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Numerical Solution of Calculus of Variation Problems via Exponential Spline Method

Reza Mohammadi[1,*], *Moosarreza Shamsyeh Zahedi*[2] *and Zahra Bayat*[2]

[1] Department of Mathematics, University of Neyshabur, P. O. Box 91136-899,Neyshabur, Iran
[2] Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran

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Abstract: In this work, an exponential spline method is developed and analyzed for approximating solutions of calculus of variations problems. The method uses a spline interpolant, which is constructed from exponential spline. It is proved to be second-order convergent. Finally some illustrative examples are included to demonstrate the applicability of the new technique. Numerical results confirm the order of convergence predicted by the analysis.

Keywords: Calculus of variation, Exponential spline, Convergence, Maximum absolute errors

1 Introduction

The calculus of variations has a long history of interaction with other branches of mathematics such as geometry and differential equations, and with physics, particularly mechanics. More recently, the calculus of variations has found applications in other fields such as economics and electrical engineering. Much of the mathematics underlying control theory, for instance, can be regarded as part of the calculus of variations [1].

The main problem that we will be investigating throughout the present paper is the following functional which is the simplest form of a variational problem as

$$J[y(x)] = \int_a^b F(x,y(x),y'(x))dx, \tag{1}$$

where J is the functional that its extremum must be found. To find the extreme value of J, the boundary points of the admissible curves are known in the following form:

$$y(a) = \alpha, \quad y(b) = \beta. \tag{2}$$

The necessary condition for $y(x)$ to extremize $J[y(x)]$ is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \tag{3}$$

with boundary conditions given in Eq. (2). The above boundary value problem, does not always have a solution and if the solution exists, it may not be unique. Note that if the solution of Euler's equation satisfies the boundary conditions, it is unique.

The general form of the variational problem in Eq. (1) is

$$J[y_1(x),\ldots,y_m(x)] = \int_a^b F(x,y_1(x),\ldots,y_m(x),y_1'(x),\ldots,y_m'(x))dx, \tag{4}$$

with the given boundary conditions for all functions

$$y_1(a) = \alpha_1,\ y_2(a) = \alpha_2,\ \ldots,\ y_m(a) = \alpha_m, \tag{5}$$

$$y_1(b) = \beta_1,\ y_2(b) = \beta_2,\ \ldots,\ y_m(b) = \beta_m. \tag{6}$$

Here the necessary condition for the extremum of the functional in Eq. (4) is to satisfy the following system of second-order differential equations

$$\frac{\partial F}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_j'}\right) = 0, \quad j = 1,2,\ldots,m, \tag{7}$$

with boundary conditions given in Eqs. (5)-(6). However, the above system of differential equations can be solved easily only for simple cases. More historical comments about variational problems are found in [1,2,3].

Many efforts are going on to develop efficient and high accuracy methods for solving calculus of variation

* Corresponding author e-mail: rez.mohammadi@gmail.com, mohammadi@neyshabur.ac.ir

problems. Gelfand [2] and Elsgolts [3] investigated the Ritz and Galerkin direct methods for solving variational problems. The Walsh series method is introduced to variational problems by Chen and Hsiao [4]. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. The authors in [5,6,7] applied some orthogonal polynomials on variational problems to find the continuous solutions for these problems.

Razzaghi and Marzban [8] introduced a new direct computational method via hybrid of Block-Pulse and Chebyshev functions to solve variational problems. Then, Razzaghi et al. [9,10] presented direct methods for solving variational problems using Legendre wavelets. The rationalized Haar functions are applied to variational problems by Razzagi and Ordokhani [11,12]. Dehghan and Tatari [13] aimed at producing approximate solutions of some variational problems, which are obtained in rapidly convergent series with elegantly computable components by the Adomian decomposition technique. Then, in their earlier research [14] the He's variational iteration method is employed for solving some problems in calculus of variations. Saadatmandi and Dehghan [15] used the Chebyshev finite difference method for finding the solution of the ordinary differential equations which arise from problems of calculus of variations. In [16] the homotopy-perturbation method has been intensively developed to obtain exact and approximate analytical solutions of variational problems by Abdulaziz and his co-authors. Dixit et al. [17] proposed a simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six. In [18], the variational iteration method was implemented to give approximate solution of the Euler-Lagrange, Euler-Poisson and Euler-Ostrogradsky equations as ordinary (or partial) differential equations which arise from the variational problems. Maleki and Mashali-Firouzi [19] proposed a direct method using nonclassical parameterization and nonclassical orthogonal polynomials, for finding the extremal of variational problems. Nazemi et al. [20] employed the differential transform method (DTM) for solving some problems in calculus of variations.

The term spline in the spline function arises from the prefabricated wood or plastic curve board, which is called spline and is used by a draftman to plot smooth curves through connecting the known points. Spline functions can be integrated and differentiated due to being piecewise polynomials and since they have basis with small support, many of the integral that occur in the numerical methods are zero. Thus, spline functions are adapted to numerical methods to get the solution of the differential equations. Numerical methods with spline functions in getting the numerical solution of the differential equations lead to band matrices which are solvable easily with some algorithms in the market with low cost computation. During last four decades, there has been a growing interest in the theory of splines and their applications (see [21,22]). For example, Rashidinia et al. [23] used cubic spline functions to develop a numerical method for the solution of second-order linear two-point boundary value problems.

In this paper, we have developed a new spline function method for solving problems in calculus of variations by using exponential spline. The main purpose is to analyze the efficiency of the exponential spline-difference method for such problems with sufficient accuracy.

The procedure is to combine a spline approximation for the second order space derivative with a difference approximation for the first order space derivative. The result is a finite difference scheme wherein the more usual explicit difference schemes are special cases. Also the combination of a finite difference and an exponential spline function techniques provide better accuracy than the finite difference methods. The method involves some parameters, which enable us to obtain the classes of methods. Our method is a modification of cubic spline method for solution of this equation. Application of our method is simple and in comparison with the existing well-known methods is accurate. The resulting spline difference scheme is analyzed for local truncation error and convergence. We have shown that by making use of the exponential spline function, the resulting exponential spline difference scheme gives a tri-diagonal system which can be solved efficiently by using a tri-diagonal solver.

The outline of this paper is as follows: In Section 2, we present the formulation of our method and useful spline difference formulas are given for discretization of the Equation (1). In Section 3, we present the convergence analysis of the introduced method. In Section 4, the numerical results obtained from applying the new method on six problems are shown. Finally a conclusion is drawn in Section 5.

2 Description of the method

2.1 Exponential spline

Let us consider a mesh with nodal points x_i on $[a,b]$ such that:

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b,$$

where $h = \frac{b-a}{N}$ and $x_i = a + ih$ for $i = 1,\ldots,N$. We also denote the function value $y(x_i)$ by y_i.

Let $S_i(x)$ be the exponential spline of the function $y(x)$ at the grid point x_i and be given by

$$S_i(x) = a_i + b_i(x - x_i) + c_i e^{\imath\tau(x-x_i)} + d_i e^{-\imath\tau(x-x_i)}, \quad (8)$$

for each $i = 0,\ldots,N$, where a_i, b_i, c_i, and d_i are unknown coefficients, τ is a free parameter and $\imath = \sqrt{-1}$. We first develop the explicit expressions for the four coefficients in (8) in terms of y_i, y_{i+1}, M_i and M_{i+1}, where:

$$S_i(x_i) = y_i, \qquad S_i(x_{i+1}) = y_{i+1},$$
$$S_i''(x_i) = M_i, \qquad S_i''(x_{i+1}) = M_{i+1}. \quad (9)$$

Now using (9), we can determine the four unknown coefficients in (8) as

$$a_i = y_i + \frac{M_i}{\tau^2},$$

$$b_i = \frac{M_{i+1}-M_i+\tau^2(y_{i+1}-y_i)}{h\tau^2},$$

$$c_i = \frac{M_i - e^{\imath\tau h}M_{i+1}}{\tau^2(e^{2\imath\tau h}-1)}$$

$$d_i = \frac{e^{\imath\tau h}(M_{i+1}-e^{\imath\tau h}M_i)}{\tau^2(e^{2\imath\tau h}-1)}.$$

The continuity of the first derivative of $S_i(x)$ at $x = x_i$ yields the following consistency relation:

$$\frac{1}{h^2}(y_{i+1}-2y_i+y_{i-1}) = (\kappa_1 M_{i-1}+2\kappa_2 M_i+\kappa_1 M_{i+1}), \quad (10)$$

where

$$\kappa_1 = \frac{1-e^{2\imath\tau h}+2\imath\tau h e^{\imath\tau h}}{\tau^2 h^2(e^{2\imath\tau h}-1)},$$
$$\kappa_2 = \frac{-1-\imath\tau h+(1-\imath\tau h)e^{2\imath\tau h}}{\tau^2 h^2(e^{2\imath\tau h}-1)}. \quad (11)$$

In the limiting case when $\tau \to 0$, then $(\kappa_1,\kappa_2) \to (\frac{1}{6},\frac{1}{3})$, and the relation defined by (10) reduces into ordinary cubic spline relation:

$$\frac{1}{h^2}(y_{i+1}-2y_i+y_{i-1}) = \frac{1}{6}\left[M_{i+1}+4M_i+M_{i-1}\right]. \quad (12)$$

Now, for analyzing the truncation error of the equation (10), we present the following lemma.

Lemma 2.1. Suppose $q(x) \in C^6[a,b]$. Then

$$T_i(h) = [\kappa_1 q''(x_{i+1})+2\kappa_2 q''(x_i)+\kappa_1 q''(x_{i-1})] - \frac{1}{h^2}[q(x_{i+1}) -2q(x_i)+q(x_{i-1})] = \frac{h^2}{12}(12\kappa_1-1)q^{(4)}(\xi_i) + \frac{h^4}{360}(30\kappa_1-1)q^{(6)}(\xi_i)+\mathscr{O}(h^6),$$

$$\xi_i \in (x_{i-1},x_{i+1}), \quad i = 1,\ldots,N-1, \quad (13)$$

where $2\kappa_1 + 2\kappa_2 = 1$.

Proof. According to the Taylor expansion, we have

$$q(x_{i+1}) = q(x_i)+hq'(x_i)+\frac{h^2}{2}q''(x_i)+\frac{h^3}{6}q'''(x_i)+\frac{h^4}{24}q^{(4)}(x_i) +\frac{h^5}{120}q^{(5)}(x_i)+\frac{h^6}{120}\int_0^1 q^{(6)}(x_i+sh)(1-s)^5 ds,$$

$$q(x_{i-1}) = q(x_i)-hq'(x_i)+\frac{h^2}{2}q''(x_i)-\frac{h^3}{6}q'''(x_i)+\frac{h^4}{24}q^{(4)}(x_i) -\frac{h^5}{120}q^{(5)}(x_i)+\frac{h^6}{120}\int_0^1 q^{(6)}(x_i-sh)(1-s)^5 ds.$$

Adding the two equalities above, we

$$\frac{1}{h^2}[q(x_{i+1})-2q(x_i)+q(x_{i-1})] = q''(x_i)+\frac{h^2}{12}q^{(4)}(x_i) +\frac{h^4}{120}\int_0^1[q^{(6)}(x_i+sh)+q^{(6)}(x_i-sh)](1-s)^5 ds. \quad (14)$$

Similarly, from the following Taylor expansions

$$q''(x_{i+1}) = q''(x_i)+hq'''(x_i)+\frac{h^2}{2}q^{(4)}(x_i)+\frac{h^3}{6}q^{(5)}(x_i) +\frac{h^4}{6}\int_0^1 q^{(6)}(x_i+sh)(1-s)^3 ds,$$

$$q''(x_{i-1}) = q''(x_i)-hq'''(x_i)+\frac{h^2}{2}q^{(4)}(x_i)-\frac{h^3}{6}q^{(5)}(x_i) +\frac{h^4}{6}\int_0^1 q^{(6)}(x_i-sh)(1-s)^3 ds,$$

we can obtain

$$\kappa_1 q''(x_{i+1})+2\kappa_2 q''(x_i)+\kappa_1 q''(x_{i-1}) = q''(x_i)+\frac{h^2}{12}\kappa_1 q^{(4)}(x_i) +\frac{h^4}{72}\kappa_1\int_0^1[q^{(6)}(x_i+sh)+q^{(6)}(x_i-sh)](1-s)^5 ds. \quad (15)$$

Subtracting (14) from (15) and using the mean value theorem of integration, we get

$$T_i(h) = [\kappa_1 q''(x_{l+1})+2\kappa_2 q''(x_i)+\kappa_1 q''(x_{i-1})] - \frac{1}{h^2}[q(x_{i+1})-2q(x_i) +q(x_{i-1})] = \frac{h^2}{12}(12\kappa_1-1)q^{(4)}(\xi_i)+\frac{h^4}{360}\int_0^1[q^{(6)}(x_i+sh)+ q^{(6)}(x_i-sh)](1-s)^3[5\kappa_1-3(1-s)^2]ds = \frac{h^2}{12}(12\kappa_1-1)q^{(4)}(\xi_i) +\frac{h^4}{360}[q^{(6)}(x_i+\hat{s}h)+q^{(6)}(x_i-\hat{s}h)]\int_0^1(1-s)^3[5\kappa_1-3(1-s)^2]ds = \frac{h^2}{12}(12\kappa_1-1)q^{(4)}(\xi_i)+\frac{h^4}{360}(30\kappa_1-1)q^{(6)}(\xi_i),$$
$$\hat{s} \in (0,1), \xi_i \in (x_{i-1},x_{i+1}).$$

This completes the proof.

2.2 Numerical method

For the sake of the simplicity, we consider a general form of equations (3) as follow

$$y'' = f(x,y,y'), \quad a < x < b, \quad (16)$$

subjected to boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta. \quad (17)$$

At the grid point (x_i), we may write differential equation (16) as

$$y_i'' = f(x_i, y_i, y_i'). \tag{18}$$

By using moment of spline in (18) we obtain

$$M_i = f(x_i, y_i, y_i'). \tag{19}$$

Following [24] now we use the following approximations for first derivative of y:

$$\begin{aligned} y_{i-1}' &= \frac{-y_{i+1}+4y_i-3y_{i-1}}{2h} - \frac{h^2}{3}y'''(\xi_i) + \mathscr{O}(h^3), \\ y_i' &= \frac{y_{i+1}-y_{i-1}}{2h} + \frac{h^2}{6}y'''(\xi_i) + \mathscr{O}(h^3), \\ y_{i+1}' &= \frac{3y_{i+1}-4y_i+y_{i-1}}{2h} - \frac{5h^2}{3}y'''(\xi_i) + \mathscr{O}(h^3). \end{aligned} \tag{20}$$

Now applying the difference formula (10) to the nonlinear equations (16) and (17) and using (20), we have

$$\begin{aligned} &\tfrac{1}{h^2}(y_{i+1}-2y_i+y_{i-1}) = [\kappa_1 f(x_{i-1}, y_{i-1}, \tfrac{-y_{i+1}+4y_i-3y_{i-1}}{2h}) \\ &+2\kappa_2 f(x_i, y_i, \tfrac{y_{i+1}-y_{i-1}}{2h}) + \kappa_1 f(x_{i+1}, y_{i+1}, \tfrac{3y_{i+1}-4y_i+y_{i-1}}{2h})], \\ &i = 1, 2, \ldots, N-1, \end{aligned} \tag{21}$$

where κ_1 and κ_2 are parameters defined in (11) which $2\kappa_1 + 2\kappa_2 = 1$.

The application of (21) at the points x_i, $i = 1, \ldots, N-1$ gives the $(N-1)\times(N-1)$ nonlinear system

$$\frac{1}{h^2}\mathscr{A}y - \mathscr{G}(y) = T(h), \tag{22}$$

where

$$\mathscr{A} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix},$$

$$\mathscr{G}(y) = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{N-2} \\ g_{N-1} \end{pmatrix}$$

and

$$\begin{aligned} g_i(y_{i-1}, y_i, y_{i+1}) = \kappa_1 f(x_{i-1}, y_{i-1}, y_{i-1}') + 2\kappa_2 f(x_i, y_i, y_i') \\ +\kappa_1 f(x_{i+1}, y_{i+1}, y_{i+1}'). \end{aligned}$$

In actual practice we use (13) and get

$$\frac{1}{h^2}\mathscr{A}\mathscr{S} - \mathscr{G}(\mathscr{S}) = 0, \tag{23}$$

where $\mathscr{S}$ is an approximation of the solution vector y.

The expressions of (22) and (23) become

$$\mathscr{A}y - h^2\mathscr{G}(y) = h^2 T(h), \tag{24}$$

$$\mathscr{A}\mathscr{S} - h^2\mathscr{G}(\mathscr{S}) = 0. \tag{25}$$

We assume that for all $x \in [a,b]$ and all $\zeta_i, \eta_i \in \mathbb{R}$, $i = 0, 1$, f satisfies the Lipschitz condition,

$$|f(x, \zeta_0, \zeta_1) - f(x, \eta_0, \eta_1)| \leq \mathscr{L}(|\zeta_0 - \eta_0| + |\zeta_1 - \eta_1|), \tag{26}$$

where $\mathscr{L} > 0$ is the Lipschitz constant.

Lemma 2.2. The matrix $\mathscr{A}$ is invertible.

Proof. See [25].

Proposition 2.3. Let the Lipschitz constant $\mathscr{L}$ satisfies the inequality $\mathscr{L}\|\mathscr{A}^{-1}\|_\infty(h^2\|\mathscr{A}_0\|_\infty + h\|\mathscr{A}_1\|_\infty) < 1$, then there exists an unique exponential spline that estimates the exact solution Y of the problem (16) with boundary conditions (17).

Proof. From Equation (25), we obtain

$$\mathscr{S} = h^2\mathscr{A}^{-1}\mathscr{G}(\mathscr{S}). \tag{27}$$

Putting $\mathscr{Z} = \mathscr{S}$, we introduce the following

$$\psi(\mathscr{Z}) = h^2\mathscr{A}^{-1}\mathscr{G}(\mathscr{Z}) = \mathscr{Z}, \tag{28}$$

where

$$\mathscr{G}(\mathscr{Z}) = \begin{pmatrix} g_1(z_0, z_1, z_2) \\ g_2(z_1, z_2, z_3) \\ \vdots \\ g_{N-2}(z_{N-3}, z_{N-2}, z_{N-1}) \\ g_{N-1}(z_{N-2}, z_{N-1}, z_N) \end{pmatrix} \text{ and } \mathscr{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N-2} \\ z_{N-1} \end{pmatrix}.$$

We will show that the function ψ has a unique fixed point, i.e. Equation (27) has a unique solution. Suppose $\mathscr{W}, \mathscr{V} \in \mathbb{R}^{N-1}$. By help of Eq. (28), we have

$$\begin{aligned} \|\psi(\mathscr{W}) - \psi(\mathscr{V})\|_\infty \leq \\ h^2\|\mathscr{A}^{-1}\|_\infty\|\mathscr{G}(\mathscr{W}) - \mathscr{G}(\mathscr{V})\|_\infty. \end{aligned} \tag{29}$$

Applying the mean value theorem and condition (26), we have

$$\begin{aligned} |f(x_i, \mathscr{W}(x_i), \mathscr{W}'(x_i)) - f(x_i, \mathscr{V}(x_i), \mathscr{V}'(x_i))| \leq \\ \mathscr{L}(|\mathscr{W}(x_i) - \mathscr{V}(x_i)| - |\mathscr{W}'(x_i) - \mathscr{V}'(x_i)|) \leq \\ \mathscr{L}(\|\mathscr{A}_0\|_\infty + h^{-1}\|\mathscr{A}_1\|_\infty)\|\mathscr{W} - \mathscr{V}\|_\infty. \end{aligned} \tag{30}$$

Therefore, we have

$$\begin{aligned} \|\mathscr{G}(\mathscr{W}) - \mathscr{G}(\mathscr{V})\|_\infty \leq \\ \mathscr{L}(\|\mathscr{A}_0\|_\infty + h^{-1}\|\mathscr{A}_1\|_\infty)\|\mathscr{W} - \mathscr{V}\|_\infty. \end{aligned} \tag{31}$$

Consequently, using Eq. (29) we obtain

$$\begin{aligned} \|\psi(\mathscr{W}) - \psi(\mathscr{V})\|_\infty \leq \\ \mathscr{L}\|\mathscr{A}^{-1}\|_\infty(h^2\|\mathscr{A}_0\|_\infty + h\|\mathscr{A}_1\|_\infty)\|\mathscr{W} - \mathscr{V}\|_\infty. \end{aligned} \tag{32}$$

Thus, if $\mathscr{L}\|\mathscr{A}^{-1}\|_\infty(h^2\|\mathscr{A}_0\|_\infty + h\|\mathscr{A}_1\|_\infty) < 1$ then ψ is a strong contraction mapping.

3 Convergence Analysis

We next discuss the convergence of the method (21) by using the method which presented in [25,26,27,28]. Substracting (24) from (25), we get

$$\mathscr{A}(y-\mathscr{S}) = h^2\left(\mathscr{G}(y)-\mathscr{G}(\mathscr{S})\right)+h^2T(h). \quad (33)$$

Proposition 3.1. Let the Lipschitz constant $\mathscr{L}$ satisfies the inequality $\mathscr{L}\|\mathscr{A}^{-1}\|_\infty\left(h^2\|\mathscr{A}_0\|_\infty + h\|\mathscr{A}_1\|_\infty\right) < \frac{1}{2}$, then there exists a constant $\mathscr{K} > 0$ which depends only on the function f such that

$$\|y-\mathscr{S}\|_\infty \le \mathscr{K}h^2. \quad (34)$$

Proof. Eq. (33) yields

$$(y-\mathscr{S}) = h^2\mathscr{A}^{-1}\left(\mathscr{G}(y)-\mathscr{G}(\mathscr{S})\right)+h^2\mathscr{A}^{-1}T(h). \quad (35)$$

From Lemmma 2.1 and (20), it is easy to see that $T(h) = \mathscr{O}(h^2)$, since there exists a constant $\mathscr{K}_1 > 0$ such that $\|T(h)\|_\infty \le \mathscr{K}_1 h^2$.

Therefore,

$$\|y-\mathscr{S}\|_\infty \le h^2\|\mathscr{A}^{-1}\|_\infty\|\mathscr{G}(y)-\mathscr{G}(\mathscr{S})\|_\infty + \mathscr{K}_1\|\mathscr{A}^{-1}\|_\infty h^4. \quad (36)$$

Now, following Eq. (26), it is easy to establish that

$$|f(x_i,Y_i,Y_i')-f(x_i,\mathscr{S}_i,\mathscr{S}_i')| \le \mathscr{L}(|Y_i-\mathscr{S}_i|-|Y_i-\mathscr{S}_i'|) \le \mathscr{L}((|Y_i-y_i|+|y_i-\mathscr{S}_i|)+(|Y_i'-y_i'|+|y_i'-\mathscr{S}_i'|)).$$

According to Stoer and Bulirsch [29], the interpolation with splines of degree 3 gives $\mathscr{O}(h^4)$ uniform norm errors for the interpolant and $\mathscr{O}(h^{4-r})$ errors for the r-th derivative of the interpolant. Thus, in our case, we have for any $y \in \mathscr{C}^4[a,b]$

$$\|D^r(Y-y)\|_\infty = \mathscr{O}(h^{4-r}), \text{ for } r = 0,1,2,3, \quad (37)$$

where D^r is the differential operator of order r.

Following (37) there exists a constant Ω_j such that

$$\|Y^{(j)}-y^{(j)}\|_\infty \le \Omega_j h^{4-j}\|Y^{(4)}\|_\infty, \text{ for } j = 0,1,2,3. \quad (38)$$

From (31), we obtain,

$$\|\mathscr{G}(y)-\mathscr{G}(\mathscr{S})\|_\infty \le \mathscr{L}\|y-\mathscr{S}\|_\infty(\|\mathscr{A}_0\|_\infty + h^{-1}\|\mathscr{A}_1\|_\infty)+ \mathscr{L}(\Omega_0h^4+\Omega_1h^3)\|y^{(4)}\|_\infty. \quad (39)$$

Now from equation (36), we get

$$[1-\mathscr{L}h^2\|\mathscr{A}^{-1}\|_\infty(\|\mathscr{A}_0\|_\infty + h^{-1}\|\mathscr{A}_1\|_\infty)]\|y-\mathscr{S}\|_\infty \le h^2\|\mathscr{A}^{-1}\|_\infty[\mathscr{L}(\Omega_0h^4+\Omega_1h^3)\|y^{(4)}\|_\infty + \mathscr{K}_1h^2]. \quad (40)$$

Consequently, if $\mathscr{L}\|\mathscr{A}^{-1}\|_\infty\left(h^2\|\mathscr{A}_0\|_\infty + h\|\mathscr{A}_1\|_\infty\right) < \frac{1}{2}$, we obtain

$$\|y-\mathscr{S}\|_\infty \le \frac{\mathscr{L}(\Omega_0h^2+\Omega_1h)\|y^{(4)}\|_\infty+\mathscr{K}_1}{\mathscr{L}(\|\mathscr{A}_0\|_\infty+(b-a)^{-1}\|\mathscr{A}_1\|_\infty)}h^2. \quad (41)$$

We investigate the convergence analysis of the main scheme (21) in the following main theorem.

Theorem 3.2. The exponential spline approximation $\mathscr{S}$ converges quadratically to the exact solution Y of the boundary value problem defined by Equations (16) and (17), i.e. $\|Y-\mathscr{S}\|_\infty = \mathscr{O}(h^2)$.

Proof. using Proposition 3.1, it can be easily verified that

$$\|y-\mathscr{S}\|_\infty \le \mathscr{K}h^2.$$

From Eq. (38) and the fact that $\|Y-\mathscr{S}\|_\infty \le \|Y-y\|_\infty + \|y-\mathscr{S}\|_\infty$, we prove the proposed result.

Remark 3.3. For $\kappa_1 = \frac{1}{6}$ and $\kappa_2 = \frac{1}{3}$, our method reduces to the cubic spline method for the solution of equation (16) with boundary conditions (17).

Remark 3.4. According to (13) and (20), for $\kappa_1 = \frac{1}{12}$ and $\kappa_2 = \frac{5}{12}$, our method is optimal second-order method.

Remark 3.5. According to (13) and (20), if we do not have the term $y'(x)$ in equation (16), then for $\kappa_1 = \frac{1}{12}$ and $\kappa_2 = \frac{5}{12}$, our method is fourth-order method.

4 Illustrative test problems

This section is devoted to computational results. In order to check the accuracy and reliability of the proposed algorithm, four examples whose exact solutions are known to us are presented to test the performance of our algorithms. All the experiments were performed in Mathematica 8. In our tests, we use the Newton method for solving the nonlinear equation. The starting vector is chosen to be zero.

Also we compare our method with existing methods for accuracy, conservation and computational cost. The Maximum absolute errors are measured by using following formula

$$L_\infty(h) = \max_{1\le i\le N-1}|Y_i-y_i|,$$

where Y and y represent the exact and approximate solutions, respectively, we calculate the classical convergence rate

$$R(h) = \frac{\ln\left(L_\infty(h)\right)-\ln\left(L_\infty(h/2)\right)}{\ln 2}.$$

Table 1: *The maximum absolute errors with various values of N and different values of parameters κ_1 and κ_2 for Example 4.1.*

N	Our method $\kappa_1=\frac{1}{6}$, $\kappa_2=\frac{1}{3}$	R	Our method $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$	R
4	2.6505×10^{-1}	—	7.0628×10^{-3}	—
8	6.9076×10^{-2}	1.9400	4.7907×10^{-4}	3.8819
16	1.7163×10^{-2}	2.0088	3.0066×10^{-5}	3.9939
32	4.2843×10^{-3}	2.0023	1.8811×10^{-6}	3.9984
64	1.0706×10^{-3}	2.0005	1.1760×10^{-7}	3.9996
128	2.6764×10^{-4}	2.0001	7.3505×10^{-9}	3.9999

Table 2: *The observed maximum absolute errors with various values of N for Example 4.2.*

N	Our method $\kappa_1=\frac{1}{6}$, $\kappa_2=\frac{1}{3}$	Our method $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$	CFD method in [15]	NP method in [19]
3	1.6167×10^{-5}	1.3574×10^{-6}	1.8×10^{-5}	6.3×10^{-6}
4	8.7687×10^{-6}	4.2965×10^{-7}	1.5×10^{-6}	6.6×10^{-7}
5	5.7169×10^{-6}	1.8052×10^{-7}	5.8×10^{-9}	1.8×10^{-9}
6	3.8265×10^{-6}	8.4549×10^{-8}	6.6×10^{-10}	2.2×10^{-10}
7	2.8760×10^{-6}	4.6916×10^{-8}	6.7×10^{-11}	—

Example 4.1. *Consider the following problem [15]*

$$\min \quad J[y(x)] = \int_0^1 (y(x)+y'(x)-4\exp(3x))^2 dx, \quad (42)$$

subjected to given boundary conditions

$$y(0)=1, \quad y(1)=\exp(3). \quad (43)$$

The corresponding Euler-Lagrange equation is

$$y''(x)-y(x)-8\exp(3x)=0, \quad (44)$$

with boundary conditions (43). The exact solution of this problem is $Y(x)=\exp(3x)$.

In Table 1, the maximum absolute errors between the exact solution $Y(x)$ and the approximate solution and the numerical convergence rates with $N=4,8,16,32,64,128$ and for different values of parameters κ_1 and κ_2. From Table 1, we see that we can achieve a good approximation for the exact solution using exponential spline method. We observe that present method is nearly of second order of convergence with respect to these error norms.

Example 4.2. *Consider the following minimization problem [15,19]*

$$\min \quad J[y(x)] = \int_0^1 \frac{1+(y(x))^2}{(y'(x))^2} dx, \quad (45)$$

with the boundary conditions

$$y(0)=0, \quad y(1)=0.5. \quad (46)$$

Note that the exact solution to this problem is $Y(x)=\sinh(0.4812118250x)$.

The corresponding Euler-Poisson equation is of the form

$$y''(x)+y''(x)(y(x))^2-y(x)(y'(x))^2=0, \quad (47)$$

with boundary conditions (46).

We solved the problem, by applying the technique described in Section 2 with $N=3,4,5,6,7$ and for different values of parameters κ_1 and κ_2. In Table 2, the results of the presented method are compared with Chebyshev finite difference method [15] and nonclassical parameterization method [19]. According to Table 2, we find that the presented method provides accurate results even for small N. From Table 2, we see that we can achieve a good approximation for the exact solution using exponential spline method and also our results are in good agreement with the methods introduced in [15,19].

Example 4.3. *In this example, consider the following variational problem [8,9,11,19]:*

$$\min \quad J[y(x)] = \int_0^1 ((y'(x))^2 + xy'(x) + (y(x))^2) dx, \quad (48)$$

with given boundary conditions

$$y(0)=0, \quad y(1)=\frac{1}{4}. \quad (49)$$

The exact solution of this problem is

$$Y(x)=\frac{1}{2}+\frac{2-e}{4(e^2-1)}e^x+\frac{e(1-2e)}{4(e^2-1)}e^{-x}.$$

The Euler-Lagrange equation of this problem is

$$y''(x)-y(x)-\frac{1}{2}=0, \quad (50)$$

with boundary conditions (49).

This example has been solved by using our scheme (21) with different values of $N=4,8,16,32,64,128$ and parameters κ_1 and κ_2. The maximum absolute errors in solution and the numerical convergence rates and comparison with [8,9,11,19] are tabulated in Tables 3 and 4 respectively. From Tables 3-5, we see that we can achieve a good approximation for the exact solution using our method and also our results are in good agreement with the methods introduced in [8,9,11,19].

Example 4.4. *Consider the problem of finding the extremal of the functional [3,14,15,19]*

$$J[y(x),z(x)] = \int_0^{\frac{\pi}{2}} ((y'(x))^2 + (z'(x))^2 + 2y(x)z(x)) dx, \quad (51)$$

let the boundary conditions be

$$y(0)=0, \quad y(\frac{\pi}{2})=1, \quad z(0)=0, \quad z(\frac{\pi}{2})=-1, \quad (52)$$

Table 3: *The maximum absolute errors with various values of N and different values of parameters κ_1 and κ_2 for Example 4.3.*

N	Our method $\kappa_1=\frac{1}{6}$, $\kappa_2=\frac{1}{3}$	R	Our method $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$	R
4	2.0168×10^{-4}	—	6.2379×10^{-7}	—
8	5.0124×10^{-5}	2.0085	3.9058×10^{-8}	3.9973
16	1.2522×10^{-5}	2.0009	2.4442×10^{-9}	3.9982
32	3.1402×10^{-6}	1.9955	1.5266×10^{-10}	4.0009
64	7.8498×10^{-7}	2.0001	9.5825×10^{-12}	3.9937
128	1.9624×10^{-7}	2.0000	5.9924×10^{-13}	3.9992

Table 4: *Estimated and exact values of $y(x)$ for Example 4.3.*

x	RH functions in [11]	x	Hybrid in [8]	Legendre wavelets method in [9]
	—	0.1	0.041949	0.041949
$x\in[0,\frac{1}{8})$	0.0396	0.2	0.079318	0.079315
$x\in[\frac{1}{8},\frac{2}{8})$	0.0761	0.3	0.112472	0.112471
$x\in[\frac{2}{8},\frac{3}{8})$	0.1146	0.4	0.141750	0.141749
$x\in[\frac{3}{8},\frac{4}{8})$	0.1482	0.5	0.167444	0.167443
$x\in[\frac{4}{8},\frac{5}{8})$	0.1817	0.6	0.189805	0.189807
$x\in[\frac{5}{8},\frac{6}{8})$	0.2078	0.7	0.209065	0.209064
$x\in[\frac{6}{8},\frac{7}{8})$	0.2267	0.8	0.225412	0.225411
$x\in[\frac{7}{8},1)$	0.2398	0.9	0.239011	0.239010
$x=1$	0.2515	1	0.249999	0.249999

Table 5: *Estimated and exact values of $y(x)$ for Example 4.3.*

x	nonclassical parameterization in [19]	Our method for $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$ and $N=10$	Exact
0.1	0.04195073	0.04195073	0.04195073
0.2	0.07931715	0.07931715	0.07931715
0.3	0.11247322	0.11247322	0.11247323
0.4	0.14175080	0.14175080	0.14175081
0.5	0.16744292	0.16744292	0.16744292
0.6	0.18980669	0.18980669	0.18980668
0.7	0.20906593	0.20906593	0.20906592
0.8	0.22541340	0.22541340	0.22541340
0.9	0.23901272	0.23901272	0.23901272
1	0.25000000	0.25000000	0.25000000

which has the following analytical solution

$$Y(x)=\sin x, \quad \text{and} \quad Z(x)=-\sin x.$$

In this case the Euler-Lagrange equations are written in the following form:

$$y''(x)-z(x)=0, \quad \text{and} \quad z''(x)-y(x)=0, \tag{53}$$

Table 6: *The observed maximum absolute errors of $y(x)$ with various values of N for Example 4.4.*

N	Our method $\kappa_1=\frac{1}{6}$, $\kappa_2=\frac{1}{3}$	Our method $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$	CFD method in [15]	NP method in [19]
4	3.4816×10^{-4}	2.7691×10^{-7}	3.7×10^{-4}	—
6	1.5686×10^{-4}	5.4501×10^{-8}	8.9×10^{-7}	1.8×10^{-7}
8	8.8659×10^{-5}	1.7223×10^{-8}	2.1×10^{-9}	—
10	5.6869×10^{-5}	7.0508×10^{-9}	2.6×10^{-12}	1.2×10^{-12}
12	3.9715×10^{-5}	3.4142×10^{-9}	2.8×10^{-15}	3.8×10^{-15}
14	2.9295×10^{-5}	1.8486×10^{-9}	1.6×10^{-16}	—

Table 7: *The observed maximum absolute errors of $z(x)$ with various values of N for Example 4.4.*

N	Our method $\kappa_1=\frac{1}{6}$, $\kappa_2=\frac{1}{3}$	Our method $\kappa_1=\frac{1}{12}$, $\kappa_2=\frac{5}{12}$	CFD method in [15]	NP method in [19]
4	3.48169×10^{-4}	2.76910×10^{-7}	3.7×10^{-4}	—
6	1.56862×10^{-4}	5.45018×10^{-8}	8.9×10^{-7}	1.8×10^{-7}
8	8.86597×10^{-5}	1.72236×10^{-8}	2.1×10^{-9}	—
10	5.68690×10^{-5}	7.05084×10^{-9}	2.6×10^{-12}	1.2×10^{-12}
12	3.97151×10^{-5}	3.41424×10^{-9}	2.8×10^{-15}	4.3×10^{-15}
14	2.92958×10^{-5}	1.84865×10^{-9}	1.6×10^{-16}	—

with boundary conditions (52).

We solved the problem, by applying the presented method with different values of $N=4,6,8,10,12,14$ and parameters κ_1 and κ_2. In Tables 5 and 6, the maximum of absolute errors of the resulting approximate solutions are compared with those obtained in Saadatmandi and Dehghan [15] and Maleki and Mashali-Firouzi [19]. From Tables 6 and 7, we see that we can achieve a good approximation for the exact solution using presented method and also our results are in good agreement with the method introduced in [15,19].

5 Conclusion

In this article, we proposed a numerical scheme, based on the exponential spline method, to solve the problems in calculus of variation. The properties of the non-polynomial splines are used to reduce the Euler-Lagrange equation to the solution of system of nonlinear algebraic equations. The solution obtained using the suggested method shows that this approach can solve the problem effectively and it needs less CPU time. Comparisons are made between the approximate and exact solutions and another methods to illustrate the validity and the great potential of the new technique. Moreover, employing the new technique only a small number of the grid points are needed to obtain a satisfactory result. One can use the presented method for solving different types of partial differential equations.

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References

[1] Bruce van Brunt, The Calculus of Variations, Springer-Verlag, New York, 2004.
[2] I. M. Gelfand, S. V. Fomin, Calculus of Variations, Prentice-Hall, NJ, 1963 (revised English edition translated and edited by R.A. Silverman).
[3] L. Elsgolts, Differential Equations and Calculus of Variations, Mir, Moscow, 1977 (translated from the Russian by G. Yankovsky).
[4] C. F. Chen, C. H. Hsiao, A walsh series direct method for solving variational problems, J. Franklin Inst. 300 (1975) 265-280.
[5] R. Y. Chang, M. L. Wang, Shifted Legendre direct method for variational problems, J. Optim. Theory Appl. 39 (1983) 299-306.
[6] I. R. Horng, J. H. Chou, Shifted Chebyshev direct method for solving variational problems, Internat. J. Systems Sci. 16 (1985) 855-861.
[7] C. Hwang, Y. P. Shih, Laguerre series direct method for variational problems, J. Optim. Theory Appl. 39 (1) (1983) 143-149.
[8] M. Razzaghi, H. R. Marzban, Direct method for variational problems via of Block-Pulse and chebyshev functions, Mathematical Problems in Engineering 6 (2000) 85-97.
[9] M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, Math. Comput. Simulation 53 (2000) 185-192.
[10] M. Razzaghi, S. Yousefi, Legendre Wavelets Method for the Solution of Nonlinear Problems in the Calculus of Variations, Mathematical and Computer Modellmg 34 (2001) 45-54.
[11] M. Razzaghi, Y. Ordokhani, An application of rationalized Haar functions for variational problems, Appl. Math. Comput. 122 (2001) 353-364.
[12] M. Razzaghi, Y. Ordokhani, Solution for a classical problem in the calculus of variations via rationalized haar functions, Kybernetika 37 (5) (2001) 575-583.
[13] M. Dehghan, M. Tatari, The use of Adomian decomposition method for solving problems in calculus of variations, Math. Probl. Eng. 2006 (2006) 1-12.
[14] M. Tatari, M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, Phys. Lett. A 362 (2007) 401-406.
[15] A. Saadatmandi, M. Dehghan, The numerical solution of problems in calculus of variation using Chebyshev finite difference method, Phys. Lett. A 372 (2008) 4037-4040.
[16] O. Abdulaziz, I. Hashim. M. S. H. Chowdhury, Solving variational problems by homotopy perturbation method, International Journal of Numerical Methods in Engngineering, 75 (2008) 709-721.
[17] S. Dixit, V. K. Singh, A. K. Singh, O. P. Singh, Bernstein Direct Method for Solving Variational Problems, International Mathematical Forum, 5 (48) (2010) 2351-2370.
[18] S. A. Yousefi, M. Dehghan, The use of He's variational iteration method for solving variational problems, Int. J. Comput. Math. 87 (6) (2010) 1299-1314.
[19] M. Maleki, M. Mashali-Firouzi, A numerical solution of problems in calculus of variation using direct method and nonclassical parameterization, Journal of Computational and Applied Mathematics 234 (2010) 1364-1373.
[20] A. R. Nazemi, S. Hesam, A. Haghbin, A fast numerical method for solving calculus of variation problems, AMO - Advanced Modeling and Optimization, 15 (2) (2013) 133-149.
[21] J. H. Ahlberg, J. H. Nilson, E. N. Walsh, The Theory of Splines and Their Applications. Academic Press, San Diego, 1967.
[22] C. De Boor, Practical Guide to Splines. Springer, Berlin, 1978.
[23] J. Rashidinia, R. Mohammadi, R. Jalilian, Cubic spline method for two-point boundary value problems, IUST International Journal of Engineering Science, 19 (5-2) (2008) 39-43.
[24] J. Rashidinia, R. Mohammadi, R. Jalilian, The numerical solution of non-linear singular boundary value problems arising in physiology, Appl. Math. Comput. 185 (2007) 360-367.
[25] P. Henrici, Discrete variable methods in ordinary differential equations, Wiley, New York, 1962.
[26] J. Rashidinia, M. Ghasemi, R. Jalilian, A collocation method for the solution of nonlinear one-dimensional parabolic equations, Mathematical Sciences, 4 (1) (2010) 87-104.
[27] R. Mohammadi, B-Spline Collocation Algorithm for Numerical Solution of the Generalized Burgers-Huxley Equation, Numer Methods Partial Differential Eq 29 (2013) 1173-1191.
[28] A. Lamnii, H. Mraoui, D. Sbibih, A. Tijini and A. Zidna, Sextic spline collocation methods for nonlinear fifth-order boundary value problems, International Journal of Computer Mathematics 88 (10) (2011) 2072-2088.
[29] J. Stoer and R. Bulrisch, An introduction to numerical analysis, Springer-Verlag, New York, 1991.

Reza Mohammadi is an Assistant Professor of Applied Mathematics in School of Basic Sciences, University of Neyshabur, Neyshabur, Iran, since 2010. He received his Ph.D. degree in Applied Mathematics (Numerical analysis) from Iran University of Science and Technology in 2010 under supervision of Professor Jalil Rashidinia. Research Interests: Applied mathematics and computational sciences Numerical solution of Ordinary Differential Equations and Partial Differential Equations, Spline approximations.