Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/100201

On Generalized Tetranacci Quaternions

Yüksel Soykan[1], *İnci Okumuş* [2] *and Melih Göcen* [1,*]

[1]Department of Mathematics,Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak, Turkey
[2]Department of Engineering Sciences, Faculty of Engineering, İstanbul University-Cerrahpasa, İstanbul, Turkey

Received: 7 Dec. 2022, Revised: 21 Jan. 2023, Accepted: 17 Feb. 2023
Published online: 1 May 2023

Abstract: In this paper, we introduce the generalized Tetranacci quaternions and present some properties of these quaternions.

Keywords: Tetranacci numbers, quaternions, Tetranacci quaternions, Tetranacci-Lucas quaternions.

1 Introduction

In this paper, we define generalized Tetranacci quaternions in the next section and give some properties of them and Tetranacci and Tetranacci-Lucas quaternions as special cases. First, we present some background about quaternions and generalized Tetranacci numbers.

A quaternion is a hyper-complex number and is defined by

$$q = a_0 + ia_1 + ja_2 + ka_3 = (a_0, a_1, a_2, a_3)$$

where a_0, a_1, a_2 and a_3 are real numbers or scalers and $1, i, j, k$ are the standard orthonormal basis in $\mathbb{R}^4$. The set of all quaternions are denoted by $\mathbb{H}$. Note that we can write

$$q = a_0 + p$$

where $p = ia_1 + ja_2 + ka_3$. a_0 and p are called the scalar part and the vector part of the quaternion q, respectively. The a_0, a_1, a_2, a_3 are called the components of the quaternion q.

Addition of quaternions is defined as componentwise and the quaternion multiplication is defined as follows:

$$i^2 = j^2 = k^2 = ijk = -1. \qquad (1)$$

Note that from (1), we have

$$ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik. \qquad (2)$$

So, multiplication on $\mathbb{H}$ is not commutative.

The product of two quaternions $q = a_0 + ia_1 + ja_2 + ka_3$ and $p = b_0 + ib_1 + jb_2 + kb_3$ is

$$\begin{aligned} qp = & (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ & + i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \\ & + j(a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1) \\ & + k(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0). \end{aligned}$$

The conjugate of the quaternion q is defined by

$$q^* = (a_0 + ia_1 + ja_2 + ka_3)^* = a_0 - ia_1 - ja_2 - ka_3.$$

For two quaternions p, q we have

$$(q^*)^* = q, \; (p+q)^* = p^* + q^*, \; (pq)^* = q^*p^* \text{ and } (p^*q)^* = q^*p.$$

The norm of a quaternion q is defined by

$$N(q) = \|q\| := qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

The norm is multiplicative:

$$N(pq) = N(p)N(q).$$

Division is uniquely defined (except by zero), thus quaternions form a division algebra. For two quaternions $p, q \in \mathbb{H}$, we have

$$(pq)^{-1} = q^{-1}p^{-1}.$$

The inverse (reciprocal) of a nonzero quaternion q is given by

$$q^{-1} = \frac{q^*}{N(q)}.$$

* Corresponding author e-mail: gocenm@hotmail.com

A generalized Tetranacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0,V_1,V_2,V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} \quad (3)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [11,16,17,19,27,28].

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} + V_{-(n-4)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (3) holds for all integer n.

The first few generalized Tetranacci numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Tetranacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$c_3 - c_2 - c_1 - c_0$
2	c_2	$2c_2 - c_3$
3	c_3	$2c_1 - c_2$
4	$c_0 + c_1 + c_2 + c_3$	$2c_0 - c_1$
5	$c_0 + 2c_1 + 2c_2 + 2c_3$	$2c_3 - 2c_2 - 2c_1 - 3c_0$
6	$2c_0 + 3c_1 + 4c_2 + 4c_3$	$c_0 + c_1 + 5c_2 - 3c_3$
7	$4c_0 + 6c_1 + 7c_2 + 8c_3$	$4c_1 - 4c_2 + c_3$
8	$8c_0 + 12c_1 + 14c_2 + 15c_3$	$4c_0 - 4c_1 + c_2$

If we consider $V_0 = 0, V_1 = 1, V_2 = 1, V_3 = 2$, then $\{V_n\}$ is the well-known Tetranacci sequence and if we take $V_0 = 4, V_1 = 1, V_2 = 3, V_3 = 7$ then $\{V_n\}$ is the well-known Tetranacci-Lucas sequence. In other words, Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$\begin{aligned} M_n &= M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \\ M_0 &= 0, M_1 = 1, M_2 = 1, M_3 = 2 \end{aligned} \quad (4)$$

and

$$\begin{aligned} R_n &= R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \\ R_0 &= 4, R_1 = 1, R_2 = 3, R_3 = 7. \end{aligned} \quad (5)$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)},$$

for $n = 1,2,3,\ldots$ respectively. So, recurrences (4) and (5) hold for all integer n. Now, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	11	...
M_n	0	1	1	2	4	8	15	29	56	108	208	401	...
M_{-n}	0	0	0	1	−1	0	0	2	−3	1	0	4	...
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	...
R_{-n}	4	−1	−1	−1	7	−6	−1	−1	15	−19	4	−1	...

For all integers n, usual Tetranaci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

(see for example [11 or 30]) or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}, \quad (6)$$

(see for example [7]) and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where α, β, γ and δ are the roots of the cubic equation $x^4 - x^3 - x^2 - x - 1 = 0$. Furthermore,

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

$$\delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

We give Binet's formula of the generalized Tetranacci sequence.

Corollary 1. *The Binet's formula of the generalized Tetranacci sequence $\{V_n\}$ is given as*

$$V_n = A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6}$$

where

$$A = \frac{\alpha-1}{5\alpha-8}(V_3\alpha^3+(V_0+V_1+V_2)\alpha^2+(V_1+V_2)\alpha+V_2),$$
$$B = \frac{\beta-1}{5\beta-8}(V_3\beta^3+(V_0+V_1+V_2)\beta^2+(V_1+V_2)\beta+V_2),$$
$$C = \frac{\gamma-1}{5\gamma-8}(V_3\gamma^3+(V_0+V_1+V_2)\gamma^2+(V_1+V_2)\gamma+V_2),$$
$$D = \frac{\delta-1}{5\delta-8}(V_3\delta^3+(V_0+V_1+V_2)\delta^2+(V_1+V_2)\delta+V_2.)$$

Proof. For a proof see [21, Corollary 1.3.].

In fact, Corollary 1 is a special case of a result in [1, Remark 2.3.].

Recall that the Binet form of a sequence satisfying (3) for non-negative integers is valid for all integers n, for a proof of this result see [13]. This result of Howard and Saidak [13] is even true in the case of higher-order recurrence relations.

Next, we present the ordinary generating function $\sum\limits_{n=0}^{\infty} a_n x^n$ of the sequence V_n.

Lemma 1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is the ordinary generating function of the generalized Tetranacci sequence $\{V_n\}_{n\geq 0}$. Then, $f_{V_n}(x)$ is given by*

$$f_{V_n}(x) = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{1-x-x^2-x^3-x^4}. \quad (7)$$

Proof. Using (3) and some calculation, we have

$$\begin{aligned} & f_{V_n}(x)-xf_{V_n}(x)-x^2f_{V_n}(x)-x^3f_{V_n}(x)-x^4f_{V_n}(x) \\ &= V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2 \\ &\quad +(V_3-V_2-V_1-V_0)x^3 \end{aligned}$$

which gives (7).

The previous Lemma gives the following results as particular examples: generating function of the Tetranacci sequence M_n is

$$f_{M_n}(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-x-x^2-x^3-x^4}$$

and generating function of the Tetranacci-Lucas sequence R_n is

$$f_{R_n}(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4}.$$

2 Generalized Tetranacci Quaternions and their Generating Functions and Binet's Formulas

In this section, we present generalized Tetranacci quaternions and give generating functions and Binet formulas for them. First, we give some information about quaternion sequences from the literature.

There are various types of quaternion sequences which have been studied by many researchers. Horadam [12] introduced nth Fibonacci and nth Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = \sum_{s=0}^{3} F_{n+s}e_s$$

and

$$R_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = \sum_{s=0}^{3} L_{n+s}e_s$$

respectively, where F_n and L_n are the nth Fibonacci and Lucas numbers respectively. He also gave generalized Fibonacci quaternion as

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3 = \sum_{s=0}^{3} H_{n+s}e_s$$

where H_n is the nth generalized Fibonacci number (which is now called Horadam number) by the recursive relation $H_1 = p$, $H_2 = p+q$, $H_n = H_{n-1}+H_{n-2}$ (p and q are arbitrary integers). Halici [8] gave the generating functions and Binet formulas for the Fibonacci and Lucas quaternions. We can list a few references of quaternion sequences. We list the references of a few second order quaternion sequences as

[4,9,18,23,25]

and we list the references of a few third order quaternion sequences as

[3,6,24,26].

Soykan [22] introduced the Tetranacci and Tetranacci-Lucas quaternions as fourth order quaternion sequences.

We now give generalized Tetranacci quaternions over the quaternion algebra $\mathbb{H}$. The nth generalized Tetranacci quaternion is

$$\widehat{V}_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}. \quad (8)$$

As special cases, the nth Tetranacci quaternion and the nth Tetranacci-Lucas quaternion are defined as

$$\widehat{M}_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}$$

and

$$\widehat{R}_n = R_n + iR_{n+1} + jR_{n+2} + kR_{n+3}$$

respectively. One can see that

$$\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3} + \widehat{V}_{n-4}. \quad (9)$$

The sequence $\{\widehat{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{V}_{-n} = -\widehat{V}_{-(n-1)} - \widehat{V}_{-(n-2)} - \widehat{V}_{-(n-3)} + \widehat{V}_{-(n-4)}.$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrence (9) holds for all integer n.

The conjugate of $\widehat{V}_n$ is given by

$$\overline{\widehat{V}_n} = V_n - iV_{n+1} - jV_{n+2} - kV_{n+3}.$$

Now, we will present Binet's formula for the generalized Tetranacci quaternions and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widehat{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, \\
\widehat{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3, \\
\widehat{\gamma} &= 1 + i\gamma + j\gamma^2 + k\gamma^3, \\
\widehat{\delta} &= 1 + i\delta + j\delta^2 + k\delta^3.
\end{aligned}$$

Theorem 1.*(Binet's Formula) For any integer n, the nth generalized Tetranacci quaternion is*

$$\widehat{V}_n = A\widehat{\alpha}\alpha^{n-6} + B\widehat{\beta}\beta^{n-6} + C\widehat{\gamma}\gamma^{n-6} + D\widehat{\delta}\delta^{n-6} \tag{10}$$

where A, B, C and D are as in Corollary 1.

Proof. Using Binet's formula of the generalized Tetranacci numbers, we have

$$\begin{aligned}
\widehat{V}_n &= V_n + iV_{n+1} + jV_{n+2} + kV_{n+3} \\
&= A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6} \\
&\quad + i(A\alpha^{n-5} + B\beta^{n-5} + C\gamma^{n-5} + D\delta^{n-5}) \\
&\quad + j(A\alpha^{n-4} + B\beta^{n-4} + C\gamma^{n-4} + D\delta^{n-4}) \\
&\quad + k(A\alpha^{n-3} + B\beta^{n-3} + C\gamma^{n-3} + D\delta^{n-3}) \\
&= A\widehat{\alpha}\alpha^{n-6} + B\widehat{\beta}\beta^{n-6} + C\widehat{\gamma}\gamma^{n-6} + D\widehat{\delta}\delta^{n-6}.
\end{aligned}$$

This proves (10).

As special cases, for any integer n, the Binet's Formula of nth Tetranacci quaternion is

$$\begin{aligned}
\widehat{M}_n &= \frac{\alpha - 1}{5\alpha - 8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta - 1}{5\beta - 8}\widehat{\beta}\beta^{n-1} \\
&\quad + \frac{\gamma - 1}{5\gamma - 8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta - 1}{5\delta - 8}\widehat{\delta}\delta^{n-1}
\end{aligned} \tag{11}$$

and the Binet's Formula of nth Tetranacci-Lucas quaternion is

$$\widehat{R}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n. \tag{12}$$

Now, we give generating function.

Theorem 2.*The generating function for the generalized Tetranacci quaternions is*

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + \widehat{V}_{-1}x^3}{1 - x - x^2 - x^3 - x^4}. \tag{13}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{V}_n x^n ,$$

be generating function of generalized Tetranacci quaternions. Then, using the definition of the generalized Tetranacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
&(1 - x - x^2 - x^3 - x^4)g(x) \\
&= \sum_{n=0}^{\infty} \widehat{V}_n x^n - x\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^2\sum_{n=0}^{\infty} \widehat{V}_n x^n \\
&\quad - x^3\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^4\sum_{n=0}^{\infty} \widehat{V}_n x^n \\
&= \sum_{n=0}^{\infty} \widehat{V}_n x^n - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+1} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+2} \\
&\quad - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+3} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+4} \\
&= \sum_{n=0}^{\infty} \widehat{V}_n x^n - \sum_{n=1}^{\infty} \widehat{V}_{n-1} x^n - \sum_{n=2}^{\infty} \widehat{V}_{n-2} x^n \\
&\quad - \sum_{n=3}^{\infty} \widehat{V}_{n-3} x^n - \sum_{n=4}^{\infty} \widehat{V}_{n-4} x^n \\
&= (\widehat{V}_0 + \widehat{V}_1 x + \widehat{V}_2 x^2 + \widehat{V}_3 x^3) \\
&\quad - (\widehat{V}_0 x + \widehat{V}_1 x^2 + \widehat{V}_2 x^3) \\
&\quad - (\widehat{V}_0 x^2 + \widehat{V}_1 x^3) - \widehat{V}_0 x^3 \\
&\quad + \sum_{n=4}^{\infty} (\widehat{V}_n - \widehat{V}_{n-1} - \widehat{V}_{n-2} - \widehat{V}_{n-3} - \widehat{V}_{n-4}) x^n \\
&= \widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 \\
&\quad + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3} + \widehat{V}_{n-4}$. Rearranging above equation, we get

$$g(x) = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3}{1 - x - x^2 - x^3 - x^4},$$

or

$$g(x) = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + \widehat{V}_{-1}x^3}{1 - x - x^2 - x^3 - x^4},$$

since $\widehat{V}_3 = \widehat{V}_2 + \widehat{V}_1 + \widehat{V}_0 + \widehat{V}_{-1}$.

As special cases, the generating functions for the Tetranacci and Tetranacci-Lucas quaternions are

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{M}_n x^n \\
&= \frac{(i + j + 2k) + (1 + j + 2k)x + (j + 2k)x^2 + (j + k)x^3}{1 - x - x^2 - x^3 - x^4}
\end{aligned} \tag{14}$$

and

$$\sum_{n=0}^{\infty}\widehat{R}_n x^n = \frac{(4+i+3j+7k)+(-3+2i+4j+8k)x}{1-x-x^2-x^3-x^4} + \frac{(-2+3i+5j+4k)x^2+(-1+4i+j+3k)x^3}{1-x-x^2-x^3-x^4} \quad (15)$$

respectively.

Next, we obtain the formula which give the summation of the first n generalized Tetranacci numbers.

Theorem 3. *For $n \geq 1$, we have*

$$\sum_{p=1}^{n} V_p = \frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3). \quad (16)$$

Proof. This is a Theorem in Soykan [21, Theorem 2.6].

Note that from above theorem, we have

$$\begin{aligned}\sum_{p=0}^{n} V_p &= V_0+\sum_{p=1}^{n} V_p \\ &= V_0+\frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3) \\ &= \frac{1}{3}(V_{n+2}+2V_n+V_{n-1}+2V_0+V_1-V_3). \quad (17)\end{aligned}$$

Note that, as special cases, for every integer $n \geq 0$, we have

$$\sum_{p=0}^{n} M_p = \frac{1}{3}(M_{n+2}+2M_n+M_{n-1}-1)$$

and

$$\sum_{p=0}^{n} R_p = \frac{1}{3}(R_{n+2}+2R_n+R_{n-1}+2).$$

Next, we have the formulas which give the summation of the first n generalized Tetranacci quaternions.

Theorem 4. *The summation formula for generalized Tetranacci quaternions is*

$$\sum_{p=0}^{n}\widehat{V}_p = \frac{1}{3}(\widehat{V}_{n+2}+2\widehat{V}_n+\widehat{V}_{n-1}+c) \quad (18)$$

where

$$\begin{aligned}c &= 2V_0+V_1-V_3+i(-V_0+V_1-V_3) \\ &\quad +j(-V_0-2V_1-V_3)+k(-V_0-2V_1-3V_2-V_3).\end{aligned}$$

Proof. Using (17), we obtain

$$\begin{aligned}\sum_{p=0}^{n}\widehat{V}_p &= \sum_{p=0}^{n} V_p+i\sum_{p=0}^{n} V_{p+1}+j\sum_{p=0}^{n} V_{p+2}+k\sum_{p=0}^{n} V_{p+3} \\ &= (V_0+...+V_n)+i(V_1+...+V_{n+1}) \\ &\quad +j(V_2+...+V_{n+2})+k(V_3+...+V_{n+3}),\end{aligned}$$

and so,

$$\begin{aligned}3\sum_{p=0}^{n}\widehat{V}_p &= (V_{n+2}+2V_n+V_{n-1}+2V_0+V_1-V_3) \\ &\quad +i(V_{n+3}+2V_{n+1}+V_n+2V_0+V_1-V_3-3V_0) \\ &\quad +j(V_{n+4}+2V_{n+2}+V_{n+1}+2V_0 \\ &\quad +V_1-V_3-3(V_0+V_1)) \\ &\quad +k(V_{n+5}+2V_{n+3}+V_{n+2}+2V_0 \\ &\quad +V_1-V_3-3(V_0+V_1+V_2)) \\ &= \widehat{V}_{n+2}+2\widehat{V}_n+\widehat{V}_{n-1}+c\end{aligned}$$

where

$$\begin{aligned}c &= 2V_0+V_1-V_3+i(2V_0+V_1-V_3-3V_0) \\ &\quad +j(2V_0+V_1-V_3-3(V_0+V_1)) \\ &\quad +k(2V_0+V_1-V_3-3(V_0+V_1+V_2)) \\ &= 2V_0+V_1-V_3+i(-V_0+V_1-V_3)+j(-V_0-2V_1-V_3) \\ &\quad +k(-V_0-2V_1-3V_2-V_3).\end{aligned}$$

Hence

$$\sum_{p=0}^{n}\widehat{V}_p = \frac{1}{3}(\widehat{V}_{n+2}+2\widehat{V}_n+\widehat{V}_{n-1}+c).$$

This proves (18).

As special cases we obtain the following summation formula for Tetranacci and Tetranacci-Lucas quaternions:

$$\sum_{p=0}^{n}\widehat{M}_p = \frac{1}{3}(\widehat{M}_{n+2}+2\widehat{M}_n+\widehat{M}_{n-1}-(1+i+4j+7k)) \quad (19)$$

and

$$\sum_{p=0}^{n}\widehat{R}_p = \frac{1}{3}(\widehat{R}_{n+2}+2\widehat{R}_n+\widehat{R}_{n-1}+(2-10i-13j-22k)). \quad (20)$$

respectively.

Next, we present the formulas which give the summation of odd and even generalized Tetranacci numbers.

Theorem 5. *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{p=1}^{n} V_{2p+1} = \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3)$.

(b) $\sum_{p=1}^{n} V_{2p} = \frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+V_0-V_1+3V_2-2V_3)$.

Proof. This is a Theorem in Soykan [21, Theorem 2.6].

Recall that from above theorem, we obtain

$$\begin{aligned}&\sum_{p=0}^{n} V_{2p+1} \\ &= V_1+\sum_{p=1}^{n} V_{2p+1} \\ &= V_1+\frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3) \\ &= \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0+2V_1-3V_2+V_3) \quad (21)\end{aligned}$$

and

$$\begin{aligned}
&\sum_{p=0}^{n} V_{2p} \\
&= V_0 + \sum_{p=1}^{n} V_{2p} \\
&= V_0 + \frac{1}{3}(2V_{2n+1} + V_{2n-1} - V_{2n-2} + V_0 - V_1 + 3V_2 - 2V_3) \\
&= \frac{1}{3}(2V_{2n+1} + V_{2n-1} - V_{2n-2} + 4V_0 - V_1 + 3V_2 - 2V_3). \quad (22)
\end{aligned}$$

Note that, as special cases, for every integer $n \geq 0$, we have

$$\begin{aligned}
&\sum_{p=0}^{n} M_{2p+1} \\
&= \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} - 2M_0 + 2M_1 - 3M_2 + M_3), \\
&\sum_{p=0}^{n} M_{2p} \\
&= \frac{1}{3}(2M_{2n+1} + M_{2n-1} - M_{2n-2} + 4M_0 - M_1 + 3M_2 - 2M_3)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{p=0}^{n} R_{2p+1} \\
&= \frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 2R_0 + 2R_1 - 3R_2 + R_3), \\
&\sum_{p=0}^{n} R_{2p} \\
&= \frac{1}{3}(2R_{2n+1} + R_{2n-1} - R_{2n-2} + 4R_0 - R_1 + 3R_2 - 2R_3).
\end{aligned}$$

Theorem 6. *For $n \geq 0$, we have the following formulas:*

(a)

$$\sum_{p=0}^{n} \widehat{V}_{2p+1} = \frac{1}{3}(2\widehat{V}_{2n+2} + \widehat{V}_{2n} - \widehat{V}_{2n-1} + d)$$

where

$$\begin{aligned}
d &= (-2V_0 + 2V_1 - 3V_2 + V_3) + i(V_0 - V_1 + 3V_2 - 2V_3) \\
&\quad + j(-2V_0 - V_1 - 3V_2 + V_3) + k(V_0 - V_1 - 2V_3),
\end{aligned}$$

(b)

$$\sum_{p=0}^{n} \widehat{V}_{2p} = \frac{1}{3}(2\widehat{V}_{2n+1} + \widehat{V}_{2n-1} - \widehat{V}_{2n-2} + e)$$

where

$$\begin{aligned}
e &= (4V_0 - V_1 + 3V_2 - 2V_3) + (-2V_0 + 2V_1 - 3V_2 + V_3)i \\
&\quad + (V_0 - V_1 + 3V_2 - 2V_3)j + (-2V_0 - V_1 - 3V_2 + V_3)k.
\end{aligned}$$

Proof. *The proof can be easily obtained by using (21) and (22), so we omit it.*

As special cases, we have the following two corollaries.

Corollary 2. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{p=0}^{n} \widehat{M}_{2p+1} = \frac{1}{3}(2\widehat{M}_{2n+2} + \widehat{M}_{2n} - \widehat{M}_{2n-1} + (1 - 2i - 2j - 5k))$.

(b) $\sum_{p=0}^{n} \widehat{M}_{2p} = \frac{1}{3}(2\widehat{M}_{2n+1} + \widehat{M}_{2n-1} - \widehat{M}_{2n-2} - (2 - i + 2j + 2k))$.

Corollary 3. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{p=0}^{n} \widehat{R}_{2p+1} = \frac{1}{3}(2\widehat{R}_{2n+2} + \widehat{R}_{2n} - \widehat{R}_{2n-1} - (8 + 2i + 11j + 11k))$.

(b) $\sum_{p=0}^{n} \widehat{R}_{2p} = \frac{1}{3}(2\widehat{R}_{2n+1} + \widehat{R}_{2n-1} - \widehat{R}_{2n-2} + (10 - 8i - 2j - 11k))$..

3 Matrices related with Generalized Tetranacci Quaternions

Let the square matrix B of order 4 as:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B = -1$.

Consider the sequence $\{U_n\}$ which is defined by the fourth-order recurrence relation

$$\begin{aligned}
U_n &= U_{n-1} + U_{n-2} + U_{n-3} + U_{n-4}, \\
U_0 &= U_1 = 0, U_2 = U_3 = 1.
\end{aligned}$$

The numbers U_n can be expressed using Binet's formula

$$\begin{aligned}
U_n &= \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}$$

Induction proof may be used to establish

$$B^n = \begin{pmatrix} U_{n+2} & U_{n+1} + U_n + U_{n-1} & U_{n+1} + U_n & U_{n+1} \\ U_{n+1} & U_n + U_{n-1} + U_{n-2} & U_n + U_{n-1} & U_n \\ U_n & U_{n-1} + U_{n-2} + U_{n-3} & U_{n-1} + U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2} + U_{n-3} + U_{n-4} & U_{n-2} + U_{n-3} & U_{n-2} \end{pmatrix} \quad (23)$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}. \quad (24)$$

Now, we define the matrices B_V as

$$B_V=\begin{pmatrix}\widehat{V}_5 & \widehat{V}_4+\widehat{V}_3+\widehat{V}_2 & \widehat{V}_4+\widehat{V}_3 & \widehat{V}_4\\ \widehat{V}_4 & \widehat{V}_3+\widehat{V}_2+\widehat{V}_1 & \widehat{V}_3+\widehat{V}_2 & \widehat{V}_3\\ \widehat{V}_3 & \widehat{V}_2+\widehat{V}_1+\widehat{V}_0 & \widehat{V}_2+\widehat{V}_1 & \widehat{V}_2\\ \widehat{V}_2 & \widehat{V}_1+\widehat{V}_0+\widehat{V}_{-1} & \widehat{V}_1+\widehat{V}_0 & \widehat{V}_1\end{pmatrix}.$$

This matrice B_V is called generalized Tetranacci quaternion matrix. As special cases, Tetranacci quaternion matrix and Tetranacci-Lucas quaternion matrix are

$$B_M=\begin{pmatrix}\widehat{M}_5 & \widehat{M}_4+\widehat{M}_3+\widehat{M}_2 & \widehat{M}_4+\widehat{M}_3 & \widehat{M}_4\\ \widehat{M}_4 & \widehat{M}_3+\widehat{M}_2+\widehat{M}_1 & \widehat{M}_3+\widehat{M}_2 & \widehat{M}_3\\ \widehat{M}_3 & \widehat{M}_2+\widehat{M}_1+\widehat{M}_0 & \widehat{M}_2+\widehat{M}_1 & \widehat{M}_2\\ \widehat{M}_2 & \widehat{M}_1+\widehat{M}_0+\widehat{M}_{-1} & \widehat{M}_1+\widehat{M}_0 & \widehat{M}_1\end{pmatrix}$$

and $$B_R=\begin{pmatrix}\widehat{R}_5 & \widehat{R}_4+\widehat{R}_3+\widehat{R}_2 & \widehat{R}_4+\widehat{R}_3 & \widehat{R}_4\\ \widehat{R}_4 & \widehat{R}_3+\widehat{R}_2+\widehat{R}_1 & \widehat{R}_3+\widehat{R}_2 & \widehat{R}_3\\ \widehat{R}_3 & \widehat{R}_2+\widehat{R}_1+\widehat{R}_0 & \widehat{R}_2+\widehat{R}_1 & \widehat{R}_2\\ \widehat{R}_2 & \widehat{R}_1+\widehat{R}_0+\widehat{R}_{-1} & \widehat{R}_1+\widehat{R}_0 & \widehat{R}_1\end{pmatrix},$$

respectively.

Theorem 7.*For $n\geq 0$, the following is valid:*

$$B_V\begin{pmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}^n =\begin{pmatrix}\widehat{V}_{n+5} & \widehat{V}_{n+4}+\widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+4}+\widehat{V}_{n+3} & \widehat{V}_{n+4}\\ \widehat{V}_{n+4} & \widehat{V}_{n+3}+\widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+3}\\ \widehat{V}_{n+3} & \widehat{V}_{n+2}+\widehat{V}_{n+1}+\widehat{V}_{n} & \widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+2}\\ \widehat{V}_{n+2} & \widehat{V}_{n+1}+\widehat{V}_{n}+\widehat{V}_{n-1} & \widehat{V}_{n+1}+\widehat{V}_{n} & \widehat{V}_{n+1}\end{pmatrix}. \quad (25)$$

Proof. We prove by mathematical induction on n. If $n=0$, then the result is clear. Now, we assume it is true for $n=k$, that is

$$B_VB^k=\begin{pmatrix}\widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4}\\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3}\\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2}\\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+1}\end{pmatrix}.$$

If we use (8), then we obtain $\widehat{V}_{k+4}=\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k$. Then, by induction hypothesis, we have

$$\begin{aligned}
&B_VB^{k+1}\\
&=(B_VB^k)B\\
&=\begin{pmatrix}\widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4}\\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3}\\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2}\\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+1}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\\
&=\begin{pmatrix}\widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5}\\ \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4}\\ \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3}\\ \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2}\end{pmatrix}\\
&=\begin{pmatrix}\widehat{V}_{k+6} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5}\\ \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4}\\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3}\\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2}\end{pmatrix}.
\end{aligned}$$

Thus, (25) holds for all non-negative integers n.

Corollary 4.*For $n\geq 0$, the following holds:*

$$\widehat{V}_{n+3}=\widehat{V}_3U_{n+2}+(\widehat{V}_2+\widehat{V}_1+\widehat{V}_0)U_{n+1}+(\widehat{V}_1+\widehat{V}_2)U_n+\widehat{V}_2U_{n-1}.$$

Proof. The proof can be seen by the coefficient of the matrix B_V and (23).

4 Conflict of Interest

The authors declare that they have no conflict of interest.

Acknowledgement

The authors would like to express sincere gratitude to the reviewers and editor for his/her valuable suggestions.

References

[1] J. B. Bacani and J. F. T. Rabago, On Generalized Fibonacci Numbers, *Appl. Math. Sci.*, **9(25)**, 3611-3622, (2015).

[2] J. Baez, The octonions, *Bull. Amer. Math. Soc.,* **39(2)**, 145-205, (2002).

[3] I. Akkus and G. Kızılaslan, On some Properties of Tribonacci Quaternions, *Analele ştiinţifice ale Universităţii "Ovidius" Constanţa. Seria Matematică*, **26(3)**, 5-20, (2018).

[4] P. Catarino, The Modified Pell and Modified k-Pell Quaternions and Octonions, *Adv. Appl. Clifford Algebras,* **26**, 577-590, (2016).

[5] G. Cerda-Morales, On a Generalization for Tribonacci Quaternions, *Mediterranean Journal of Mathematics*, **14(239)**, 1–12, (2017).

[6] G. Cerda-Morales, Identities for Third Order Jacobsthal Quaternions, *Adv. Appl. Clifford Algebras,* **27**, 1043-1053, (2017).

[7] G. P. Dresden and Z. Du, A Simplified Binet Formula for k-Generalized Fibonacci Numbers, *J. Integer Seq.,* **17**, art. 14.4.7, 9 pp., (2014).

[8] S. Halici, On Fibonacci Quaternions, *Adv. Appl. Cliff ord Algebras,* **22**, 321–327, (2012).

[9] S. Halici and A. Karataş, On a Generalization for Fibonacci Quaternions, *Chaos Solitons and Fractals,* **98**, 178–182, (2017).

[10] T. L. Hankins, *Sir William Rowan Hamilton*, Johns Hopkins University Press, Baltimore, (1980).

[11] G. S. Hathiwala and D. V. Shah, Binet–Type Formula For The Sequence of Tetranacci Numbers by Alternate Methods, *Mathematical Journal of Interdisciplinary Sciences,* **6(1)**, 37–48, (2017).

[12] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci quaternions, *Amer. Math. Monthly,* **70**, 289–291, (1963).

[13] F.T. Howard and F. Saidak, Zhou's Theory of Constructing Identities, *Congress Numer.,* **200**, 225-237, (2010).

[14] D. W. Lewis, Quaternion Algebras and the Algebraic Legacy of Hamilton's Quaternions, *Irish Math. Soc. Bulletin,* **57**, 41–64, (2006).
[15] R. S. Melham and A. G. Shannon, A Generalization of a Result of D'Ocagne, *The Fibonacci Quarterly*, **33(2)**, 135-138, (1995).
[16] R. S. Melham, Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, *Fibonacci Quarterly*, 305-311, (1999).
[17] L. R. Natividad, On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, *International Journal of Mathematics and Computing*, **3(2)**, (2013).
[18] E .Polatlı, A Generalization of Fibonacci and Lucas Quaternions, *Adv. Appl. Clifford Algebras*, **26 (2)**, 719-730, (2016).
[19] B. Singh, P. Bhadouria, O. Sikhwal and K. Sisodiya, A Formula for Tetranacci-Like Sequence, *Gen. Math. Notes*, **20(2)**, 136-141, (2014).
[20] N. J. A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/
[21] Y. Soykan, Gaussian Generalized Tetranacci Numbers, *Journal of Advances in Mathematics and Computer Science*, **31(3)**, 1-21, (2019).
[22] Y. Soykan, Tetranacci and Tetranacci-Lucas Quaternions, *Asian Research Journal of Mathematics*, **15(1)**, 1–24, (2019).
[23] A. Szynal-Liana and I. Wloch, The Pell quaternions and the Pell octonions, *Adv. Appl. Clifford Algebras*, **26(1)**, 435-440, (2016).
[24] A. Szynal-Liana and I. Wloch, Some Properties of Generalized Tribonacci Quaternions, Scientfic Issues, Jan Dlugosz University in Czestochowa, *Mathematics,* **XXII**, 73-81, (2017).
[25] D.Tasci,, On k-Jacobsthal and k-Jacobsthal-Lucas Quaternions, *Journal of Science and Arts*, **3(40)**, pp. 469-476, (2017).
[26] D. Tasci, Padovan and Pell-Padovan Quaternions, *Journal of Science and Arts*, **1(42)**, 125-132, (2018).
[27] M. E. Waddill, Another Generalized Fibonacci Sequence, *Fibonacci Quarterly*, **5(3)**, 209-227, (1967).
[28] M. E. Waddill, The Tetranacci Sequence and Generalizations, *The Fibonacci Quarterly*, 9-20, (1992).
[29] J. P. Ward, *Quaternions and Cayley Numbers: Algebra and Applications*, Kluwer Academic Publishers, London, (1997).
[30] M. N. Zaveri and J. K. Patel, Binet's Formula for the Tetranacci Sequence, *International Journal of Science and Research*, **5(12)**, 1911-1914, (2016).