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A New Approach on Isomorphism Theorems in Fuzzy Topological Partial Groups

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Abstract: In this paper, we introduce the notion of the fuzzy normal partial group, discussing its key properties. Subsequently, the notion of the fuzzy quotient partial group is constructed, where the quotient is over a fuzzy normal partial group, and the fuzzy kernel of the fuzzy partial group homomorphism is then formulated. Finally, we present the isomorphism theorems of fuzzy topological partial groups using this new framework, as our central result.

Keywords: Fuzzy Topology; Normal Group; Fuzzy Partial Group; Fuzzy Topological Group; Topological Partial Group

1 Introduction

The fuzzy set theory [1] has been widely applied to generalize various topological [2–4] and algebraic [5–7] structures. In [8], the fuzzy topological partial group was introduced, building on the definition of topological partial groups established by Abd-Allah et al [9]-that is, a partial group acquiring a topology under which the partial group operations are continuous.

The quotient fuzzy topological partial group was presented in [8], where the underlying set is the quotient of a fuzzy partial group ξ in a partial group X, over a crisp normal partial group N. The quotient, ξ/N, is a fuzzy partial group of the quotient partial group X/N, whose elements are the cosets xN. However, this approach relies on classical normal partial groups, which does not align with the central theme of the fuzzy theory, where classical structures are generally replaced by fuzzy structures.

In this paper, we introduce a novel approach by defining the quotient fuzzy partial group as the quotient of a fuzzy partial group over a fuzzy normal partial group. This new notion of the fuzzy normal partial group acts as the foundational concept of this paper.

The fuzzy normal group was defined in [10–13], with these definitions essentially sharing a common condition of normality, which is xyx^{-1} and y having the same membership degree in the fuzzy group. We follow the approach of [13], generalizing the normality property of fuzzy groups to fuzzy partial groups. We begin by investigating the basic properties of fuzzy cosets of a fuzzy partial group. We then define the fuzzy normal partial group, providing examples and establishing its main properties. The newly defined notion of the fuzzy normal partial group is then used to redefine the fuzzy quotient partial group.

We also study the fuzzy kernel of fuzzy partial group homomorphisms, thereby fuzzifying the classical kernel. Finally, we prove the isomorphism theorems in fuzzy partial groups and fuzzy topological partial groups, replacing the crisp kernel used in [8] with the fuzzy kernel. Furthermore, we establish an isomorphism involving the product fuzzy topological partial group.

2 Preliminaries

We provide basic definitions, used notations and necessary results utilised throughout the paper. We consider $I = [0,1]$.

Definition 2.1 *[1] Let X be a set and $\alpha : X \longrightarrow I$ be a mapping. We call α a fuzzy set, that associates a membership degree $\alpha(x)$ to every element $x \in X$.*

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Definition 2.2 *[4] Let X be a set and δ be a collection of fuzzy subsets of X. We call (X,δ) a fuzzy topological space if:*

1. *$I_c \in \delta, \forall c \in I$, where I_c is a constant fuzzy set,*
2. *If $\mu, \nu \in \delta$, then $\mu \cap \nu \in \delta$,*
3. *If $\mu_j \in \delta, \forall j \in J$, then $\bigcup_{j\in J} \mu_j \in \delta$.*

We refer to the elements of δ as fuzzy open sets, and the fuzzy topological space will be called FTS for short.

Definition 2.3 *[3] Let $\{(X_j,\delta_j), j \in J\}$ be a collection of FTS's, $X = \prod_{j\in J} X_j$, and let δ be the smallest topology on X such that the projection maps P_j are fuzzy continuous, $\forall j \in J$. Then, we call (X,δ) the product FTS.*

Definition 2.4 *[6] Let (X,δ) be an FTS, and let α be a fuzzy subset of X. Then, $\delta_\alpha = \{\mu \cap \alpha : \mu \in \delta\}$ is called the induced fuzzy topology on α, and we call (α,δ_α) a fuzzy subspace of (X,δ).*

Proposition 2.1 *[6] Let $\{(X_i,\delta_i), i = 1,2,...,n\}$, be a collection of FTS's, and let (X,δ) be their product FTS. Let α_i be a fuzzy subset of X_i, and δ_{α_i} be the induced fuzzy topology on $\alpha_i, \forall i = 1,2,...,n$. Then, the product of the fuzzy subspaces $\{(\alpha_i,\delta_{\alpha_i}), i = 1,2,...,n\}$, is a fuzzy subspace of (X,δ).*

Definition 2.5 *[6] Let $f : X \longrightarrow Y$, where (X,δ) is an FTS. The fuzzy image topology $f[\delta] = \{\mu \in Y : f^{-1}[\mu] \in \delta\}$ is the largest fuzzy topology on Y such that f is fuzzy continuous.*

Definition 2.6 *[5] Let γ be a fuzzy subset of X, where X is a group. We call γ a fuzzy group if:*

1. *$\gamma(xy) \geq \min\{\gamma(x),\gamma(y)\}$, $\forall x,y \in X$;*
2. *$\gamma(x^{-1}) \geq \gamma(x)$, $\forall x \in X$.*

Definition 2.7 *[13] Let γ be a fuzzy group in a group X. γ is called fuzzy normal if $\gamma(xyx^{-1}) = \gamma(y)$, $\forall x,y \in X$.*

Definition 2.8 *[5] Let γ be a fuzzy group in a group X and φ be a mapping on X. Then, γ is called φ-invariant if $\forall x_1,x_2 \in X$ such that $\varphi(x_1) = \varphi(x_2)$, then $\gamma(x_1) = \gamma(x_2)$.*

Proposition 2.2 *[5] Let $\varphi : X \longrightarrow Y$ be a group homomorphism and let γ be a fuzzy group in X. Then, $\varphi[\gamma]$ is a fuzzy group in Y provided that γ is φ-invariant.*

Definition 2.9 *[6] Let γ be a fuzzy group in a group X, and let $a \in X$. Then, the two mappings $R_a : \gamma \longrightarrow \gamma; x \longmapsto xa$ and $L_a : \gamma \longrightarrow \gamma; x \longmapsto ax$, are the right transformation and the left transformation, respectively.*

Definition 2.10 *[9] Let X be a semigroup. Then, we call X a partial group if:*

1. *For all $x \in X$ there exists a partial identity e_x,*
2. *For all $x \in X$ there exists a partial inverse x^{-1}.*
3. *The mapping $e_X : X \longrightarrow X; x \longmapsto e_x$ is a homomorphism.*
4. *The mapping $\sigma_X : X \longrightarrow X; x \longmapsto x^{-1}$ is an antihomomorphism.*

Let $E(X) = \{e_x : x \in X\}$, $X_a = \{x \in X : e_x = e_a\}$.

Proposition 2.3 *[9] Let X be a partial group. Then, the set $E(X)$ is central and commutative.*

Proposition 2.4 *[9] Let X be a partial group and let $a,b \in X$. Then,*

1. *X_a is a group with identity e_a,*
2. *X_a and X_b are either equal or disjoint.*

Proposition 2.5 *[9] A partial group is a disjoint union of a collection of groups X_i, indexed by a semilattice Y of the identities e_i of these groups, where $e_i \geq e_j, i,j \in Y$ if $e_ie_j = e_j$.*

Definition 2.11 *[9] Let N be a subsemigroup of a partial group X. Then, N is a subpartial group of X if $x^{-1}, e_x \in N$, $\forall x \in N$.*

Definition 2.12 *[9] A subpartial group N of a partial group X is called normal if $E(X) \subseteq N$ and $xyx^{-1} \in N$, $\forall x \in X, y \in N$.*

Definition 2.13 *[9] Let $f : X \longrightarrow Y$ be a homomorphism of partial groups. Then, $\ker f = \{x \in X : f(x) = e_{f(x)}\}$.*

Proposition 2.6 *[9] The kernel of a partial group homomorphism is normal.*

Definition 2.14 *[9] Let $f : X \longrightarrow Y$ be a homomorphism of partial groups. Then, f is said to be idempotent separating if $f(e_x) = f(e_y)$ implies that $e_x = e_y$, $\forall x,y \in X$.*

Definition 2.15 *[8] Let X be a partial group, and ξ be a fuzzy subset of X. We call ξ a fuzzy partial group in X if $\forall x,y \in X$,*

1. *$\xi(xy) \geq \min\{\xi(x),\xi(y)\}$;*
2. *$\xi(x^{-1}) = \xi(x)$;*
3. *$\xi(e_x) = k$, where k is a constant.*

The third condition can be stated differently by saying that ξ is constant on $E(X)$, moreover, it implies that $\xi(e_x) = \sup_{x\in X} \xi(x)$, $\forall x \in X$.

Proposition 2.7 *[8] Let $\{X_i, i = 1,2,\ldots,n\}$ be a collection of partial groups and $X = \prod_{i=1}^{n} X_i$ be the product partial group. Let $\xi = \prod_{i=1}^{n} \xi_i$, where ξ_i is a fuzzy partial group in $X_i, \forall i = 1,2,\ldots,n$. Then, ξ is a fuzzy partial group in X.*

Definition 2.16 *[8] Let (X,δ) be an FTS, and (ξ,δ_ξ) be the fuzzy subspace. Then, we call ξ a fuzzy topological partial group (FTPG for short) if the following mappings are fuzzy continuous:*

1. $\beta_\xi : \xi \times \xi \longrightarrow \xi; (x,y) \longmapsto xy$.
2. $\sigma_\xi : \xi \longrightarrow \xi; x \longmapsto x^{-1}$.
3. $e_\xi : \xi \longrightarrow \xi; x \longmapsto e_x$.

Proposition 2.8 *[8] Let $\varphi : X \longrightarrow Y$ be a partial group homomorphism, and let (X,δ), (Y,ω) be two FTS's, such that $\omega = \varphi[\delta]$, and φ is fuzzy open. Let ξ be an FTPG in X. Then, $\varphi[\xi]$ is an FTPG in Y provided that ξ is φ-invariant.*

Proposition 2.9 *[8] Let (X_i,δ_i) be an FTS and ξ_i be a fuzzy partial group in partial group X_i, $\forall i = 1,2,\ldots,n$. Let $X = \prod_{i=1}^{n} X_i, \xi = \prod_{i=1}^{n} \xi_i$ have respective product fuzzy topologies δ and δ_ξ. If $\forall i = 1,2,\ldots,n$, ξ_i is an FTPG in X_i, then the product fuzzy partial group ξ is an FTPG in X.*

3 Fuzzy Normal Partial Groups

In the current section, we present the concept of the fuzzy normal partial group and examine the fundamental properties of this new concept.

From now on throughout the paper, X denotes a partial group.

Definition 3.1 *Let η be a fuzzy set in X, let $a \in X$ and let R_a, L_a be the right and the left transformation maps, respectively. We call $a\eta$ a fuzzy left coset, which is defined as $L_a[\eta]$ and ηa a fuzzy right coset, defined as $R_a[\eta]$, where* $a\eta(y) = \begin{cases} \sup\limits_{z \in {L_a}^{-1}(y)} \eta(z) & \text{if } {L_a}^{-1}(y) \neq \phi \\ 0 & \text{if } {L_a}^{-1}(y) = \phi \end{cases}$ *and* $\eta a(y) = \begin{cases} \sup\limits_{z \in {R_a}^{-1}(y)} \eta(z) & \text{if } {R_a}^{-1}(y) \neq \phi \\ 0 & \text{if } {R_a}^{-1}(y) = \phi \end{cases}$

Proposition 3.1 *Let η be a fuzzy set in X and let $x,y \in X$. Then,* $\eta x(y) = \begin{cases} \sup\limits_{zx=y} \eta(z) & \text{if } e_x \geq e_y \\ 0 & \text{if } e_x < e_y \end{cases}$ *where $e_x, e_y \in E(X)$. Similarly for $x\eta(y)$.*

Proof. Let ${R_x}^{-1}(y) \neq \phi$. Then, $\eta x(y) = \sup\limits_{z \in {R_x}^{-1}(y)} \eta(z) = \sup\limits_{zx=y} \eta(z)$. Let $cx = y$ for some $c \in X$. We have two cases:

1. If $e_x \leq e_c$, then $cx = y \in X_x$, and hence $e_x = e_y$.
2. If $e_x > e_c$, then $cx = y \in X_c$, and hence $e_x > e_y$.

Now, let ${R_x}^{-1}(y) = \phi$. Then, $\nexists z \in X$ that satisfies $zx = y$. Which implies that $e_x < e_y$.

Definition 3.2 *Let η be a fuzzy partial group in X. Then, η is called fuzzy normal if $\eta(xyx^{-1}) = \eta(y)$, $\forall x,y \in X$.*

η will denote a fuzzy normal partial group in X from now on.

Example 3.1 *Let $X = \{p,q,r,s,t\}$ be a partial group with the multiplication table shown, such that $X = X_p \cup X_r$, where X_p and X_r are isomorphic to the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively, such that $e_p \leq e_r$. Let $\eta = \{p_{0.8}, q_{0.3}, r_{0.8}, s_{0.8}, t_{0.8}\}$. Then, it can be verified easily that η is indeed a fuzzy normal partial group in X.*

$\cdot$	p	q	r	s	t
p	p	q	p	p	p
q	q	p	q	q	q
r	p	q	r	s	t
s	p	q	s	t	r
t	p	q	t	r	s

Proposition 3.2 *The following properties are true $\forall x,y,z \in X$.*

1. $\eta(xy) = \eta(yx)$,
2. $x\eta(y) = \eta(yx^{-1}), \eta x(y) = \eta(x^{-1}y)$ *if* $e_x \geq e_y$,
3. $x\eta = \eta x$,
4. $x\eta x^{-1} = \eta$ *if* $e_x \geq e_w$, $\forall w \in X$,
5. $xy\eta(z) = x\eta(zy^{-1}) = y\eta(x^{-1}z)$ *if* $e_x, e_y \geq e_z$,
6. $x\eta(x) = y\eta(y) = k$*, where k is a constant,*
7. $y\eta(xy) = \eta(x)$,
8. $\eta(xe_y) = \eta(x)$,
9. $e_x\eta(y) = e_x\eta(y^{-1})$,
10. $x\eta(y) = y\eta(x)$ *if* $e_x = e_y$*, and either* $x\eta(y) = 0$ *or* $y\eta(x) = 0$ *if* $e_x \neq e_y$.

Proof.

1. $\eta(xy) = \eta(xe_xy) = \eta(xye_x) = \eta(xyxx^{-1}) = \eta(yx)$.
2. $x\eta(y) = \sup\limits_{xz=y} \eta(z) = \sup\limits_{xz=y} \eta(xzx^{-1}) = \eta(yx^{-1})$. Similarly, $\eta x(y) = \eta(x^{-1}y)$.
3. $x\eta(y) = \eta(yx^{-1}) = \eta(x^{-1}y) = \eta x(y)$ if $e_x \geq e_y$. Also, $x\eta(y) = \eta x(y) = 0$ if $e_x < e_y$.
4. $x\eta x^{-1}(y) = \sup\limits_{xz=y} \eta x^{-1}(z) = \sup\limits_{xz=y} \eta(xz) = \eta(y)$.
5. $xy\eta(z) = \sup\limits_{xyw=z} \eta(w) = \sup\limits_{xywy^{-1}=zy^{-1}} \eta(ywy^{-1}) = \sup\limits_{xm=zy^{-1}} \eta(m) = x\eta(zy^{-1})$. Similarly, $xy\eta(z) = y\eta(x^{-1}z)$.
6. $x\eta(x) = \eta(xx^{-1}) = \eta(e_x) = \eta(e_y) = \eta(yy^{-1}) = y\eta(y) =$ constant.
7. $y\eta(xy) = \eta(xyy^{-1}) = \eta(y^{-1}xy) = \eta(x)$.
8. $\eta(xe_y) = \eta(xy^{-1}y) = \eta(yxy^{-1}) = \eta(x)$.
9. $e_x\eta(y) = \eta(ye_x) = \eta(y) = \eta(y^{-1}) = \eta(y^{-1}e_x) = e_x\eta(y^{-1})$ if $e_x \geq e_y$. Also, if $e_x < e_y$, then $e_x\eta(y) = e_x\eta(y^{-1}) = 0$.
10. Let $e_x = e_y$. Then, $x\eta(y) = \eta(yx^{-1}) = \eta((yx^{-1})^{-1}) = \eta(xy^{-1}) = y\eta(x)$. Now, if $e_x \ngeq e_y$, then $y\eta(x) = 0$, and if $e_x \nleq e_y$, then $x\eta(y) = 0$.

Proposition 3.3 *$\forall x,y \in X$, we have the following:*

1. If $x\eta = y\eta$, then $e_x = e_y$,
2. If $x\eta = y\eta$, then $xy^{-1}\eta = e_y\eta$,

3. If $x\eta = y\eta$, then $\eta(x) = \eta(y)$,
4. $\eta(x) = \eta(e_x)$ if and only if $x\eta = e_x\eta$,
5. If $\eta(x) = \eta(e_x)$, then $y\eta(x) = y\eta(e_x)$.

Proof.

1. If $e_x \ngeq e_y$, then $y\eta(x) = 0$. Since $x\eta(x) = \eta(e_x)$, then $y\eta \neq x\eta$.
2. Let $x\eta = y\eta$. Then, $x\eta(z) = y\eta(z)$, $\forall z \in X$. We have two cases:
   (a) If $e_x, e_y \geq e_z$, then $xy^{-1}\eta(z) = x\eta(zy) = y\eta(zy) = \eta(z) = e_y\eta(z)$,
   (b) If $e_x, e_y < e_z$, then $xy^{-1}\eta(z) = e_y\eta(z) = 0$,
3. If $x\eta = y\eta$, then $\eta(x) = y\eta(xy) = x\eta(xy) = \eta(xyx^{-1}) = \eta(y)$,
4. Let $\eta(x) = \eta(e_x)$. If $z \in X$, then:
   (a) If $e_x \geq e_z$, then $e_x\eta(z) = \eta(ze_x) = \eta(zx^{-1}x) \geq \min\{\eta(zx^{-1}), \eta(x)\} = \eta(zx^{-1}) = x\eta(z) = \eta(zx^{-1}) \geq \min\{\eta(z), \eta(x^{-1})\} = \eta(z) = e_x\eta(z)$, and so it is proved that $e_x\eta(z) = x\eta(z)$,
   (b) If $e_x < e_z$, then $e_x\eta(z) = x\eta(z) = 0$.
   Conversely, if $x\eta = e_x\eta$, then $\eta(x) = \eta(e_x)$ from (iii).
5. Let $\eta(x) = \eta(e_x)$. Then:
   (a) If $e_y \geq e_x$, then $y\eta(e_x) = \eta(e_xy^{-1}) = \eta(x^{-1}xy^{-1}) \geq \min\{\eta(x^{-1}), \eta(xy^{-1})\} = \eta(xy^{-1}) = y\eta(x) = \eta(xy^{-1}) \geq \min\{\eta(x), \eta(y^{-1})\} = \eta(y^{-1}) = \eta(y^{-1}e_x) = y\eta(e_x)$, which implies that $y\eta(e_x) = y\eta(x)$.

   (b) If $e_y < e_x$, then $y\eta(e_x) = y\eta(x) = 0$.

Remark 3.1 *The converse for 1, 2, 3 and 5 in Proposition 3.3 is not necessarily true. This is demonstrated in the next example.*

Example 3.2 *Let $X = \{p, q, r, s, t, u, v\}$ be a partial group with the multiplication table shown, such that $X = X_p \cup X_q$ with $e_q \leq e_p$, where X_p is the trivial group of one element, and $X_q = \{q, r, s, t, u, v\}$ is the permutation group S_3. Let $\eta = \{p_{0.9}, q_{0.9}, r_{0.7}, s_{0.7}, t_{0.5}, u_{0.5}, v_{0.5}\}$. Then, η is a fuzzy normal partial group in X.*

$\cdot$	p	q	r	s	t	u	v
p	p	q	r	s	t	u	v
q	q	q	r	s	t	u	v
r	r	r	s	q	u	v	t
s	s	s	q	r	v	t	u
t	t	t	v	u	q	s	r
u	u	u	t	v	r	q	s
v	v	v	u	t	s	r	q

We find that $r\eta = \{p_0, q_{0.7}, r_{0.9}, s_{0.7}, t_{0.5}, u_{0.5}, v_{0.5}\}$ and $s\eta = \{p_0, q_{0.7}, r_{0.7}, s_{0.9}, t_{0.5}, u_{0.5}, v_{0.5}\}$, and hence $r\eta \neq s\eta$, but we note that $e_r = e_s$ and $\eta(r) = \eta(s)$, which are the conditions in 1 and 3 respectively. For the converse of 2, we have that $pq^{-1}\eta = q\eta$ and $e_q\eta = q\eta$, and hence they are equal whereas $p\eta \neq q\eta$. Lastly for 5, we notice that $r\eta(s) = r\eta(e_s)$ while $\eta(s) \neq \eta(e_s)$.

4 Fuzzy Quotient Partial Group

Throughout this section, we study the quotient of a fuzzy partial group over a fuzzy normal partial group. From now on, we consider ξ a fuzzy partial group in X.

Definition 4.1 *Let X/η be the set of all fuzzy cosets of X with respect to η. The set X/η is called the quotient of X over η.*

Proposition 4.1 *X/η is a partial group with respect to the operation $X/\eta \times X/\eta \longrightarrow X/\eta; (x\eta, y\eta) \longmapsto xy\eta$.*

Proof. We first show that it is a semigroup. Let $x_1, x_2, y_1, y_2 \in X$ such that $(x_1\eta, y_1\eta) = (x_2\eta, y_2\eta)$. Then, $x_1\eta = x_2\eta$ and $y_1\eta = y_2\eta$. Now, $x_1y_1\eta(z) = x_1\eta(zy_1{}^{-1}) = x_2\eta(zy_1{}^{-1}) = \eta(zy_1{}^{-1}x_2{}^{-1}) = \eta(x_2{}^{-1}zy_1{}^{-1}) = y_1\eta(x_2{}^{-1}z) = y_2\eta(x_2{}^{-1}z) = x_2y_2\eta(z)$, where $e_{x_1}, e_{y_1} \geq e_z$. Also, if $e_{x_1} < e_z$ or $e_{y_1} < e_z$, then it is obvious that $x_1y_1\eta(z) = x_2y_2\eta(z) = 0$. This proves that the operation is well defined. Associativity is straightforward. Now, we show that it is a partial group.

1. To show $e_x\eta$ is the partial identity of $x\eta$, we note that:
   (a) $x\eta e_x\eta = xe_x\eta = x\eta$ and similarly, $e_x\eta x\eta = x\eta$.
   (b) Let $e\eta \in X/\eta$ be such that $e\eta x\eta = x\eta, x\eta e\eta = x\eta$. Then, $ex\eta = x\eta$, and so $exx^{-1}\eta = e_x\eta$, and hence $ee_x\eta = e_x\eta$. That means that $e\eta e_x\eta = e_x\eta$. Similarly, we can show that $e_x\eta e\eta = e_x\eta$.
2. The proof that $x^{-1}\eta$ is the partial inverse of $x\eta$ is straightforward.
3. We have $e_{x\eta y\eta} = e_x\eta e_y\eta$.
4. Finally, we have that $(x\eta y\eta)^{-1} = y^{-1}\eta x^{-1}\eta$.

Definition 4.2 *Let $p : X \longrightarrow X/\eta; x \longmapsto x\eta$ be the quotient map. Then, the map $p : \xi \longrightarrow p[\xi]$ is called the fuzzy quotient map, and $p[\xi]$, denoted by ξ/η, is called the fuzzy quotient of ξ over η.*

Remark 4.1 *It is clear from the definition that $(\xi/\eta)(x\eta) = \sup\limits_{z \in p^{-1}(x\eta)} \xi(z) = \sup\limits_{z\eta = x\eta} \xi(z) \geq \xi(x)$.*

Proposition 4.2 *ξ/η is a fuzzy partial group in X/η.*

Proof.

1. Let $x, x_o, y, y_o \in X$ be such that $(\xi/\eta)(x\eta) = \xi(x_o)$ and $(\xi/\eta)(y\eta) = \xi(y_o)$, where $x_o\eta = x\eta, y_o\eta = y\eta$. Since $x_oy_o\eta = xy\eta$, we have that
$$\begin{aligned}(\xi/\eta)(xy\eta) &= \sup_{z\eta = xy\eta} \xi(z)\\ &\geq \xi(x_oy_o)\\ &\geq \min\{\xi(x_o), \xi(y_o)\}\\ &= \min\{(\xi/\eta)(x\eta), (\xi/\eta)(y\eta)\}\end{aligned}$$
2. $(\xi/\eta)(x\eta) = \sup\limits_{z\eta = x\eta} \xi(z) = \sup\limits_{z^{-1}\eta = x^{-1}\eta} \xi(z^{-1}) = (\xi/\eta)(x^{-1}\eta)$

3. $(\xi/\eta)(e_x\eta) = \sup_{z\eta=e_x\eta} \xi(z) = \sup_{\eta(z)=\eta(e_x)} \xi(z) = \xi(e_x) = k$, where $k = \sup_{z\eta\in X/\eta} (z\eta)$.

ξ/η will be called the quotient fuzzy partial group.

Remark 4.2 *We note that ξ/η is not necessarily a fuzzy normal partial group in X/η. This is shown in the next example.*

Example 4.1 *With reference to Example 3.2, let $\xi = \{p_{0.9}, q_{0.9}, r_{0.3}, s_{0.3}, t_{0.3}, u_{0.3}, v_{0.8}\}$ be a fuzzy partial group in X. We can see that $\xi/\eta = \{p\eta_{0.9}, q\eta_{0.9}, r\eta_{0.3}, s\eta_{0.3}, t\eta_{0.3}, u\eta_{0.3}, v\eta_{0.8}\}$, which is not fuzzy normal since $(\xi/\eta)(tvt^{-1}\eta) \neq (\xi/\eta)(v\eta)$.*

Proposition 4.3 *ξ/η is fuzzy normal if ξ is fuzzy normal.*

Proof.
$$
\begin{aligned}
(\xi/\eta)(y\eta) &= \sup_{z\eta=y\eta} \xi(z)\\
&= \sup_{xzx^{-1}\eta=xyx^{-1}\eta} \xi(xzx^{-1})\\
&= (\xi/\eta)(xyx^{-1}\eta)
\end{aligned}
$$

5 Isomorphism Theorems

Throughout this section, $f : X \longrightarrow Y$ is a homomorphism of partial groups, and ξ, ζ are fuzzy partial groups in X, Y, respectively. The map $f : \xi \longrightarrow \zeta$ will be called a fuzzy partial group homomorphism. In this section, we discuss the fundamental theorems of isomorphism, based on a new definition of the fuzzy kernel of fuzzy partial group homomorphism.

Definition 5.1 *The fuzzy kernel of $f : \xi \longrightarrow \zeta$, denoted by κ_f, is defined by*

$$\kappa_f(x) = \begin{cases} \xi(e_x) & \text{if } x \in \ker f \\ 0 & \text{if } x \notin \ker f \end{cases}$$

Proposition 5.1 *The fuzzy kernel κ_f is a fuzzy partial group in X.*

Proof.

1. Let $x, y \in X$, where either x or $y \notin \ker f$. So, either $\kappa_f(x) = 0$ or $\kappa_f(y) = 0$, which implies that $\kappa_f(xy) \geq \min\{\kappa_f(x), \kappa_f(y)\}$. If both x and $y \in \ker f$, then it is trivial.
2. If $x \in \ker f$, then $x^{-1} \in \ker f$. Hence, $\kappa_f(x) = \kappa_f(x^{-1})$, $\forall x \in X$.
3. Since $e_x \in \ker f$, $\forall x \in X$, then $\kappa_f(e_x) = \xi(e_x)$ is constant.

Remark 5.1 *The fuzzy kernel κ_f is not necessarily fuzzy normal, as seen in the next example.*

Example 5.1 *With reference to Example 3.1, let $f : X \longrightarrow X$ be the identity map and let $\xi = I_{0.5}$. Then, $f : \xi \longrightarrow \xi$ is a fuzzy partial group homomorphism and its fuzzy kernel is $\kappa_f = \{p_{0.5}, q_0, r_{0.5}, s_0, t_0\}$. It follows that κ_f is not fuzzy normal since $(\kappa_f)(qsq^{-1}) = (\kappa_f)(p) \neq (\kappa_f)(s)$.*

Let $\eta_e = \{x \in X : \eta(x) = \eta(e_x)\}$. Clearly, η_e is a normal partial group in X.

Definition 5.2 *We say that the map $f : X \longrightarrow Y$ is fuzzy kernel normalizing if $\ker f = \eta_e$. η is said to be the fuzzy normalizer of f.*

Proposition 5.2 *The fuzzy kernel κ_f is fuzzy normal if the map $f : X \longrightarrow Y$ is fuzzy kernel normalizing.*

Proof. Take $x, y \in X$. First, if $y \in \ker f$, then $\kappa_f(xyx^{-1}) = \kappa_f(y)$ since $xyx^{-1} \in \ker f$. Whereas if $y \notin \ker f$, given that η is the fuzzy normalizer of f, then $y \notin \eta_e$. And so we have $\eta(y) \neq \eta(e_y)$, but we know that $\eta(y) = \eta(xyx^{-1})$. Hence, we get $\eta(xyx^{-1}) \neq \eta(e_y) = \eta(e_{xyx^{-1}})$, which implies that $xyx^{-1} \notin \eta_e = \ker f$. So, in both cases, we can conclude that $\kappa_f(xyx^{-1}) = \kappa_f(y)$.

Remark 5.2 *The kernel of the quotient map $p : X \longrightarrow X/\eta$, is $\ker p = \{x \in X : x\eta = e_x\eta\} = \{x \in X : \eta(x) = \eta(e_x)\} = \eta_e$.*

Proposition 5.3 *The quotient map $p : X \longrightarrow X/\eta$ is fuzzy kernel normalizing.*

Proof. Direct from the previous remark.

Remark 5.3 *The fuzzy kernel of the fuzzy quotient map $p : \xi \longrightarrow \xi/\eta$ is defined as $\kappa_p(x) = \begin{cases} \xi(e_x) & \text{if } x \in \eta_e \\ 0 & \text{if } x \notin \eta_e \end{cases}$. From Proposition 5.2, we can see that κ_p is a fuzzy normal partial group.*

Definition 5.3 *Let $f : X \longrightarrow Y$ be fuzzy kernel normalizing. Let $\bar{\kappa}_f$ be a fuzzy set in X defined as $\bar{\kappa}_f(x) = \begin{cases} m & \text{if } x \in \ker f \\ 0 & \text{if } x \notin \ker f \end{cases}$, $m \leq \xi(e_x)$.*

Remark 5.4 *$\bar{\kappa}_f$ is a fuzzy subset of κ_f. Obviously, $\bar{\kappa}_f$ is a fuzzy normal partial group in X.*

By using the newly defined notion of the fuzzy kernel, the isomorphism theorems of fuzzy partial groups follow next. From now on, we will denote the fuzzy kernel of a fuzzy kernel normalizing map f by η_ε, where η is the fuzzy normalizer of f.

Theorem 5.1 *Let $\varphi : X \longrightarrow Y$ be an idempotent separating, fuzzy kernel normalizing homomorphism of partial groups and let $\nu = \bar{\kappa}_\varphi$ Then, $\exists$ a unique fuzzy partial group monomorphism $\psi : \xi/\nu \longrightarrow \zeta$.*

Proof. We need to prove first that $\exists$ a unique partial group monomorphism $\psi : X/\nu \longrightarrow Y$. We first show that ψ is well defined. Let $x, y \in X$ such that $x\nu = y\nu$. Then, $e_x = e_y$ and $xy^{-1}\nu = e_y\nu$. It follows that $\nu(xy^{-1}) = \nu(e_y)$, which implies that $xy^{-1} \in \ker\varphi$. So, $\varphi(xy^{-1}) = e_{\varphi(xy^{-1})}$, $\varphi(x)\varphi(y^{-1}) = \varphi(e_x e_y)$, multiplying both sides by $\varphi(y)$, we get $\varphi(x)\varphi(e_y) = \varphi(e_x y)$, $\varphi(x) = \varphi(y)$. Hence, $\psi(x\nu) = \psi(y\nu)$. It is easy to show that ψ is a unique partial group homomorphism, we will only prove that it is injective. Let $\psi(x\nu) = \psi(y\nu)$. Then, $\varphi(x) = \varphi(y)$, which implies that $e_x = e_y$. Take $z \in X$ such that $e_x \geq e_z$. Then, $x\nu(z) = \nu(zx^{-1})$. Now, if $zx^{-1} \in \ker\varphi$, then $x\nu(z) = y\nu(z) =$ constant, since $zx^{-1} \in \ker\varphi$ if and only if $zy^{-1} \in \ker\varphi$. And if $zx^{-1} \notin \ker\varphi$, then $x\nu(z) = y\nu(z) = 0$. Also, in case $e_x < e_z$, we get $x\nu(z) = y\nu(z) = 0$. We arrive at the conclusion that $x\nu = y\nu$, which proves that ψ is a monomorphism. Since ξ/ν and ζ are fuzzy partial groups in X/ν and Y respectively, then $\psi : \xi/\nu \longrightarrow \zeta$ is a fuzzy partial group monomorphism.

Theorem 5.2 *Let α, β be fuzzy normal partial groups in X, where $\alpha \subseteq \beta_\varepsilon$, and α is constant on β_e. Then, $\exists$ a fuzzy partial group isomorphism $\psi : (\xi/\alpha)/(\beta/\alpha)_\varepsilon \longrightarrow \xi/\beta$.*

Proof. The quotient map $p_\beta : X \longrightarrow X/\beta$ is idempotent separating, since $p_\beta(e_x) = p_\beta(e_y)$ implies that $e_x\beta = e_y\beta$, and so $e_x = e_y$. Since β_ε is the fuzzy kernel of p_β, then α is precisely $\bar{\kappa}_{p_\beta}$. From Theorem 5.1, since p_β is a fuzzy kernel normalizing partial group homomorphism that is also surjective, then $\varphi : X/\alpha \longrightarrow X/\beta$ is a unique partial group isomorphism. Let $\varphi(e_x\alpha) = \varphi(e_y\alpha)$, then $e_x\beta = e_y\beta$, so $e_x = e_y$ and hence $e_x\alpha = e_y\alpha$, which shows that φ is idempotent separating. Since

$$\begin{aligned}\ker\varphi &= \{x\alpha \in X/\alpha : \varphi(x\alpha) = e_x\beta\}\\ &= \{x\alpha \in X/\alpha : x\beta = e_x\beta\}\\ &= \{x\alpha \in X/\alpha : \beta(x) = \beta(e_x)\}\\ &= \{x\alpha \in X/\alpha : x \in \beta_e\}\\ &= \beta_e/\alpha = (\beta/\alpha)_e,\end{aligned}$$

then β/α is the fuzzy normalizer of φ, and so φ is fuzzy kernel normalizing, which implies that its fuzzy kernel $(\beta/\alpha)_\varepsilon$ is a fuzzy normal partial group in X/α. From Theorem 5.1, $\exists$ a unique fuzzy partial group isomorphism $\psi : (\xi/\alpha)/(\beta/\alpha)_\varepsilon \longrightarrow \xi/\beta$.

Now, we proceed with the isomorphism theorems in FTPGs, but we first define the quotient FTPG.

Definition 5.4 *Let α, β by fuzzy sets in X. α is said to be constant on β if $\beta(x) = \beta(y)$ implies that $\alpha(x) = \alpha(y)$ $\forall x, y \in X$.*

Proposition 5.4 *Let $p : \xi \longrightarrow \xi/\eta$. Then, ξ is p-invariant if ξ is constant on η.*

Proof. If $p(x) = p(y)$, then $\eta(x) = \eta(y)$. Since ξ is constant on η, then $\xi(x) = \xi(y)$.

Assume that δ is a fuzzy topology on X and ξ is an FTGP with the induced fuzzy topology δ_ξ. Then, from Proposition 2.8, ξ/η is also an FTPG with the induced fuzzy topology $p[\delta]_{\xi/\eta}$ provided that ξ is constant on η, where $p[\delta]$ is the fuzzy image topology which we call the quotient fuzzy topology. We call ξ/η the quotient FTPG. Let ζ be an FTPG in Y.

Theorem 5.3 *Let $\varphi : X \longrightarrow Y$ be an idempotent separating, fuzzy kernel normalizing, continuous partial group homomorphism and let ξ be constant on ν, where $\nu = \bar{\kappa}_\varphi$. Then, $\exists$ a unique fuzzy continuous partial group monomorphism $\psi : \xi/\nu \longrightarrow \zeta$.*

Proof. From Theorem 5.1, $\exists$ a partial group monomorphism $\psi : X/\nu \longrightarrow Y$. Clearly, ξ/ν with the induced fuzzy quotient topology is the quotient FTPG, and so, $\psi : \xi/\nu \longrightarrow \zeta$ is a fuzzy continuous partial group monomorphism.

Theorem 5.4 *Let α, β be fuzzy normal partial groups in X, where $\alpha \subseteq \beta_\varepsilon$ and α is constant on β_e. Let ξ be constant on α. Then, $\exists$ a unique fuzzy homeomorphism $\psi : (\xi/\alpha)/(\beta/\alpha)_\varepsilon \longrightarrow \xi/\beta$.*

Proof. First, it is clear that the fuzzy quotient partial groups $\xi/\alpha, \xi/\beta$ and $(\xi/\alpha)/(\beta/\alpha)_\varepsilon$ are FTPGs with the respective induced fuzzy quotient topologies, since ξ is constant on both α and β, (ξ/α) is constant on $(\beta/\alpha)_\varepsilon$. Secondly, $p_\beta : X \longrightarrow X/\beta$ is an idempotent separating, fuzzy kernel normalizing, continuous partial group homomorphism that is fuzzy open and surjective, hence, from Theorem 5.3, $\exists$ a fuzzy homeomorphism $\varphi : \xi/\alpha \longrightarrow \xi/\beta$. Finally, since $\varphi : X/\alpha \longrightarrow X/\beta$ is an idempotent separating, fuzzy kernel normalizing, homeomorphism, then $\exists$ a unique fuzzy homeomorphism $\psi : (\xi/\alpha)/(\beta/\alpha)_\varepsilon \longrightarrow \xi/\beta$.

We now study the isomorphism of product of FTPGs. But we first need to discuss the product of fuzzy normal partial groups. Let η_i be a fuzzy normal partial group in partial group X_i, $\forall i = 1, 2, \ldots, n$. Let X be the product of X_i and η be the product of $\eta_i, i = 1, 2, \ldots, n$, defined by $\eta(x) = \min\{\eta_1(x_1), \eta_2(x_2), \ldots, \eta_n(x_n)\}$, where $x = (x_1, x_2, \ldots, x_n) \in X$.

Proposition 5.5 *$\eta = \prod_{i=1}^{n} \eta_i$ is a fuzzy normal partial group in $X = \prod_{i=1}^{n} X_i$.*

Proof. It is already known from Proposition 2.7 that η is a fuzzy partial group in X. Let $x, y \in X$. Then,

$$\begin{aligned}\eta(xyx^{-1}) &= \min\{\eta_1(x_1 y_1 x_1{}^{-1}), \eta_2(x_2 y_2 x_2{}^{-1}), \ldots,\\ &\qquad \eta_n(x_n y_2 x_n{}^{-1})\}\\ &= \min\{\eta_1(y_1), \eta_2(y_2), \ldots, \eta_n(y_n)\} = \eta(y).\end{aligned}$$

Let X_i have the fuzzy topology δ_i, $\forall i = 1,2,\ldots,n$, and let $\prod_{i=1}^{n} X_i$ have the product fuzzy topology δ. Let ξ_i with the induced fuzzy topology be an FTPG in X_i, $\forall i = 1,2,\ldots,n$. From Proposition 2.9, the product fuzzy partial group $\prod_{i=1}^{n} \xi_i$ with the product induced fuzzy topology δ_ξ is an FTPG in $\prod_{i=1}^{n} X_i$.

Proposition 5.6 *Let ξ_i be constant on η_i, $\forall i = 1,2,\ldots,n$. Then, $\theta : \prod_{i=1}^{n} \xi_i / \prod_{i=1}^{n} \eta_i \longrightarrow \prod_{i=1}^{n} (\xi_i/\eta_i)$ is a fuzzy homeomorphism.*

Proof. Similar to Proposition 6.3 in [8].

6 Conclusions

The fuzzy normal partial group was defined and it was shown that it acquires some important properties. Moreover, we demonstrated a new suitable approach to the concept of the fuzzy quotient partial group, incorporating the fuzzy aspect of the fuzzy normal partial group. Also, the definition of the fuzzy kernel was presented, and it was proved that the fuzzy kernel, while not generally normal, is normal under a specific condition, hence ensuring the normality of the kernel of the fuzzy quotient map. Consequently as key results, the isomorphism theorems of fuzzy partial groups and FTPGs are established, and also it was proved that the product of the quotient is homeomorphic to the quotient of product.

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