\lambda_1 = \alpha_1 = \frac{2+\sqrt{3}}{2}h$. From relation $\sum_{i=1}^{m} \lambda_i = 1$ we find $\lambda_{m-1} = \frac{6-\sqrt{2}}{2}h.$

Therefore, we obtain the following coefficients of the optimal quadrature formula

$$A_0 = \frac{1}{\sqrt{2}}h, \; A_1 = \frac{6-\sqrt{2}}{2}h, \;\; A_k = 2h, \;\; k = \overline{2, m-2},$$

$$A_{m-1} = \frac{2+\sqrt{3}}{2}h.$$

In the next part of this section we will calculate the coefficients A_i, $i = \overline{0, m}$ and the nodes a_i, $i = \overline{1, m-1}$ such that the following quadrature formula which degree of exactness 1

$$\int_0^1 f(x)dx = \sum_{i=0}^{m} A_i f(a_i) + \mathcal{R}[f], \qquad (13)$$

where $0 = a_0 < a_1 < \cdots < a_m = 1$, to be optimal, considering that the remainder term is evaluate in sense of (4).

Theorem 7 *For $f \in W_\infty^2[0,1]$ the quadrature formula of the form (13), optimal to the error has the following nodes and coefficients*

$$a_1 = \frac{\sqrt{3(2+\sqrt{2})}}{2}h, \;\; a_{m-1} = 1 - \frac{\sqrt{3(2+\sqrt{2})}}{2}h, \quad (14)$$

$$a_k = 1 - \frac{4(m-k-1) + \sqrt{3(2+\sqrt{2})}}{2}h, \;\; k = \overline{2, m-2},$$

$$A_0 = A_m = \frac{\sqrt{6(2+\sqrt{2})}}{8}h, \; A_k = 2h, \; k = \overline{2, m-2} \quad (15)$$

$$A_1 = A_{m-1} = \frac{(4-\sqrt{2})\sqrt{3(2+\sqrt{2})}+8}{8}h,$$

where $h = \dfrac{1}{2(m-2) + \sqrt{3(2+\sqrt{2})}}.$

The remainder term has the following evaluation

$$|\mathcal{R}[f]| \le \frac{h^2}{8}\|f''\|_\infty.$$

Proof. The remainder term of quadrature formula (13) can be evaluate in the following way

$$|\mathcal{R}[f]| = \left| \int_0^1 K(t) f''(t) dt \right| \le \|f''\|_\infty \cdot \int_0^1 |K(t)| dt,$$

where

$$K(t) = \mathcal{R}\left[(x-t)_+\right] = \frac{(1-t)^2}{2} - \sum_{i=0}^{m} A_i (a_i - t)_+.$$

The quadrature formula is optimal with regard to the error if $\int_0^1 |K(t)|dt \to$ minimum.

If we consider the substitution $1 - t = u$, the function K can be written

$$K(t) = K(1-u) = \tilde{K}(u) = \frac{1}{2}u^2 - \sum_{i=0}^{m} A_i \left[u - (1-a_i)\right]_+.$$

Denote by $1 - a_i = u_{m-i}$, $i = \overline{0, m}$. Then the nodes u_k, $k = \overline{1, m}$ verifies

$$0 = u_0 < u_1 < u_2 < \cdots < u_m = 1, \text{ and}$$

$$\begin{aligned} \tilde{K}(u) &= \frac{1}{2}u^2 - \sum_{i=0}^{m} A_i (u - u_{m-i})_+ \\ &= \frac{1}{2}u^2 - \sum_{i=0}^{m} A_{m-i} (u - u_i)_+. \end{aligned}$$

If we denote $A_{m-i} = \lambda_i, i = \overline{0, m}$, then

$$\tilde{K}(u) = \frac{1}{2}u^2 - \sum_{i=0}^{m} \lambda_i (u - u_i)_+.$$

Since the quadrature formula has degree of exactness 1 it follows

$$\sum_{i=0}^{m} A_i = 1, \; \sum_{i=0}^{m} A_i a_i = \frac{1}{2}, \text{ namely}$$

$$\sum_{i=0}^{m} \lambda_i = 1, \; \sum_{i=0}^{m} \lambda_i u_i = \frac{1}{2}, \qquad (16)$$

and the function $\tilde{K}$ can be written

$$\tilde{K}(u)=\begin{cases}\frac{1}{2}u^2-\lambda_0 u,\ u_0\le u\le u_1,\\ \frac{1}{2}u^2-\sum_{i=0}^{k}\lambda_i(u-u_i),\ u_k\le u\le u_{k+1},\\ k=\overline{1,m-2},\\ \frac{1}{2}u^2-\sum_{i=0}^{m-1}\lambda_i(u-u_i),\ u_{m-1}\le u\le u_m.\end{cases}$$

If we denote $\sum_{i=0}^{k}\lambda_i=\alpha_k,\ -\sum_{i=0}^{k}\lambda_i u_i=\beta_k,\ k=\overline{1,m-2}$, it follows

$$\tilde{K}(u)=\begin{cases}\frac{1}{2}u^2-\lambda_0 u,\ u_0\le u\le u_1,\\ \frac{1}{2}u^2-\alpha_k u-\beta_k,\ u_k\le u\le u_{k+1},\\ k=\overline{1,m-2},\\ \frac{1}{2}(1-u)^2-\lambda_m(1-u),\ u_{m-1}\le u\le u_m.\end{cases}$$

Let us consider $[u_k,u_{k+1}]=[a-h,a+h]$. The parameters of the optimal formula can be obtained by identifying the function $\tilde{K}\Big|_{[u_k,u_{k+1}]}$ with the Chebyshev orthogonal polynomial of the second kind of degree 2, on the interval $[u_k,u_{k+1}]$, with the coefficient of u^2 equal with $1/2$, namely

$$\tilde{K}(u)=\frac{1}{2}u^2-\alpha_k u-\beta_k=\frac{1}{2}h^2\tilde{U}_2\left(\frac{u-a}{h}\right),$$

where $\tilde{U}_2(x)=x^2-\frac{1}{4}$ is the Chebyshev polynomial of the second kind of degree 2, on the interval $[-1,1]$.

By identifying the coefficients we obtain

$$\alpha_k=a=\frac{u_k+u_{k+1}}{2},\ \beta_k=\frac{h^2}{8}-\frac{a^2}{2}=\frac{h^2}{8}-\frac{(u_k+u_{k+1})^2}{8}.$$

If we denote by $[u_{k+1},u_{k+2}]=[b-h_1,b+h_1]$, we have $\tilde{K}(u_{k+1}-0)=\tilde{K}(a+h)=\frac{1}{2}h^2\tilde{U}_2(1)=\frac{3h^2}{8}$ and $\tilde{K}(u_{k+1}+0)=\tilde{K}(b-h_1)=\frac{1}{2}h_1^2\tilde{U}_2(-1)=\frac{3h_1^2}{8}$. Since $\tilde{K}\in C[0,1]$ it follows that $u_1,\dots,u_m$ are equidistant nodes and $u_{k+1}-u_k=2h,\ k=\overline{1,m-1}$.

The quadrature formula is optimal with regard to the error if

$$I=\int_0^1\left|\tilde{K}(u)\right|du\to\text{minimum},\ \text{where}$$

$$I=\sum_{k=0}^{m-1}I_k,\ I_0=\int_0^{u_1}\left|\frac{u^2}{2}-\lambda_0 u\right|du,$$

$$I_k=\int_{u_k}^{u_{k+1}}\left|\frac{1}{2}u^2-\alpha_k u-\beta_k\right|du,\ k=\overline{1,m-1}.$$

We have

$$\begin{aligned}I_0&=\int_0^{u_1}\left|\frac{u^2}{2}-\lambda_0 u\right|du\\&=\int_0^{2\lambda_0}\left(\lambda_0 u-\frac{1}{2}u^2\right)du+\int_{2\lambda_0}^{u_1}\left(\frac{1}{2}u^2-\lambda_0 u\right)du\\&=\frac{4}{3}\lambda_0^3-\frac{1}{2}u_1^2\lambda_0+\frac{1}{6}u_1^3,\end{aligned}$$

$$\begin{aligned}I_k&=\int_{u_k}^{u_{k+1}}\left|\frac{1}{2}u^2-\alpha_k u-\beta_k\right|du\\&=\int_{a-h}^{a+h}\left|\frac{1}{2}u^2-\alpha_k u-\beta_k\right|du\\&=\frac{h^3}{2}\int_{-1}^{1}\left|t^2-\frac{1}{4}\right|dt=\frac{h^3}{4},\ k=\overline{1,m-2},\end{aligned}$$

$$\begin{aligned}I_{m-1}&=\int_{u_{m-1}}^{1}\left|\frac{u^2}{2}-\lambda_m u\right|du\\&=\int_0^{1-u_{m-1}}\left|\frac{t^2}{2}-\lambda_m t\right|dt\\&=\frac{4}{3}\lambda_m^3-\frac{1}{2}(1-u_{m-1})^2\lambda_m+\frac{1}{6}(1-u_{m-1})^3.\end{aligned}$$

Using the condition that I_0, respectively I_{m-1} to attains minimum value, we obtain $\lambda_0=\frac{1}{2\sqrt{2}}u_1$, respectively $\lambda_m=\frac{1}{2\sqrt{2}}(1-u_{m-1})$.

From the condition $\tilde{K}(u_1-0)=\tilde{K}(u_1+0)=\frac{3h^2}{8}$ we have $u_1=\frac{\sqrt{3(2+\sqrt{2})}}{2}h$, and

$$u_k=\frac{4(k-1)+\sqrt{3(2+\sqrt{2})}}{2}h,k=\overline{2,m-2}.$$

In a similar way we find $u_{m-1}=1-\frac{\sqrt{3(2+\sqrt{2})}}{2}h$. To obtain the parameter h can be used the relation

$$u_1+(u_2-u_1)+\dots+(1-u_{m-1})=1$$

and we have

$$h=\frac{1}{2(m-2)+\sqrt{3(2+\sqrt{2})}}.$$

Therefore, we obtain the nodes of the optimal quadrature formula given in (14).

Also, it follows

$$\lambda_0=\frac{1}{2\sqrt{2}}u_1=\frac{\sqrt{6(2+\sqrt{2})}}{8}h,$$

$$\lambda_m=\frac{1}{2\sqrt{2}}(1-u_{m-1})=\frac{\sqrt{6(2+\sqrt{2})}}{8}h=\lambda_0.$$

Since $\sum_{i=0}^{k} \lambda_i = \alpha_k$, we have $\lambda_0 + \lambda_1 = \alpha_1$, namely

$$\lambda_1 = u_1 + h - \frac{1}{2\sqrt{2}} u_1 = \frac{(4-\sqrt{2})\sqrt{3(2+\sqrt{2})}+8}{8} h.$$

Also, from the above relation we find

$$\lambda_k = \alpha_k - \alpha_{k-1} = 2h, \ k = \overline{2, m-2}.$$

From relation $\sum_{i=0}^{m} \lambda_i = 1$ we obtain $\lambda_{m-1} = \lambda_1$.

Therefore, it follows the coefficients of the optimal quadrature formula given in (15).

4. Numerical examples

The main purpose of this section is to derive corrected rules of the second optimal quadrature formula obtained in previous section. We will show that the corrected formula improves the original formula.

Using the algorithm described in the second section we obtain the following corrected quadrature formula of rule (13)

$$\int_0^1 f(x)dx = \sum_{i=0}^{m} A_i f(a_i) + A\left(f'(1) - f'(0)\right) + \tilde{\mathcal{R}}[f], \quad (17)$$

where the nodes a_i, respectively the coefficients A_i, $i = \overline{0, m}$ are given in relations (14), respectively (15) and

$$A = \int_0^1 K(t)dt = \int_0^1 \tilde{K}(u)du$$
$$= \frac{h^3}{48}\left[4(m-2) + 3(1-\sqrt{2})\sqrt{3(2+\sqrt{2})}\right].$$

Remark 8 *If we consider in Theorem 7 the particular case m=2 we obtained the following optimal quadrature formula of close type with 3-points*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \mathcal{R}[f], \quad (18)$$

and the corrected rule of this quadrature formulas is given by

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \frac{4-3\sqrt{2}}{96}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}[f]. \quad (19)$$

The optimal quadrature (18) and the corrected rule (19) were obtained by N. Ujević and L. Mijić in [15]. This result motivate us to seek the optimal quadrature formulas with more than 3-points and their corrected rules.

In the next part of this paper we consider a corrected version of the optimal quadrature with 4-points and we show that this rule provides a better approximation than the original rule.

Considering $m = 3$ in Theorem 7 we have the following optimal quadrature formula

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(a_2) + A_3 f(1) + \mathcal{R}[f], \quad (20)$$

where

$$A_0 = A_3 = \frac{\sqrt{6(2+\sqrt{2})}}{8} h,$$
$$A_1 = A_2 = \frac{(4-\sqrt{2})\sqrt{3(2+\sqrt{2})}+8}{8} h,$$
$$a_1 = \frac{\sqrt{3(2+\sqrt{2})}}{2} h, \ a_2 = 1 - a_1, \ h = \frac{1}{2+\sqrt{3(2+\sqrt{2})}}.$$

The remainder term has the following representation $\mathcal{R}[f] = \int_0^1 K(t) f''(t)dt$, where

$$K(t) = \begin{cases} \frac{1}{2}t^2 - A_0 t, \ 0 \le t \le a_1, \\ \frac{1}{2}t^2 - (A_0 + A_1)t + A_1 a_1, \ a_1 \le t \le a_2, \\ \frac{1}{2}(1-t)^2 - A_3(1-t), \ a_2 \le t \le 1. \end{cases}$$

Using (17) we obtain the following corrected quadrature formula of (20)

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(a_2) + A_3 f(1) + A\left(f'(1) - f'(0)\right) + \tilde{\mathcal{R}}[f], \quad (21)$$

where $A = \frac{h^3}{48}\left[4 + 3(1-\sqrt{2})\sqrt{3(2+\sqrt{2})}\right]$ and

$$\tilde{\mathcal{R}}[f] = \int_0^1 \tilde{K}(t) f''(t)dt, \text{ with } \tilde{K}(t) = K(t) - A.$$

Theorem 9 *Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$ and there exist real number $m[f], M[f]$ such that $m[f] \le f''(t) \le M[f]$, $t \in [0,1]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le 0.00462291793614 \cdot \frac{M[f] - m[f]}{2}.$$

Theorem 10 *Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number $m[f]$ such that $m[f] \le f''(t)$, $t \in [0,1]$, then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \frac{1}{48} \frac{32 + (15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3} \times \left(f'(1) - f'(0) - m[f]\right).$$

Theorem 11 *Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number $M[f]$ such that $f''(t) \le M[f]$, $t \in [0,1]$, then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \frac{1}{48}\frac{32+(15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3} \times (M[f]-f'(1)+f'(0)).$$

Theorem 12 *Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le C \cdot \sigma(f''; 0,1) \text{ where} \tag{22}$$

$$C = \left(\frac{2374+(12\sqrt{2}+1080)\sqrt{6+3\sqrt{2}}+783\sqrt{2}}{11520(2+\sqrt{6+3\sqrt{2}})^6}\right)^{1/2}.$$

The inequality (22) is sharp in the sense that the constant C cannot be replaced by a smaller ones.

Remark 13 *Considering that the remainder term of original, respectively corrected quadrature formula is evaluate in sense of (2) with $p \in \{1,2\}$ we obtain the following inequalities*

$$|\mathcal{R}[f]| \le \|K\|_\infty \cdot \|f''\|_1 = \frac{3}{8}\cdot\frac{1}{(2+\sqrt{6+3\sqrt{2}})^2}\cdot\|f''\|_1 \approx 0.01386614276036 \cdot \|f''\|_1,$$

$$\left|\tilde{\mathcal{R}}[f]\right| \le \|\tilde{K}\|_\infty \cdot \|f''\|_1 = \frac{1}{48}\cdot\frac{32+(15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3}\cdot\|f''\|_1 \approx 0.01386273025465 \cdot \|f''\|_1,$$

$$|\mathcal{R}[f]| \le \|K\|_2 \cdot \|f''\|_2 = \left(-\frac{-92+(9\sqrt{2}-54)\sqrt{6+3\sqrt{2}}}{1920(2+\sqrt{6+3\sqrt{2}})^5}\right)^{1/2}\cdot\|f''\|_2 \approx 0.00553941461773 \cdot \|f''\|_2,$$

$$\left|\tilde{\mathcal{R}}[f]\right| \le \|\tilde{K}\|_2 \cdot \|f''\|_2 = \left(\frac{2374+(12\sqrt{2}+1080)\sqrt{6+3\sqrt{2}}+783\sqrt{2}}{11520(2+\sqrt{6+3\sqrt{2}})^6}\right)^{1/2}\cdot\|f''\|_2 \approx 0.00553941356661 \cdot \|f''\|_2.$$

We can remark that the estimations of the remainder term in corrected rule are better than in original quadrature formula.

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