

# Symbolic Dynamics of Odd Discontinuous Bimodal Maps

Henrique M. Oliveira\*

Center of Mathematical Analysis, Geometry and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

Received: 15 Jan. 2015, Revised: 16 Apr. 2015, Accepted: 17 Apr. 2015

Published online: 1 Sep. 2015

**Abstract:** Iterations of odd piecewise continuous maps with two discontinuities, i.e., symmetric discontinuous bimodal maps, are studied. Symbolic dynamics is introduced. The tools of kneading theory are used to study the homology of the discrete dynamical systems generated by the iterations of that type of maps. When there is a Markov matrix, the spectral radius of this matrix is the inverse of the least root of the kneading determinant.

**Keywords:** Symbolic dynamics, kneading theory, homology of dynamical systems, Markov matrices. AMS Classif.: 37E05, 37B10, 37B40

This paper is dedicated to the memory of Professor José Sousa Ramos.

## 1 Introduction

In this paper we apply techniques of Markov partitions and kneading theory to the study of iterates of discontinuous maps of the interval (or the real line) in itself. We show that these systems can be studied with a proper framework, which is related to kneading theory and Markov matrices.

We cite, as examples of discontinuous one dimensional cases, the Lorenz maps, Newton maps, circle and tree maps, see [1, 2, 3, 7] among other literature.

In [9] Lampreia and Sousa Ramos studied symbolic dynamics of continuous bimodal maps on the compact interval. Using similar techniques, we study in this paper the case of symmetric (odd) discontinuous maps in the real line or some suitable interval with two discontinuity points and three maximal intervals (laps) of continuity, which are as well maximal intervals of monotonicity. We call to this type of mapping a *symmetric bimodal discontinuous map* because of the existence of exactly three laps as in the continuous bimodal case.

In section two, we introduce the notation, the main definitions and revision of basic results. We include as well, the notions of symbolic dynamics, kneading theory

and Markov partitions. We relate these concepts with lap growth number. We tried to define with great detail all the concepts presented. Since good definitions are essential for the constructive proof of the main result, which is actually done along the full length of the paper, section two is relatively long.

In section three, we present the main result of the paper, i.e., the spectral radius of the Markov matrix is the inverse of the least root of the kneading determinant for that kind of maps. We point out that the introduction of the linear operator  $\mu$  in section three is one of the main ideas of this paper along with the matrix  $\Theta$  relating the kneading and the Markov data. The linear transformation  $\mu$ , representing the symmetry of this type of non-continuous maps, is completely different of its continuous counterpart [9]. We think that the proof of the result can be instructive giving methods that can be applied to other non-continuous mappings.

### 1.1 Motivation

The iterates of the complex tangent family  $\lambda \tan z$ , introduced in [5] and [8], when the parameter  $\lambda = i\beta$  is pure imaginary and the initial condition  $x_0$  is a real number can be identified with the iterates of the real

\* Corresponding author e-mail: [holiv@math.tecnico.ulisboa.pt](mailto:holiv@math.tecnico.ulisboa.pt)

alternating map  $[f_{1,\beta}, f_{2,\beta}]$  [4] in the real line

$$\begin{aligned} x_1 &= f_{1,\beta}(x_0) = \beta \tan(x_0) \\ x_2 &= f_{2,\beta}(x_1) = -\beta \tanh(x_1) \\ x_3 &= f_{1,\beta}(x_2) \\ x_4 &= f_{1,\beta}(x_3) \\ &\vdots \end{aligned}$$

The composition map  $g_\beta$  is

$$g_\beta : \mathbb{R} \mapsto \mathbb{R} \\ x \mapsto -\beta \tanh(\beta \tan(x)), \tag{1}$$

which can be interpreted as the second return map to the real axis for the mapping  $\lambda \tan(z)$ . Knowing  $x_0$  and  $g_\beta$ , we obtain all the even iterates of the original system. To obtain the odd iterates knowing the even iterates is easy

$$x_{2n+1} = \beta \tan(x_{2n}).$$

The geometric behavior of the maps  $g_\beta$  in this family depends on the parameter  $\beta$ . The map is periodic and the real line is mapped on the interval  $I = (-\beta, \beta)$ . We restrict the map only to the interval  $I$ . When  $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$  the maps  $g_\beta$  have two discontinuities. The study of the real projection of the complex tangent map is a good clue to the dynamics in the complex plane, similarly to the case of quadratic maps.

In this paper, we center our study on the symbolic dynamics of the iterates of maps  $F$  with the same geometrical properties of  $g_\beta$ . Considering that the tangent family was an initial motivation and a good example, we point out that the results are independent on the choice of the family.

## 2 Basics

### 2.1 Bimodal symmetric discontinuous map

**Definition 1.** Bimodal symmetric discontinuous map of type  $(-, -, -)$ .

Let  $I = (-a, a)$  be a real interval (where  $a$  can be  $+\infty$ ) and  $F : I \rightarrow I$ , such that:

1.  $F$  is odd  $F(x) = -F(-x)$
2.  $F$  is piecewise continuous having two discontinuities  $c_1 < c_2$ ,  $c_1 = -c_2$ , where  $\lim_{x \rightarrow c_i^\pm} F(x) = \pm a$ , and  $\lim_{x \rightarrow \pm a} F(x) = \pm b$ , where  $b$  is a real number.
3.  $F$  is decreasing in every interval of continuity  $(-a, c_1)$ ,  $(c_1, c_2)$  and  $(c_2, a)$ .

*Example 1.* The family of maps  $g_\beta$  defined in (1) is a family of bimodal symmetric discontinuous maps.

Definition 1 applies to maps with infinite jumps at  $c_1$  and  $c_2$  as we can see in the next example. Actually, as we see in the next example, any such map is smoothly conjugated to a map with finite jumps via a diffeomorphism.

*Example 2.* Consider  $u : \mathbb{R} \mapsto \mathbb{R}$  such that

$$u(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{2} \\ 0 & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \end{cases}$$

the family

$$G_\alpha : \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto \frac{x}{4x^2-1} - \alpha u(x)$$

is a family of bimodal symmetric discontinuous maps with  $b = -\alpha$ ,  $c_1 = -\frac{1}{2}$ ,  $c_2 = \frac{1}{2}$  and  $a = +\infty$ . Any map in this family satisfies definition 1 and is smoothly conjugated to a map with finite jumps using for instance the diffeomorphism  $h(x) = \arctan(x)$  such that

$$\tilde{G}_\alpha(x) = h \circ G_\alpha \circ h^{-1}(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The map  $\tilde{G}_\alpha$  can be prolonged by continuity to the endpoints  $\pm\frac{\pi}{2}$  of the interval, since

$$\lim_{x \rightarrow \pm\frac{\pi}{2}} h \circ G_\alpha \circ h^{-1}(x) = \lim_{x \rightarrow \pm\infty} h \circ G_\alpha(x) = \mp \arctan(\alpha).$$

### 2.2 Symbolic dynamics

For sake of completeness and readability we introduce here briefly notions well known like orbit, periodic orbit and symbolic itinerary among other concepts, see for instance [6].

**Definition 2.** We define the orbit of a real point  $x_0$  as a sequence of numbers  $O(x_0) = \{x_j\}_{j=0,1,\dots}$  such that  $x_j = F^j(x_0)$  where  $F^j$  is the  $j$ -th composition of  $F$  with itself.

**Definition 3.** Any point  $x$  is periodic with period  $n > 0$  if the condition  $F^n(x) = x$  is fulfilled with  $n$  minimal.

Because of condition 1 in definition 1, the orbit of any point  $x$  is symmetric relative to the orbit of  $-x$ . To avoid ambiguities in the definition of the orbit of the pre discontinuity points we adopt the convention that  $F(c_1) = F(c_1^-) = -a$  and  $F(c_2) = F(c_2^+) = +a$ .

**Definition 4.** [11] Consider the alphabet  $\mathcal{A} = \{L, A, M, B, R\}$  the address  $A(x)$  of a real point  $x$  is defined such that

$$A(x) = \begin{cases} L & \text{if } x < c_1 \\ A & \text{if } x = c_1 \\ M & \text{if } c_1 < x < c_2 \\ B & \text{if } c_2 = x \\ R & \text{if } c_2 < x \end{cases}$$

We can apply this function to an orbit of a given real point  $x_0$ , we associate to that orbit one infinite symbolic sequence.

**Definition 5.** Consider the sequence of symbols in  $\mathcal{A}$

$$\text{It}(x_0) = A(x_0)A(x_1)A(x_2)\dots A(x_n)\dots$$

this infinite sequence is the symbolic itinerary of  $x_0$ .

The orbit  $O(-a)$  is

$$\{x_j^{(1)} : x_j^{(1)} = F^j(-a), j = 0, 1, \dots\}.$$

The orbit of  $O(+a)$  is

$$\{x_j^{(2)} : x_j^{(2)} = F^j(+a), j = 0, 1, \dots\},$$

with  $F(+a) = b$ .

**Definition 6.**[11] Kneading sequences and kneading pairs. The kneading sequences are defined as the symbolic itineraries of the orbits of  $a$  and  $-a$ . The kneading pair is the ordered pair formed by these two symbolic sequences

$$(\text{It}(a), \text{It}(-a)).$$

**Definition 7.** Order relation in  $\mathcal{A}$ . The order on  $\mathcal{A}$  is naturally induced from the order in the real axis

$$L \prec A \prec M \prec B \prec R.$$

**Definition 8.**[11] Parity function  $\rho(S)$ . Given any finite sequence  $S$  with length  $p$ ,  $\rho(S)$  is such that

$$\rho(S) = (-1)^p.$$

**Definition 9.**[11] Let  $\mathcal{A}^{\mathbb{N}}$  denote the set of all sequences written with the alphabet  $\mathcal{A}$ . We define an ordering  $\prec$  on the set  $\mathcal{A}^{\mathbb{N}}$  such that: given two symbolic sequences  $P = P_0P_1P_2\dots$  and  $Q = Q_0Q_1Q_2\dots$  let  $n$  be the first integer such that  $P_n \neq Q_n$ . Denote by  $S = S_0S_1S_2\dots S_{n-1}$  the common first subsequence of both  $P$  and  $Q$ . Then, we say that  $P \prec Q$  if  $P_n \prec Q_n$  and  $\rho(S) = +1$  or  $Q_n \prec P_n$  if  $\rho(S) = -1$ . If no such  $n$  exists then  $P = Q$ .

This ordering is originated by the fact that when  $x < y$  then  $\text{It}(x) \preceq \text{It}(y)$ .

To state the rules of admissibility the shift operator  $\sigma$  will be used, defined as usual.

**Definition 10.** Shift operator  $\sigma$ . The shift operator is defined

$$\sigma(P_0P_1P_2\dots) = P_1P_2\dots$$

When we have a finite sequence  $S$  the shift operator acts such that

$$\sigma(S_0S_1S_2\dots S_{n-1}) = S_1S_2\dots S_{n-1}S_0.$$

The orbit of  $+a$  has the symbolic itinerary  $\text{It}(+a)$ . The sequence  $\text{It}(+a)$  is maximal (resp.  $\text{It}(-a)$  is minimal) in the ordering defined in this section. Maximal in the sense that every shift of the sequence  $\text{It}(+a)$  is less or equal than  $\text{It}(+a)$ . Every orbit with initial condition  $x_0$  is symmetric to the orbit with initial condition  $-x_0$ . Thus any orbit beginning by  $+a$  is accompanied by a symmetric orbit started by  $-a$ . Therefore, we shall focus the admissibility rules for kneading sequences only on the itineraries with the first symbol  $R$  (corresponding to  $+a$ ).

**Definition 11.**[11] Operator  $\tau$ . The operator  $\tau : \mathcal{A}^{\mathbb{N}} \mapsto \mathcal{A}^{\mathbb{N}}$  is defined such that

$$\tau L = R, \tau A = B, \tau M = M, \tau B = A, \tau R = L.$$

Given a sequence  $Q = Q_0Q_1Q_2\dots$ ,  $\tau$  acts such that

$$\tau Q = \tau Q_0\tau Q_1\tau Q_2\dots$$

The operator  $\tau$  interchanges the symbols  $L$  and  $R$ , leaving the symbols  $M$  unchanged. For instance  $\tau((RLMR)^\infty) = (LRML)^\infty$ .

**Proposition 1**[11]  $\text{It}(x_0) = \tau \text{It}(-x_0)$ .

**Proof.** Is a direct consequence of condition 1 in definition 1.  $\square$

Given any itinerary of  $+a$  denoted by  $S$ , the corresponding itinerary of  $-a$  is  $\tau S$ . The kneading pair is  $(S, \tau S)$ . To know the kneading sequence  $S$ , corresponding to the orbit of  $+a$ , is to know the kneading pair. By some abuse of notation, sometimes (mainly in the examples) we use only the kneading sequence  $S$  instead of the kneading pair.

**Definition 12.**[11] Admissibility rules: Let  $S$  be a given sequence of symbols and  $(S, \tau S)$  be a pair of sequences.  $(S, \tau S)$  is a kneading pair and  $S$  is a kneading sequence, if  $S$  satisfies the admissibility condition:  $\tau S \preceq \sigma^k S \preceq S$ , for every integer  $k$ . The set of the admissible sequences is denoted by  $\Sigma \subset \mathcal{A}^{\mathbb{N}}$ .

**Definition 13.** Given a finite sequence  $P$  with length  $p$ , the sequence  $S = P^\infty$  is called a  $p$ -periodic sequence.

We will work sometimes only with  $P$  instead of  $P^\infty$  when there is no danger of confusion.

**Definition 14.**[11] A bistable periodic orbit contains both the orbit of  $+a$  and the orbit of  $-a$ . Any bistable orbit has an itinerary  $S = P^\infty = (Q\tau Q)^\infty$  or shortly  $P = Q\tau Q$

As a consequence of the previous definition bistable orbits and associated symbolic itineraries must have even period.

### 2.3 Kneading theory

In [10] were introduced the concepts of invariant coordinate, kneading increments, kneading matrix and kneading determinant. We will use the definitions of the cited work with the convenient adaptations for the discontinuous case. We present here a brief exposition of the results obtained applying kneading theory to this type of maps.

**Definition 15.**Invariant coordinate of an initial condition  $\theta_{x_0}(t)$ . Is defined using the sequence  $X = X_0X_1X_2 \dots = \text{It}(x_0)$ . Is the formal power series

$$\theta_{x_0}(t) = \sum_{k=0}^{+\infty} (-1)^k X_k t^k.$$

With the notation  $\theta_{c_i^\pm}(t) = \lim_{x \rightarrow c_i^\pm} \theta_x(t)$ , for each discontinuity point, the kneading increment is defined.

**Definition 16.**Kneading increment and kneading matrix. The kneading increment is

$$v_i(t) = \theta_{c_i^+}(t) - \theta_{c_i^-}(t).$$

This quantity is a formal power series measuring the discontinuity. After collecting the terms associated to each symbol, and remarking that, in this case,  $c_1^-$  corresponds to  $L$ ,  $c_1^+$  corresponds to  $M$ ,  $c_2^-$  corresponds to  $M$  and  $c_2^+$  corresponds to  $R$ , the decomposition

$$v_i(t) = N_{i1}(t)L + N_{i2}(t)M + N_{i3}(t)R$$

is obtained. The kneading matrix is

$$N = \begin{bmatrix} N_{11}(t) & N_{12}(t) & N_{13}(t) \\ N_{21}(t) & N_{22}(t) & N_{23}(t) \end{bmatrix}.$$

**Definition 17.**Kneading determinant. Omitting the  $j$ -th column of the kneading matrix we compute the determinant  $D_j$ . The kneading determinant is

$$D(t) = \frac{(-1)^{j+1} D_j}{(1+t)}.$$

The denominator in the kneading determinant results from the fact that  $F$  is decreasing in the three intervals where this map is defined [10]. Note that  $D_1 = -D_2 = D_3$ .

**Definition 18.**[11] Given a sequence  $X = X_1X_2 \dots$  we define a function  $\Phi : \mathcal{A} \mapsto \{-1, 0, 1\}$ , such that

$$\Phi(X_i) = \begin{cases} -1 & \text{if } X_i = L, A \\ 0 & \text{if } X_i = M \\ 1 & \text{if } X_i = R, B \end{cases}.$$

**Definition 19.**[11] Given a sequence  $X = X_1X_2 \dots$  from  $\mathcal{A}^{\mathbb{N}}$  we define a formal power series  $u(t)$ , such that  $u(t) = \sum_{k=1}^{+\infty} (-1)^k \Phi(X_k) t^k$ . When  $X$  is finite with length  $p$  we define the formal polynomial  $u_p(t) = \sum_{k=1}^p (-1)^k \Phi(X_k) t^k$ .

Let  $S = S_1S_2 \dots \in \Sigma$  be a kneading sequence with the kneading pair  $(S, \tau S)$ , then the kneading determinant is given by.

$$D(t) = \frac{1 + 2u(t)}{t + 1}. \tag{2}$$

When  $S = P^\infty$  is  $p$ -periodic, the expression of the kneading determinant simplifies into  $D(t)(t + 1) = 1 + \frac{2u_p(t)}{1 - (-1)^p t^p}$ . When the kneading sequence  $S$  is bistable:  $S = P^\infty$  with  $P = Q\tau Q$  of period  $p = 2q$ , with the associated kneading pair  $(S, \tau S) = (Q\tau Q, (\tau Q)Q)$  the kneading determinant is  $\left(1 + \frac{2u_q(t)}{1 + (-1)^q t^q}\right) \frac{1}{1+t}$ .

### 2.4 Growth number

The kneading determinant is essential in the computation of the growth number of laps.

**Definition 20.**Lap number  $\ell(F^n)$  is the number of maximal intervals of continuity of each composition of  $F$  with itself.

**Definition 21.**The growth number is defined

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\ell(F^n)}. \tag{3}$$

*Remark.*[11] The growth number of  $F$  can be computed using the relation

$$\rho = \frac{1}{t_0},$$

where  $t_0$  is the least root in the unit interval of the kneading determinant  $D(t)$ . The proof is provided defining the power series  $\Lambda(t) = \sum_{n \geq 1} \ell(F^n) t^{n-1}$ , where each coefficient is the lap number of the iterate  $F^n$ . This new power series is closely related to the kneading determinant because of the relation  $\Lambda(t) = \frac{1}{t(1-t^2)D(t)} - \frac{1}{t}$ .

*Example 3.*The kneading sequence  $(RMR)^\infty$  corresponds to the kneading determinant  $D(t) = \frac{1-2t-t^3}{(t+1)(1+t^3)}$ , which is realized for instance by  $g_\beta(x)$  with  $\beta$  approximately 3.1588 or by  $G_\alpha$  (from example 2) with  $\alpha = \frac{1}{4}(\sqrt{5}-1)$ . We obtain

$\Lambda(t) = 3 + 7t + 17t^2 + 39t^3 + 87t^4 + 193t^5 + \dots$ , in this case  $\ell(F) = 3$ ,  $\ell(F^2) = 7$ ,  $\ell(F^3) = 17 \dots$

It will be an interesting work to see if the usual relationship between the topological entropy and growth number still remains valid in the case of discontinuous maps.

### 2.5 Markov partition

Whenever we can define Markov matrices, the method of Markov transition matrices in the case of continuous maps is an equivalent approach to the computation the roots of the kneading determinant. To each  $p$ -periodic kneading pair we associate a Markov transition matrix, see [9] and related references on that paper. Now, denote by

$$x_j^{(2)} = F^j(c_2^+), j = 0, 1, \dots, p-1,$$

$$x_j^{(1)} = F^j(c_1^-), j = 0, 1, \dots, p-1,$$

the orbits of the discontinuity points. An ordered sequence  $(z_k)_{k=1, \dots, 2p}$  is obtained reordering the elements  $x_j^{(m)}$ ,  $m = 1, 2$ , and getting a partition

$$I_k = (z_k, z_{k+1}) \text{ with } k = 1, \dots, 2p-1.$$

The discontinuity points are present in the above partition. We call  $z_{k_1} = c_1$  and  $z_{k_2} = c_2$ . To compute the Markov matrix note that  $I_{k_1-1} = (z_{k_1-1}, c_1^-)$  and  $I_{k_1} = (c_1^+, z_{k_1+1})$  and similarly with the two intervals adjacent to the discontinuity point  $c_2$ . With this precision made, the Markov transition matrix can be defined.

**Definition 22.** The Markov transition matrix  $\Psi = [\psi_{ij}]$  is defined by the rule:

$$\psi_{ij} = \begin{cases} 1 & \text{if } I_j \subset F(I_i), \\ 0 & \text{otherwise.} \end{cases}$$

In [9] the relationship between Markov partitions and kneading theory is explained for bimodal continuous maps. It is also presented the proof of the equality of the reciprocal of  $t_0$  and the spectral radius of the matrix  $\Psi$ . In this paper we will prove the same equivalence of definitions in the case of bimodal symmetric discontinuous maps.

In the next example we obtain this equivalence for a particular case, computing directly both the kneading determinant and the characteristic polynomial of the Markov matrix.

*Example 4.* The kneading pair

$$((RMR)^\infty, (LML)^\infty)$$

corresponds to a pair of orbits satisfying

$$x_1^{(1)} < x_0^{(1)} = c_1 < x_2^{(2)} < x_2^{(1)} < x_0^{(2)} = c_2 < x_1^{(2)}, \quad (4)$$

renaming the elements of the partition we get

$$z_1 = x_1^{(1)}, z_2 = x_0^{(1)}, z_3 = x_2^{(2)},$$

$$z_4 = x_2^{(1)}, z_5 = x_0^{(2)}, z_6 = x_1^{(2)}.$$

The Markov matrix is

$$\Psi = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The smallest solution  $t_0$  of the equation

$$\det(I - \Psi t) = (1 - t + t^2)(1 - 2t - t^3) = 0$$

in the unit interval gives the growth number  $\rho = \frac{1}{t_0} = \frac{1}{0.4534} = 2.2056$ , exactly the same root obtained with the kneading determinant of the example 3.

### 2.6 Relation between Markov partition and the orbits of the discontinuity points

Giving the  $p$ -periodic orbits

$$O(c_2^+) = \{x_j^{(1)} : x_j^{(1)} = F^j(c_2^+), j = 0, 1, \dots, p-1\}$$

and

$$O(c_1^-) = \{x_j^{(2)} : x_j^{(2)} = F^j(c_1^-), j = 0, 1, \dots, p-1\},$$

we define the vector

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ y_{p+1} \\ \vdots \\ y_{2p} \end{bmatrix} = \begin{bmatrix} x_0^{(2)} \\ \vdots \\ x_{p-1}^{(2)} \\ x_0^{(1)} \\ \vdots \\ x_{p-1}^{(1)} \end{bmatrix}.$$

Let  $z$  be the vector  $\{z_i\}_{i=1, \dots, 2p}$  where  $z_{i-1} < z_i < z_{i+1}$  are the ordered elements of  $y$ . There is a  $2p \times 2p$  permutation matrix  $\pi$  such that

$$z = \pi y.$$

Let  $x_k^{(2)} = \text{It}(x_k^{(2)})$ , the symbolic itinerary of  $x_k^{(2)}$ , for  $k = 0, \dots, p-1$ . It is clear that  $x_1^{(2)} = S_1 S_2 \dots = S$  is the kneading sequence of  $+a$ . Let  $x_k^{(1)} = \text{It}(x_k^{(1)})$  for  $k = 0, \dots, p-1$ . By symmetry  $x_1^{(1)} = \tau S$  is the kneading sequence of  $-a$ . It is also clear that  $x_0^{(2)} = \sigma^{p-1}(S)$  and

$x_0^{(1)} = \sigma^{p-1}(\tau S)$ . To each  $k = 1, \dots, p$  corresponds a symbolic sequence

$$x_k^{(2)} = \sigma^{k-1}(S).$$

To each  $k = 1 \dots, p$ , corresponds another sequence

$$x_k^{(1)} = \sigma^{k-1}(\tau S).$$

Naturally, we have

$$x_k^{(1)} = \tau(x_k^{(2)}).$$

To each  $z_j$  corresponds the symbolic itinerary  $w_j = \text{It}(z_j)$ . We define  $v_j = \text{It}(y_j)$ .

*Example 5.* Given the kneading sequence  $(RMB)^\infty$ , equivalent to  $(RMR)^\infty$  already used before. The kneading pair is  $((RMB)^\infty, (LMA)^\infty)$ ,  $x_1^{(2)} = RMB = v_2$ ,  $x_2^{(2)} = MBR = v_3$ ,  $x_0^{(2)} = BRM = v_1$ ,  $x_1^{(1)} = LMA = v_5$ ,  $x_2^{(1)} = MAL = v_6$ , and  $x_0^{(1)} = ALM = v_4$ . We get  $w_1 = v_5$ ,  $w_2 = v_4$ ,  $w_3 = v_3$ ,  $w_6 = v_4$ ,  $w_5 = v_1$  and  $w_6 = v_2$ .

The matrix  $\pi$  is

$$\pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Example 6.* Given the kneading sequence  $(RLMB)^\infty$ , the kneading determinant  $D(t)$  is such that

$$(1+t)D(t) = 1 + 2 \frac{-t - t^2 + t^4}{1 - t^4} = \frac{1 - 2t - 2t^2 + t^4}{1 - t^4}.$$

The kneading pair is  $((RLMB)^\infty, (LRMA)^\infty)$ ,  $x_1^{(2)} = RLMB = v_2$ ,  $x_2^{(2)} = LMBR = v_3$ ,  $x_3^{(2)} = MBRL = v_4$ ,  $x_0^{(2)} = BRML = v_1$ ,  $x_1^{(1)} = LRMA = v_6$ ,  $x_2^{(1)} = RMAL = v_7$ ,  $x_3^{(1)} = MALR = v_8$  and  $x_0^{(1)} = ALRM = v_5$ . We get  $w_1 = v_6$ ,  $w_2 = v_3$ ,  $w_3 = v_5$ ,  $w_4 = v_4$ ,  $w_5 = v_8$ ,  $w_6 = v_1$ ,  $w_7 = v_7$  and  $w_8 = v_2$ .

The Markov matrix is

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix  $\pi$  is

$$\pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix  $\pi$  can also be used to reorder the shifts of the kneading sequences, giving the vector  $v = \{v_1, \dots, v_{2p}\}$  and the vector  $w = \{w_1, \dots, w_{2p}\}$ , we have  $w = \pi v$ .

### 3 Main Result

#### 3.1 The Markov and kneading endomorphism in spaces of chain complexes

Let  $C_0$  be the vector space of the 0-chains spanned by the shifts of the kneading sequences  $\{v_j\}_{j=1, \dots, 2p}$  this space is isomorphic of the space of the 0-chains spanned by the points of the orbit  $\{y_j\}_{j=1, \dots, 2p}$ . The space  $\pi(C_0)$  is spanned by  $\{w_k\}_{k=1, \dots, 2p}$  which is isomorphic to the space of the 0-chains spanned by  $\{z_j\}_{j=1, \dots, 2p}$ . Let  $C_1$  be the space of the 1-chains spanned by  $\{I_k\}_{k=1, \dots, 2p-1}$ , isomorphic to the linear space of the 1-chains spanned by  $\{I'_k\}_{k=1, \dots, 2p-1}$  where  $I'_k$  is the set of all the admissible sequences  $w: w_k \preceq w \preceq w_{k+1}$ . In what follows we identify  $I'_k$  with  $I_k$  and use the same symbol both for sequences and intervals and call both the linear transformations and the corresponding matrix representations by the same letters.

The border of a 1-chain is obtained using the linear transformation  $\partial: C_1 \rightarrow D_0$  such that  $\partial(I_k) = w_{k+1} - w_k$ ,  $\partial(C_1) = D_0$  where  $D_0$  is spanned by

$$\{w_{k+1} - w_k\}_{k=1, \dots, 2p-1}.$$

It is clear that  $D_0 \subset \pi(C_0)$ .

We define the linear transformation  $\partial_s: C_1 \rightarrow D_0$  such that

$$\begin{aligned} \partial_s(I_k) &= \partial(\mathbf{1} - \tau)I_k \\ &= \partial(I_k - I_{2p-k}) \\ &= \partial(I_k) - \partial(I_{2p-k}). \end{aligned}$$

The image of  $I_k$  by  $\partial_s$  is

$$\partial_s(I_k) = w_{k+1} - w_k - (w_{2p+1-k} - w_{2p-k})$$

and is an element of  $D_0$ . We can define another linear transformation that acts on  $\pi(C_0)$  with matrix

representation

$$\mu = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 1 & -1 \\ 0 & -1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & -1 & \cdots & -1 & 1 & 0 \\ 1 & -1 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

where  $\mu_{ij} = \delta_{i+1,j} - \delta_{i,j} - \delta_{2p+1-i,j} + \delta_{2p-i,j}$ ,  $i = 1, \dots, 2p - 1$ ,  $j = 1, \dots, 2p$ , and  $\delta$  is the Kronecker delta symbol. This linear transformation represents the order relation of the points of the real line and the symmetry of the original mapping. It is immediate from the above definitions that  $Image(\partial_s) = B_0$  is a proper subspace of  $D_0$  such that  $D_0 = B_0 \oplus \tilde{B}$ , where  $\tilde{B}$  has dimension one, and  $B_0$  is isomorphic to  $\mu\pi(C_0)$ . We define  $\eta = \mu\pi$  and the endomorphism  $\omega$  acting on  $C_0$  with matrix representation

$$\omega = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix},$$

where  $\sigma$  is the shift operator with matrix representation  $p \times p$

$$\sigma = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Let  $\alpha$  be the endomorphism induced in  $B_0$  by the rotation  $\omega$  in  $C_0$  which results from the commutativity of the diagram

$$\begin{array}{ccccc} C_0 & \xrightarrow{\eta} & B_0 & \xleftarrow{\partial_s} & C_1 \\ \omega \downarrow & & \downarrow \alpha & & \downarrow \alpha \\ C_0 & \xrightarrow{\eta} & B_0 & \xleftarrow{\partial_s} & C_1 \end{array}.$$

It is easy to see that  $\alpha = -\Psi$ , where  $\Psi$  is the Markov matrix. Note that  $\eta\omega = \alpha\eta$ . Every entry in the matrix  $\alpha$  is non-positive, because the images of the intervals are obtained by the images of the boundary points and  $F$  is reverse order in any of each interval of continuity (lap).

Example 7. With the matrices of the examples 5 we have

$$\eta = \begin{bmatrix} 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

and

$$\eta\omega = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix},$$

which is precisely  $-\Psi\eta$ .

### 3.2 Matrix $\Theta$

Giving the right  $p$ -periodic kneading sequence  $S$ , the symbolic itinerary of the right discontinuity point is  $\sigma^{p-1}S = S_0S_1S_2 \dots = S_0S$ . We construct the vector

$$s(S) = \begin{bmatrix} \Phi(S_0) \\ \Phi(S_1) \\ \vdots \\ \Phi(S_{p-1}) \\ \Phi(\tau S_0) \\ \Phi(\tau S_1) \\ \vdots \\ \Phi(\tau S_{p-1}) \end{bmatrix} = \begin{bmatrix} 1 \\ \Phi(S_1) \\ \vdots \\ \Phi(S_{p-1}) \\ -1 \\ \Phi(\tau S_1) \\ \vdots \\ \Phi(\tau S_{p-1}) \end{bmatrix},$$

where  $\Phi$  was defined in definition 18. When applied to the other kneading sequence  $\tau S$  the vector takes the form

$$s(\tau S) = \sigma^p s(S).$$

Let  $\Gamma$  be a square matrix which columns 1 and  $p + 1$  are  $s$  and  $\sigma^p s$ , respectively, and the other elements are zeros.

Now, we introduce the matrices  $\gamma = \Gamma - I$  and  $\Theta = \gamma\omega$ . The matrix  $\Theta$  has the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\Phi(S_0) & 0 & \cdots & 0 & 0 \\ 0 & \Phi(S_1) & -1 & \cdots & 0 & 0 & 0 & -\Phi(S_1) & 0 & \cdots & 0 & 0 \\ 0 & \Phi(S_2) & 0 & \cdots & 0 & 0 & 0 & -\Phi(S_2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Phi(S_{p-2}) & 0 & \cdots & 0 & -1 & 0 & -\Phi(S_{p-2}) & 0 & \cdots & 0 & 0 \\ -1 & \Phi(S_{p-1}) & 0 & \cdots & 0 & 0 & 0 & -\Phi(S_{p-1}) & 0 & \cdots & 0 & 0 \\ 0 & -\Phi(S_0) & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\Phi(S_1) & 0 & \cdots & 0 & 0 & 0 & \Phi(S_1) & -1 & \cdots & 0 & 0 \\ 0 & -\Phi(S_2) & 0 & \cdots & 0 & 0 & 0 & \Phi(S_2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\Phi(S_{p-2}) & 0 & \cdots & 0 & 0 & 0 & \Phi(S_{p-2}) & 0 & \cdots & 0 & -1 \\ 0 & -\Phi(S_{p-1}) & 0 & \cdots & 0 & 0 & -1 & \Phi(S_{p-1}) & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Example 8. With the same kneading sequences of the example 5, we have

$$\gamma = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

**Proposition 2** *The following diagram commutes*

$$\begin{array}{ccc} & \eta & \\ C_0 & \longrightarrow & B_0 \\ \gamma \downarrow & & \downarrow -I \\ C_0 & \longrightarrow & B_0 \\ & \eta & \end{array}$$

**Proof.** We must show that  $\eta\gamma = -\eta$  or  $\eta(\Gamma - I) = -\eta$ , in other words that  $\eta\Gamma = 0$ , or that  $s(S), s(\tau S) \in \text{kernel}(\eta)$ . But  $\eta = \mu\pi$ , and  $\pi$  reorders  $s(S)$  in terms of the order of the real line, giving

$$\pi s(S) = \begin{bmatrix} v(\mathbb{I}(z_1)) \\ v(\mathbb{I}(z_2)) \\ \vdots \\ v(\mathbb{I}(z_{2p-1})) \\ v(\mathbb{I}(z_{2p})) \end{bmatrix},$$

knowing that  $\mathbb{I}(z_j) = \tau\mathbb{I}(z_{2p-j}), j = 1, \dots, p$ , we have  $v(\mathbb{I}(z_j)) = -v(\mathbb{I}(z_{2p-j}))$ . It is obvious that  $\mu\pi s(S) = \mu\pi s(\tau S) = 0$ .  $\square$

**Theorem 1.** *The characteristic polynomial of the matrix  $\Theta = \gamma\omega$  is*

$$P_\Theta(t) = \det(I - t\Theta) = (1 - (-1)^p t^p)^2 (1 + t)D(t),$$

where  $D(t)$  is the kneading determinant.

**Proof.** The determinant of the matrix  $I - t\Theta$  is

$$\begin{vmatrix} 1 & 0 & 0 \dots 0 & 0 & 0 & t\Phi(S_0) & 0 \dots 0 & 0 \\ 0 & 1 - t\Phi(S_1) & t \dots 0 & 0 & 0 & t\Phi(S_1) & 0 \dots 0 & 0 \\ 0 & -t\Phi(S_2) & 1 \dots 0 & 0 & 0 & t\Phi(S_2) & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots \dots \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -t\Phi(S_{p-2}) & 0 \dots 1 & t & 0 & t\Phi(S_{p-2}) & 0 \dots 0 & 0 \\ t & -t\Phi(S_{p-1}) & 0 \dots 0 & 1 & 0 & t\Phi(S_{p-1}) & 0 \dots 0 & 0 \\ 0 & t\Phi(S_0) & 0 \dots 0 & 0 & 1 & 0 & 0 \dots 0 & 0 \\ 0 & t\Phi(S_1) & 0 \dots 0 & 0 & 0 & 1 - t\Phi(S_1) & t \dots 0 & 0 \\ 0 & t\Phi(S_2) & 0 \dots 0 & 0 & 0 & -t\Phi(S_2) & 1 \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & t\Phi(S_{p-2}) & 0 \dots 0 & 0 & 0 & -t\Phi(S_{p-2}) & 0 \dots 1 & t \\ 0 & t\Phi(S_{p-1}) & 0 \dots 0 & 0 & t & -t\Phi(S_{p-1}) & 0 \dots 0 & 1 \end{vmatrix}$$

We first multiply the row 1 by  $-t$  and add the result to the row  $p$ , we do the same with the rows  $p + 1$  and  $2p$ . Then we develop the determinant by the columns 1 and  $p$  getting

a  $(2p - 1) \times (2p - 1)$  determinant:

$$\begin{vmatrix} 1 - t\Phi(S_1) & t & \dots & 0 & 0 & 0 & t\Phi(S_1) & 0 & \dots & 0 & 0 \\ -t\Phi(S_2) & 1 & \dots & 0 & 0 & 0 & t\Phi(S_2) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -t\Phi(S_{p-2}) & 0 & \dots & 1 & t & 0 & t\Phi(S_{p-2}) & 0 & \dots & 0 & 0 \\ -t\Phi(S_{p-1}) & 0 & \dots & 0 & 1 & t\Phi(S_{p-1}) - t^2\Phi(S_0) & 0 & 0 & \dots & 0 & 0 \\ t\Phi(S_1) & 0 & \dots & 0 & 0 & 0 & 1 - t\Phi(S_1) & t & \dots & 0 & 0 \\ t\Phi(S_2) & 0 & \dots & 0 & 0 & 0 & -t\Phi(S_2) & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t\Phi(S_{p-2}) & 0 & \dots & 0 & 0 & 0 & -t\Phi(S_{p-2}) & 0 & \dots & 1 & t \\ t\Phi(S_{p-1}) - t^2\Phi(S_0) & 0 & \dots & 0 & 0 & 0 & -t\Phi(S_{p-1}) & 0 & \dots & 0 & 1 \end{vmatrix},$$

then we multiply the row  $p - 1$  by  $-t$  and add the result to the row  $p - 2$ . We do the same with the last two rows. Then we develop the determinant by the columns  $p - 1$  and the last one getting a  $(2p - 2) \times (2p - 2)$  determinant:

$$\begin{vmatrix} 1 - t\Phi(S_1) & t & \dots & 0 & t\Phi(S_1) & 0 & \dots & 0 \\ -t\Phi(S_2) & 1 & \dots & 0 & t\Phi(S_2) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1(t) & 0 & \dots & 1 & r_2(t) & 0 & \dots & 0 \\ t\Phi(S_1) & 0 & \dots & 0 & 1 - t\Phi(S_1) & t & \dots & 0 \\ t\Phi(S_2) & 0 & \dots & 0 & -t\Phi(S_2) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_2(t) & 0 & \dots & 0 & r_1(t) & 0 & \dots & 1 \end{vmatrix},$$

where  $r_1(t) = -t\Phi(S_{p-2}) + t^2\Phi(S_{p-1})$  and  $r_2(t) = t\Phi(S_{p-2}) - t^2\Phi(S_{p-1}) + t^3\Phi(S_0)$ . Repeating this reducing process we get a  $2 \times 2$  determinant

$$\begin{vmatrix} 1 - \sum_{k=1}^{p-1} (-1)^k t^k \Phi(S_k) & -\sum_{k=1}^p (-1)^k t^k \Phi(S_k) \\ -\sum_{k=1}^p (-1)^k t^k \Phi(S_k) & 1 - \sum_{k=1}^{p-1} (-1)^k t^k \Phi(S_k) \end{vmatrix}.$$

Remembering that

$$u_p(t) = \sum_{k=1}^p (-1)^k t^k \Phi(S_k),$$

$$S_0 = S_p = B \text{ and } (-1)^p t^p \Phi(B) = (-1)^p t^p,$$

the previous determinant is equal to

$$\begin{vmatrix} (1 - (-1)^p t^p) + u_p(t) & -u_p(t) \\ -u_p(t) & (1 - (-1)^p t^p) + u_p(t) \end{vmatrix},$$

which gives

$$P_\Theta(t) = (1 - (-1)^p t^p)^2 \left( 1 + 2 \frac{u_p(t)}{1 - (-1)^p t^p} \right)$$

and this is precisely

$$P_\Theta(t) = (1 - (-1)^p t^p)^2 (1 + t)D(t),$$

as desired.  $\square$



*Example 9.* We use the kneading sequences of the example 6 to illustrate this last result, the matrix  $\Theta$  is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

with characteristic polynomial

$$(1 - t^4)(1 - 2t - 2t^2 + t^4),$$

which agrees with the value of the kneading determinant  $(1 + t)D(t) = \frac{1-2t-2t^2+t^4}{1-t^4}$ .

We have now all the ingredients to state the main results of this paper.

**Theorem 2.** *The following diagram commutes*

$$\begin{array}{ccc} & \eta & \\ C_0 & \xrightarrow{\quad} & B_0 \\ \downarrow \Theta & & \downarrow \Psi \\ C_0 & \xrightarrow{\quad} & B_0 \\ & \eta & \end{array}$$

and  $P_\Theta(t) = (1 + t)\det(I - t\Psi)$ .

**Proof.** Noticing that  $\Theta = \gamma\omega$  and  $\Psi = -\alpha$  the result is only a direct consequence of

$$\begin{array}{ccc} & \eta & \\ C_0 & \xrightarrow{\quad} & B_0 \\ \downarrow \omega & & \downarrow \alpha \\ C_0 & \xrightarrow{\quad} & B_0 \\ \downarrow \gamma & & \downarrow -I \\ C_0 & \xrightarrow{\quad} & B_0 \end{array}$$

conjugated with the fact that the two rows in the next diagram

$$\begin{array}{ccccccc} & & \text{inj} & & \eta & & \\ 0 & \longrightarrow & \tilde{B} & \longrightarrow & C_0 & \longrightarrow & B_0 \longrightarrow 0 \\ & & \downarrow -I & & \text{inj} & \downarrow \Theta & \eta & \downarrow \Psi \\ 0 & \longrightarrow & \tilde{B} & \longrightarrow & C_0 & \longrightarrow & B_0 \longrightarrow 0 \end{array}$$

are exact sequences, where inj is the natural embedding.  $\square$

**Corollary 1.** *The inverse of the least root in the unit interval of the periodic kneading determinant is the spectral radius of the Markov matrix.*

**Proof.** Is an immediate consequence of the relation

$$(1 - (-1)^p t^p)^2 D(t) = \det(I - t\Psi),$$

obtained in the last theorem.  $\square$

We think that the results of this work can be extended to general discontinuous maps with finite number of discontinuities. That is a natural extension of this work.

## Acknowledgement

The author thanks some precious comments of the referee that simplified and improved a great deal the scope and clarity of this work. Namely, the possibility of infinite jumps at the discontinuity points. The author was partially funded by FCT/Portugal through project PEst-OE/EEI/LA0009/2013.

## References

- [1] L. Alsedá and F. Mañosas. Kneading theory and rotation intervals for a class of circle maps of degree one. *Nonlinearity*, 3(2):413–452, 1990.
- [2] J. F. Alves and J. Sousa-Ramos. Kneading theory for tree maps. *Ergodic Theory and Dynamical Systems*, 24(4):957–985, 2004.
- [3] J. H. Curry, L. Garnett, and D. Sullivan. On the iteration of a rational function: computer experiments with newton’s method. *Communications in mathematical physics*, 91(2):267–277, 1983.
- [4] E. D’Aniello and H. M. Oliveira. Pitchfork bifurcation for non-autonomous interval maps. *Difference Equations and Applications*, 15(3):291–302, 2009.
- [5] R. Devaney and L. Keen. *Dynamics of Tangent*, volume 1342 of *Lecture Notes in Mathematics*, pages 105–111. Springer, Berlin, New York, 1988.
- [6] W. de Melo and S. Strien. *One-dimensional dynamics*. Springer, Berlin, Heidelberg, 1993.
- [7] P. Glendinning and C. Sparrow. Prime and renormalisable kneading invariants and the dynamics of expanding lorenz maps. *Physica D: Nonlinear Phenomena*, 62(1):22–50, 1993.
- [8] L. Keen and J. Kotus. Dynamics of the family  $\lambda \tan z$ . *Conformal Geometry and Dynamics*, 1(1):28–57, 1997.
- [9] J. P. Lampreia and J. Sousa-Ramos. Symbolic dynamics of bimodal maps. *Portugaliae Mathematica*, 54(1):1–18, 2013.
- [10] J. Milnor and W. Thurston. *On iterated maps of the interval*, volume 1342 of *Lecture Notes in Mathematics*, pages 465–563. Springer, Berlin, 1988.
- [11] H. Oliveira and J. Sousa-Ramos. Iterates of transcendent meromorphic maps. *Grazer Mathematische Berichte*, (346):313–321, 2004.



**Henrique M. Oliveira,** prepared his PhD under the supervision of José Sousa Ramos in discrete dynamical systems. At the present time, he his “auxiliar professor” with tenure at Lisbon University, Portugal. He has been publishing in discrete dynamical

systems, bifurcation theory and mathematical biology and supervised PhD and master students on discrete dynamical systems and mathematics of music. He has published research articles in reputed international journals of pure and applied mathematics. He is referee and has been guest editor of mathematical journals and research books.