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Note on Corrected Optimal Quadrature Formulas in Sense Nikolski

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Abstract: The optimal 2-points quadrature formulas of open type are derived and the error estimates in terms of a variety of norms involving the second derivative are considered. Also, the corrected formulas of the optimal quadrature rules are obtained.

Keywords: optimal quadrature formula, corrected quadrature formula, Peano's Theorem, remainder term, optimal in sense Nikolski.

1 Introduction

In the last years the problem to construct the optimal quadrature formulas was studied by many authors ([3], [7], [8], [11], [12], [13], [15], [16]).

Denote

$$W_p^n[a,b]:=\left\{ f\in C^{n-1}[a,b],\, f^{(n-1)} \text{ abs. cont.}, \left\| f^{(n)}\right\|_p < \infty \right\},$$

where

$$\|f\|_p:=\left\{\int_a^b |f(x)|^p\,dx\right\}^{\frac{1}{p}}, \text{for} 1\le p<\infty, \|f\|_\infty:=\sup_{x\in[a,b]}|f(x)|.$$

Let

$$\int_a^b f(x)dx=\sum_{k=0}^m A_{m,k}f(a_k)+\mathscr{R}_n^{[p]}[f],\;\; f\in W_p^n[a,b] \qquad (1)$$

be a quadrature formula with degree of exactness equal $n-1$, where the nodes verify $a\le a_0<a_1<\cdots<a_m\le b$.

If $f\in W_p^n[a,b]$, by using Peano's theorem, the remainder term can be written

$$\mathscr{R}_n^{[p]}[f]=\int_a^b K_n(t)f^{(n)}(t)dt,$$

where

$$K_n(t)=\mathscr{R}_n^{[p]}\left[\frac{(x-t)_+^{n-1}}{(n-1)!}\right].$$

For the remainder term we have the evaluation

$$\left|\mathscr{R}_n^{[p]}[f]\right|\le\left[\int_a^b\left|f^{(n)}(t)\right|^p dt\right]^{\frac{1}{p}}\left[\int_a^b|K_n(t)|^q\,dt\right]^{\frac{1}{q}},\frac{1}{p}+\frac{1}{q}=1, \qquad (2)$$

with the remark that in the cases $p=1$ and $p=\infty$ this evaluation is

$$\left|\mathscr{R}_n^{[1]}[f]\right|\le\int_a^b\left|f^{(n)}(t)\right|dt\cdot\sup_{t\in[a,b]}|K_n(t)|, \qquad (3)$$

$$\left|\mathscr{R}_n^{[\infty]}[f]\right|\le\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|\cdot\int_a^b|K_n(t)|\,dt. \qquad (4)$$

Definition 1. *The quadrature formula (1) is called optimal in the sense Nikolski in the space $W_p^n[a,b]$, if*

$$\mathscr{F}(A,X)=\begin{cases}\displaystyle\int_a^b|K_n(t)|^q\,dt,\;\frac{1}{p}+\frac{1}{q}=1,\,1<p\le\infty,\\ \displaystyle\sup_{t\in[a,b]}|K_n(t)|,\;p=1,\end{cases}$$

attains the minimum value with regard to A and X, where $A=\{A_{m,k}\}_{k=0}^m$ are the coefficients and $X=(a_0,a_1,\cdots,a_m)$ are the nodes of the quadrature formula.

For the particular case $n=2$, the coefficients $A=\{A_{m,k}\}_{k=0}^m$ and the nodes $X=(a_0,a_1,\cdots,a_m)$ of the

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quadrature formula optimal in sense Nikolski are calculated in [14]. In the next section we will recall the algorithm to derive a quadrature formula with 2-points which is optimal in sense Nikolski, namely we will calculate the coefficients and the nodes such that the quadrature formula to be optimal, considering that the remainder term is evaluated in sense of (2) in the cases $p=1$, $p=2$ and $p=\infty$. Also, the error estimations using different norms and involving the second derivative are given.

In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [4], [5], [6], [9], [10], [17]). By a corrected quadrature rule we mean the formula which involves not only the values of the function at certain points, but also the values of the first derivative at the endpoints of the interval. These formulas have a higher degree of exactness than the original rule. The error estimates in the corrected rule are better than in the original rule, in general.

Let $I\subset\mathbb{R}$ be an open interval such that $[0,1]\subset I$ and let $f:I\to\mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. In [18], N. Ujević and L. Mijić constructed the following formula of close type with 3 nodes

$$\int_0^1 f(t)dt=\frac{\sqrt{2}}{8}f(0)+\left(1-\frac{\sqrt{2}}{4}\right)f\left(\frac{1}{2}\right)+\frac{\sqrt{2}}{8}f(1)+\mathscr{R}[f],$$

where $|\mathscr{R}[f]|\le\dfrac{2-\sqrt{2}}{48}\|f''\|_\infty$.

This problem was generalized in [1] considering the quadrature formula of close type with more than 3 nodes.

In [2], using classical methods to obtain an optimal quadrature formula, we calculated the coefficients $A_i, i=\overline{0,2}$ and the node $a_1\in(0,1)$ such that the quadrature formula

$$\int_0^1 f(t)dt=A_0f(0)+A_1f(a_1)+A_2f(1)+\mathscr{R}[f]$$

to be optimal, considering that the remainder term is evaluated in sense of (2), in the cases $p=1$, $p=2$ and $p=\infty$. Also, we constructed the corrected quadrature formulas for all three cases. For $p=\infty$ we find the Ujević and Mijić' s result.

The main purpose of this paper is to derive corrected rules of the optimal 2-points quadrature formulas of open type. We will show that the corrected formula improves the original one. The remainder term is evaluated in sense of (2), in the cases $p=1$, $p=2$ and $p=\infty$.

2 The optimal 2-points quadrature formulas in sense Nikolski

Let

$$\int_0^1 f(x)dx=A_1f(a_1)+A_2f(a_2)+\mathscr{R}_2^{[p]}[f] \tag{5}$$

be a quadrature formula with degree of exactness equal 1.

Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathscr{R}_2^{[p]}[e_i]=0$, $e_i(x)=x^i$, $i=0,1$, namely

$$A_1+A_2=1,\ A_1a_1+A_2a_2=\frac{1}{2}, \tag{6}$$

and using Peano's theorem the remainder term has the following integral representation

$$\mathscr{R}_2^{[p]}[f]=\int_0^1 K_2(t)f''(t)dt,\ \text{where} \tag{7}$$

$$K_2(t)=\mathscr{R}_2^{[p]}[(x-t)_+]=\begin{cases}\frac{1}{2}t^2,\ 0\le t<a_1,\\ \frac{(1-t)^2}{2}+A_2t-a_2A_2, a_1\le t\le a_2,\\ \frac{1}{2}(1-t)^2,\ a_2<t\le 1.\end{cases} \tag{8}$$

Theorem 1. *For $f\in W_\infty^2[0,1]$, the quadrature formula of the form (5), optimal with regard to the error, is*

$$\int_0^1 f(x)dx=\frac{1}{2}\left[f\left(\frac{2\sqrt{3}-3}{2}\right)+f\left(\frac{5-2\sqrt{3}}{2}\right)\right]+\mathscr{R}_2^{[\infty]}[f], \tag{9}$$

with

$$\mathscr{R}_2^{[\infty]}[f]=\int_0^1 K_2(t)f''(t)dt, \tag{10}$$

where

$$K_2(t)=\begin{cases}\frac{1}{2}t^2,\ 0\le t<\frac{2\sqrt{3}-3}{2},\\ \frac{(1-t)^2}{2}+\frac{1}{2}t-\frac{5-2\sqrt{3}}{4},\ \frac{2\sqrt{3}-3}{2}\le t\le\frac{5-2\sqrt{3}}{2},\\ \frac{1}{2}(1-t)^2,\ \frac{5-2\sqrt{3}}{2}<t\le 1.\end{cases}$$

Proof. The remainder term (7) can be evaluated in the following way

$$\left|\mathscr{R}_2^{[\infty]}[f]\right|\le\|f''\|_\infty\int_0^1|K_2(t)|dt.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1|K_2(t)|dt\to\text{minimum}.$$

The parameters of the optimal quadrature formula can be obtained by identifying the function $K_2|_{[a_1,a_2]}$ with the Chebyshev orthogonal polynomial of the second kind of degree 2, on the interval $[a_1,a_2]$, with the coefficient of t^2 equal to $\frac{1}{2}$, namely

$$\frac{(1-t)^2}{2}+A_2t-a_2A_2=\frac{1}{2}t^2-A_1t+a_1A_1$$
$$=\frac{1}{2}\left(\frac{a_2-a_1}{2}\right)^2\tilde{U}_2\left(\frac{t-\frac{a_1+a_2}{2}}{\frac{a_2-a_1}{2}}\right),$$

where $\tilde{U}_2(x) = x^2 - \frac{1}{4}$ is the Chebyshev polynomial of the second kind of degree 2, on the interval $[-1,1]$. By identifying the coefficients we obtain

$$A_1 = \frac{a_1+a_2}{2}, A_1a_1 = \frac{1}{8}(a_1+a_2)^2 - \frac{1}{32}(a_2-a_1)^2.$$

Since $K_2 \in C[0,1]$, namely $K_2(a_1+0) = K_2(a_1-0)$ and $K_2(a_2-0) = K_2(a_2+0)$, we get the following relations

$$\frac{a_1^2}{2} = \frac{3(a_2-a_1)^2}{32}, \frac{(1-a_2)^2}{2} = \frac{3(a_2-a_1)^2}{32}.$$

From the above relations we obtain $A_1 = A_2 = \frac{1}{2}$, $a_1 = \frac{2\sqrt{3}-3}{2}, a_2 = \frac{5-2\sqrt{3}}{2}$.

*Remark.*For the remainder term of the quadrature formula (9) the following estimations can be established

$$\left|\mathscr{R}_2^{[\infty]}[f]\right| \le \|f''\|_\infty \int_0^1 |K_2(t)|dt = \frac{7-4\sqrt{3}}{8}\|f''\|_\infty$$
$$\approx 0.0089\|f''\|_\infty,\ f \in W_\infty^2[0,1],$$
$$\left|\mathscr{R}_2^{[\infty]}[f]\right| \le \left[\int_0^1 (K_2(t))^2\,dt\right]^{\frac{1}{2}} \|f''\|_2$$
$$= \frac{\sqrt{2400\sqrt{3}-4155}}{120}\|f''\|_2$$
$$\approx 0.0156\|f''\|_2,\ f \in W_2^2[0,1],$$
$$\left|\mathscr{R}_2^{[\infty]}[f]\right| \le \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{3(7-4\sqrt{3})}{8}\|f''\|_1$$
$$\approx 0.0269\|f''\|_1,\ f \in W_1^2[0,1].$$

Theorem 2.*For $f \in W_2^2[0,1]$, the quadrature formula of the form (5), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f\left(\frac{\sqrt{6}-2}{2}\right) + f\left(\frac{4-\sqrt{6}}{2}\right)\right] + \mathscr{R}_2^{[2]}[f], \tag{11}$$

with

$$\mathscr{R}_2^{[2]}[f] = \int_0^1 K_2(t)f''(t)dt, \tag{12}$$

where

$$K_2(t) = \begin{cases} \frac{1}{2}t^2,\ 0 \le t < \frac{\sqrt{6}-2}{2}, \\ \frac{(1-t)^2}{2} + \frac{1}{2}t - \frac{4-\sqrt{6}}{4},\ \frac{\sqrt{6}-2}{2} \le t \le \frac{4-\sqrt{6}}{2}, \\ \frac{1}{2}(1-t)^2,\ \frac{4-\sqrt{6}}{2} < t \le 1. \end{cases}$$

*Proof.*The remainder term (7) can be evaluated in the following way

$$\left|\mathscr{R}_2^{[2]}[f]\right| \le \left[\int_0^1 (K_2(t))^2\,dt\right]^{\frac{1}{2}} \|f''\|_2.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1 (K_2(t))^2\,dt \to \text{minimum}.$$

The parameters of the optimal quadrature formula can be obtained by identifying the function $K_2|_{[a_1,a_2]}$ with the Legendre orthogonal polynomial of degree 2, on the interval $[a_1,a_2]$, with the coefficient of t^2 equal to $\frac{1}{2}$, namely

$$\frac{(1-t)^2}{2} + A_2t - a_2A_2 = \frac{1}{2}\left(\frac{a_2-a_1}{2}\right)^2 \tilde{X}_2\left(\frac{t-\frac{a_1+a_2}{2}}{\frac{a_2-a_1}{2}}\right), \tag{13}$$

where $\tilde{X}_2(x) = x^2 - \frac{1}{3}$ is the Legendre orthogonal polynomial of degree 2, on the interval $[-1,1]$. By identifying the coefficients we obtain

$$A_1 = \frac{a_1+a_2}{2},\ A_1a_1 = \frac{1}{8}(a_1+a_2)^2 - \frac{1}{24}(a_2-a_1)^2.$$

Since $K_2 \in C[0,1]$, namely $K_2(a_1+0) = K_2(a_1-0)$ and $K_2(a_2-0) = K_2(a_2+0)$, we get the following relations

$$\frac{a_1^2}{2} = \frac{(a_2-a_1)^2}{12}, \frac{(1-a_2)^2}{2} = \frac{(a_2-a_1)^2}{12}.$$

From the above relations we obtain $A_1 = A_2 = \frac{1}{2}$, $a_1 = \frac{\sqrt{6}-2}{2}, a_2 = \frac{4-\sqrt{6}}{2}$.

*Remark.*For the remainder term of the quadrature formula (11) the following estimations can be established

$$\left|\mathscr{R}_2^{[2]}[f]\right| \le \left[\int_0^1 (K_2(t))^2\,dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{(5-2\sqrt{6})\sqrt{5}}{20}\|f''\|_2$$
$$\approx 0.0113\|f''\|_2,\ f \in W_2^2[0,1],$$
$$\left|\mathscr{R}_2^{[2]}[f]\right| \le \int_0^1 |K_2(t)|dt\|f''\|_\infty = \frac{(1+\sqrt{2})(9\sqrt{6}-22)}{12}\|f''\|_\infty$$
$$\approx 0.0091\|f''\|_\infty,\ f \in W_\infty^2[0,1],$$
$$\left|\mathscr{R}_2^{[2]}[f]\right| \le \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{5-2\sqrt{6}}{4}\|f''\|_1$$
$$\approx 0.0252\|f''\|_1,\ f \in W_1^2[0,1].$$

Theorem 3.*For $f \in W_1^2[0,1]$, the quadrature formula of the form (5), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f\left(\frac{\sqrt{2}-1}{2}\right) + f\left(\frac{3-\sqrt{2}}{2}\right)\right] + \mathscr{R}_2^{[1]}[f], \tag{14}$$

with

$$\mathscr{R}_2^{[1]}[f] = \int_0^1 K_2(t)f''(t)dt, \tag{15}$$

where

$$K_2(t)=\begin{cases}\frac{1}{2}t^2,\ 0\le t<\frac{\sqrt{2}-1}{2},\\ \frac{(1-t)^2}{2}+\frac{1}{2}t-\frac{3-\sqrt{2}}{4},\ \frac{\sqrt{2}-1}{2}\le t\le\frac{3-\sqrt{2}}{2},\\ \frac{1}{2}(1-t)^2,\ \frac{3-\sqrt{2}}{2}<t\le 1.\end{cases}$$

*Proof.*The remainder term (7) can be evaluated in the following way

$$\left|\mathscr{R}_2^{[1]}[f]\right|\le\|f''\|_1\cdot\sup_{0\le t\le 1}|K_2(t)|.$$

The quadrature formula is optimal with regard to the error if

$$\sup_{0\le t\le 1}|K_2(t)|\to\text{minimum}.$$

The parameters of the optimal quadrature formula can be obtained by identifying the function $K_2|_{[a_1,a_2]}$ with the Chebyshev orthogonal polynomial of the first kind of degree 2, on the interval $[a_1,a_2]$, with the coefficient of t^2 equal with $\frac{1}{2}$, namely

$$\frac{1}{2}t^2-A_1t+a_1A_1=\frac{1}{2}\left(\frac{a_2-a_1}{2}\right)^2\tilde{T}_2\left(\frac{t-\frac{a_1+a_2}{2}}{\frac{a_2-a_1}{2}}\right),\quad(16)$$

where $\tilde{T}_2(x)=x^2-\frac{1}{2}$ is the Chebyshev orthogonal polynomial of the first kind of degree 2, on the interval $[-1,1]$. By identifying the coefficients we obtain

$$A_1=\frac{a_1+a_2}{2},\ A_1a_1=\frac{1}{8}(a_1+a_2)^2-\frac{1}{16}(a_2-a_1)^2.$$

Since $K_2\in C[0,1]$, namely $K_2(a_1+0)=K_2(a_1-0)$ and $K_2(a_2-0)=K_2(a_2+0)$, we get the following relations

$$\frac{a_1^2}{2}=\frac{(a_2-a_1)^2}{16},\ \frac{(1-a_2)^2}{2}=\frac{(a_2-a_1)^2}{16}.$$

From the above relations we obtain $A_1=A_2=\frac{1}{2}$, $a_1=\frac{\sqrt{2}-1}{2}, a_2=\frac{3-\sqrt{2}}{2}$.

*Remark.*For the remainder term of the quadrature formula (14) the following estimations can be established

$$\left|\mathscr{R}_2^{[1]}[f]\right|\le\sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{3-2\sqrt{2}}{8}\|f''\|_1\approx 0.0214\|f''\|_1,\ f\in W_1^2[0,1].$$

$$\left|\mathscr{R}_2^{[1]}[f]\right|\le\int_0^1|K_2(t)|dt\cdot\|f''\|_\infty=\frac{32\sqrt{2}-45}{24}\|f''\|_\infty\approx 0.0106\|f''\|_\infty,\ f\in W_\infty^2[0,1],$$

$$\left|\mathscr{R}_2^{[1]}[f]\right|\le\left[\int_0^1(K_2(t))^2dt\right]^{\frac{1}{2}}\|f''\|_2=\frac{\sqrt{4245-3000\sqrt{2}}}{120}\|f''\|_2\approx 0.0128\|f''\|_2,\ f\in W_2^2[0,1].$$

3 The corrected quadrature formulas

In this section we will construct the corrected quadrature formulas of the optimal quadrature formulas in sense Nikolski and show that the estimations using different norms are better in the corrected formula than in the original one.

Let

$$\int_0^1 f(x)dx=A_1f(a_1)+A_2f(a_2)+A\left[f'(1)-f'(0)\right]+\tilde{\mathscr{R}}_2^{[p]}[f],\quad(17)$$

where

$$\tilde{\mathscr{R}}_2^{[p]}[e_i]=0,\ i=0,1,\ \text{and}\ A=\int_0^1K_2(t)dt$$

be the corrected quadrature formula of the rule (5).

Since the remainder term has degree of exactness 1, we can write

$$\tilde{\mathscr{R}}_2^{[p]}[f]=\int_0^1\tilde{K}_2(t)f''(t)dt,\ \text{where}\quad(18)$$

$$\tilde{K}_2(t)=\tilde{\mathscr{R}}_2^{[p]}[(x-t)_+]=K_2(t)-A.\quad(19)$$

From the relation (19) we remark that $\int_0^1\tilde{K}_2(t)dt=0$. If we consider $f(x)=\frac{x^2}{2}$ in the optimal quadrature (5), where $A_1=A_2=\frac{1}{2}$, we find

$$A=\frac{1}{2}a_1a_2-\frac{1}{12}.\quad(20)$$

Using relations (19) and (20) we construct the following corrected quadrature formula of (9), (11), respectively (14):

$$\int_0^1 f(x)dx=\frac{1}{2}\left[f\left(\frac{2\sqrt{3}-3}{2}\right)+f\left(\frac{5-2\sqrt{3}}{2}\right)\right]+\frac{48\sqrt{3}-83}{24}\left[f'(1)-f'(0)\right]+\tilde{\mathscr{R}}_2^{[\infty]}[f],\quad(21)$$

where

$$\tilde{\mathscr{R}}_2^{[\infty]}[f]=\int_0^1\tilde{K}_2(t)f''(t)dt,\quad(22)$$

$$\tilde{K}_2(t)=\begin{cases}\frac{1}{2}t^2-\frac{48\sqrt{3}-83}{24},\ 0\le t<\frac{2\sqrt{3}-3}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{65-36\sqrt{3}}{24},\ \frac{2\sqrt{3}-3}{2}\le t\le\frac{5-2\sqrt{3}}{2},\\ \frac{1}{2}(1-t)^2-\frac{48\sqrt{3}-83}{24},\ \frac{5-2\sqrt{3}}{2}<t\le 1,\end{cases}$$

$$\int_0^1 f(x)dx=\frac{1}{2}\Big[f\Big(\frac{\sqrt{6}-2}{2}\Big)+f\Big(\frac{4-\sqrt{6}}{2}\Big)\Big]+\frac{9\sqrt{6}-22}{12}\big[f'(1)-f'(0)\big]+\tilde{\mathscr{R}}_2^{[2]}[f], \quad (23)$$

where

$$\tilde{\mathscr{R}}_2^{[2]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt, \quad (24)$$

$$\tilde{K}_2(t)=\begin{cases}\frac{1}{2}t^2-\frac{9\sqrt{6}-22}{12}, & 0\le t<\frac{\sqrt{6}-2}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{8-3\sqrt{3}}{6}, & \frac{\sqrt{6}-2}{2}\le t\le\frac{4-\sqrt{6}}{2},\\ \frac{1}{2}(1-t)^2-\frac{9\sqrt{6}-22}{12}, & \frac{4-\sqrt{6}}{2}<t\le 1,\end{cases}$$

respectively

$$\int_0^1 f(x)dx=\frac{1}{2}\Big[f\Big(\frac{\sqrt{2}-1}{2}\Big)+f\Big(\frac{3-\sqrt{2}}{2}\Big)\Big]+\frac{12\sqrt{2}-17}{24}\big[f'(1)-f'(0)\big]+\tilde{\mathscr{R}}_2^{[1]}[f], \quad (25)$$

where

$$\tilde{\mathscr{R}}_2^{[1]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt, \quad (26)$$

$$\tilde{K}_2(t)=\begin{cases}\frac{1}{2}t^2-\frac{12\sqrt{2}-17}{24}, & 0\le t<\frac{\sqrt{2}-1}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{11-6\sqrt{2}}{24}, & \frac{\sqrt{2}-1}{2}\le t\le\frac{3-\sqrt{2}}{2},\\ \frac{1}{2}(1-t)^2-\frac{12\sqrt{2}-17}{24}, & \frac{3-\sqrt{2}}{2}<t\le 1.\end{cases}$$

*Remark.*For the remainder term of the quadrature formula (21) the following estimations can be established

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| &\le \|f''\|_\infty\int_0^1|K_2(t)|dt\\ &=\frac{1}{54}\Big[-62\sqrt{108\sqrt{3}-186}+108\sqrt{36\sqrt{3}-62}\\ &\quad+144\sqrt{48\sqrt{3}-83}-83\sqrt{144\sqrt{3}-249}\Big]\|f''\|_\infty\\ &\approx 0.0088\|f''\|_\infty,\ f\in W_\infty^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| &\le \left[\int_0^1 (K_2(t))^2dt\right]^{\frac{1}{2}}\|f''\|_2\\ &=\frac{\sqrt{50400\sqrt{3}-87295}}{60}\|f''\|_2\\ &\approx 0.006\|f''\|_2,\ f\in W_2^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| &\le \sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{73-42\sqrt{3}}{12}\|f''\|_1\\ &\approx 0.0211\|f''\|_1,\ f\in W_1^2[0,1].\end{aligned}$$

*Remark.*For the remainder term of quadrature formula (23) the following estimations can be established

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \left[\int_0^1 (K_2(t))^2dt\right]^{\frac{1}{2}}\|f''\|_2\\ &=\frac{\sqrt{9000\sqrt{6}-22045}}{60}\|f''\|_2\\ &\approx 0.0106\|f''\|_2,\ f\in W_2^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \int_0^1|K_2(t)|dt\|f''\|_\infty\\ &=\frac{1}{54}\Big[-29\sqrt{36\sqrt{6}-87}+36\sqrt{24\sqrt{6}-58}\\ &\quad+108\sqrt{9\sqrt{6}-22}-44\sqrt{54\sqrt{6}-132}\Big]\|f''\|_\infty\\ &\approx 0.0084\|f''\|_\infty,\ f\in W_\infty^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{37-15\sqrt{6}}{12}\|f''\|_1\\ &\approx 0.0214\|f''\|_1,\ f\in W_1^2[0,1].\end{aligned}$$

*Remark.*For the remainder term of the quadrature formula (25) the following estimations can be established

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| &\le \sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{13-9\sqrt{2}}{12}\|f''\|_1\\ &\approx 0.0226\|f''\|_1,\ f\in W_1^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| &\le \int_0^1|K_2(t)|dt\cdot\|f''\|_\infty\\ &=\frac{2(3-2\sqrt{2})\sqrt{9\sqrt{2}-12}}{27}\|f''\|_\infty\\ &\approx 0.0108\|f''\|_\infty,\ f\in W_\infty^2[0,1],\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| &\le \left[\int_0^1 (K_2(t))^2dt\right]^{\frac{1}{2}}\|f''\|_2\\ &=\frac{\sqrt{1800\sqrt{2}-2545}}{60}\|f''\|_2\\ &\approx 0.0097\|f''\|_2,\ f\in W_2^2[0,1].\end{aligned}$$

*Remark.*The estimates of the error in the corrected rules (21) and (23), respectively, are better then in the original rules (9) and (11), respectively.

The corrected quadrature formulas (21), (23) and (25), respectively, have degree of exactness 3, which is higher than the original rule, namely for $p\in\{\infty,2,1\}$, $\tilde{R}_2^{[p]}[e_i]=0$, $i=\overline{0,3}$ and $\tilde{R}_2^{[p]}[e_4]\neq 0$, where $e_i(x)=x^i$, $i=\overline{0,4}$. Using Peano's Theorem, the remainder term can be written

$$\tilde{\mathscr{R}}_2^{[p]}[f]=\int_0^1\overline{K}_2(t)f^{(4)}(t),\ \overline{K}_2(t)=\tilde{\mathscr{R}}_2^{[p]}\left[\frac{(x-t)_+^3}{3!}\right]. \quad (27)$$

In the next part of this paper, using relation (27), we will give new estimations of the remainder term in the quadrature formulas (21), (23), and (25), respectively.

Theorem 4. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (21) has the integral representation*

$$\tilde{\mathscr{R}}_2^{[\infty]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt, \text{ where}$$

$$\overline{K}_2^{[\infty]}(t) = \begin{cases} \frac{1}{24}t^2\left(t^2 - \frac{48\sqrt{3}-83}{2}\right), 0 \le t \le \frac{2\sqrt{3}-3}{2}, \\ \frac{1}{24}(1-t)^4 - \frac{1}{12}\left(\frac{5-2\sqrt{3}}{2} - t\right)^3 - \frac{48\sqrt{3}-83}{48}(1-t)^2, \\ \frac{2\sqrt{3}-3}{2} \le t \le \frac{5-2\sqrt{3}}{2}, \\ \frac{1}{24}(1-t)^2\left[(1-t)^2 - \frac{48\sqrt{3}-83}{2}\right], \frac{5-2\sqrt{3}}{2} < t \le 1. \end{cases}$$

and the following estimations hold

$$\begin{aligned} \left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| &\le \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 \left[f^{(4)}(t)\right]^2 dt} \\ &= \frac{\sqrt{2166615360\sqrt{3}-3752687855}}{40320}\|f^{(4)}\|_2 \\ &\approx 1.335 \times 10^{-4}\|f^{(4)}\|_2, \end{aligned}$$

$$\begin{aligned} &\left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| \le \int_0^1 \left|\overline{K}_2(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\ &= -\frac{9}{4}\sqrt{3} + \frac{22447}{5760} + \sqrt{-62 - \sqrt{8397-4848\sqrt{3}} + 36\sqrt{3}} \\ &\times \left(-\frac{9727}{720} + \frac{39}{5}\sqrt{3} + \sqrt{8397-4848\sqrt{3}}\left(\frac{\sqrt{3}}{20} - \frac{31}{360}\right)\right)\|f^{(4)}\|_\infty \\ &\approx 0.938 \times 10^{-4}\|f^{(4)}\|_\infty, \end{aligned}$$

$$\begin{aligned} \left|\tilde{\mathscr{R}}_2^{[\infty]}[f]\right| &\le \sup_{t\in[0,1]} |\overline{K}_2(t)| \cdot \int_0^1 |f^{(4)}(t)|dt \\ &= \frac{384\sqrt{3}-665}{384} \cdot \|f^{(4)}\|_1 \approx 2.7997 \times 10^{-4} \cdot \|f^{(4)}\|_1. \end{aligned}$$

Theorem 5. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (23) has the integral representation*

$$\tilde{\mathscr{R}}_2^{[2]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt, \text{ where}$$

$$\overline{K}_2(t) = \begin{cases} \frac{1}{24}t^2[t^2 - (9\sqrt{6}-22)], 0 \le t \le \frac{\sqrt{6}-2}{2}, \\ \frac{1}{24}(1-t)^4 - \frac{1}{12}\left(\frac{4-\sqrt{6}}{2} - t\right)^3 - \frac{9\sqrt{6}-22}{24}(1-t)^2, \\ \frac{\sqrt{6}-2}{2} \le t \le \frac{4-\sqrt{6}}{2}, \\ \frac{1}{24}(1-t)^2[(1-t)^2 - (9\sqrt{6}-22)], \frac{4-\sqrt{6}}{2} < t \le 1. \end{cases}$$

and the following estimations hold

$$\begin{aligned} \left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{27305005-11147220\sqrt{6}}}{10080}\|f^{(4)}\|_2 \\ &\approx 1.972 \times 10^{-4}\|f^{(4)}\|_2, \end{aligned}$$

$$\begin{aligned} \left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \int_0^1 \left|\overline{K}_2(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\ &= \frac{1}{1440}\left(64(485-198\sqrt{6})\sqrt{9\sqrt{6}-22}\right. \\ &\left. +630\sqrt{6}-1543\right) \cdot \|f^{(4)}\|_\infty \\ &\approx 1.337 \times 10^{-4} \cdot \|f^{(4)}\|_\infty, \\ \left|\tilde{\mathscr{R}}_2^{[2]}[f]\right| &\le \sup_{t\in[0,1]} |\overline{K}_2^{[2]}(t)| \cdot \int_0^1 |f^{(4)}(t)|dt \\ &= \frac{96\sqrt{6}-235}{384} \cdot \|f^{(4)}\|_1 \\ &\approx 3.993 \times 10^{-4} \cdot \|f^{(4)}\|_1. \end{aligned}$$

Theorem 6. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (25) has the integral representation*

$$\tilde{\mathscr{R}}_2^{[1]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt, \text{ where}$$

$$\overline{K}_2(t) = \begin{cases} \frac{1}{24}t^2\left(t^2 - \frac{12\sqrt{2}-17}{2}\right), 0 \le t \le \frac{\sqrt{2}-1}{2}, \\ \frac{1}{24}(1-t)^4 - \frac{1}{12}\left(\frac{3-\sqrt{2}}{2} - t\right)^3 - \frac{12\sqrt{2}-17}{48}(1-t)^2, \\ \frac{\sqrt{2}-1}{2} \le t \le \frac{3-\sqrt{2}}{2}, \\ \frac{1}{24}(1-t)^2\left[(1-t)^2 - \frac{12\sqrt{2}-17}{2}\right], \frac{3-\sqrt{2}}{2} < t \le 1, \end{cases}$$

and the following estimations hold

$$\begin{aligned} \left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| &\le \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{30974545-21902160\sqrt{2}}}{40320}\|f^{(4)}\|_2 \\ &\approx 3.622 \times 10^{-4}\|f^{(4)}\|_2, \\ \left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| &\le \int_0^1 \left|\overline{K}_2(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\ &= \frac{600\sqrt{2}-847}{5760} \cdot \|f^{(4)}\|_\infty \\ &\approx 2.653 \times 10^{-4} \cdot \|f^{(4)}\|_\infty, \end{aligned}$$

$$\left|\tilde{\mathscr{R}}_2^{[1]}[f]\right| \leq \sup_{t\in[0,1]} |\overline{K}_2(t)| \cdot \int_0^1 |f^{(4)}(t)|dt$$
$$= \left(-\frac{15}{128}+\frac{\sqrt{2}}{12}\right)\cdot \|f^{(4)}\|_1$$
$$\approx 6.636\times 10^{-4}\cdot \|f^{(4)}\|_1.$$

4 Conclusions

We constructed the corrected formulas of the optimal quadrature formulas in sense Nikolski. On the above discussions, the estimations using different norms are better in the corrected formula than in the original one. This aspect is revealed in the below table where are calculated the upper bounds of the absolute values of remainder term $\tilde{\mathscr{R}}_2^{[p]}$, $p \in \{2,\infty\}$ in corrected formula, respectively, $\mathscr{R}_2^{[p]}$ in the original one:

Table 1: The upper bounds of the remainder term

remainder term	$f \in W_\infty^2[0,1]$	$f \in W_2^2[0,1]$	$f \in W_1^2[0,1]$
$\tilde{\mathscr{R}}_2^{[\infty]}[f]$	$0.0088\|f''\|_\infty$	$0.0060\|f''\|_2$	$0.0211\|f''\|_1$
$\mathscr{R}_2^{[\infty]}[f]$	$0.0089\|f''\|_\infty$	$0.0156\|f''\|_2$	$0.0269\|f''\|_1$
$\tilde{\mathscr{R}}_2^{[2]}[f]$	$0.0084\|f''\|_\infty$	$0.0106\|f''\|_2$	$0.0214\|f''\|_1$
$\mathscr{R}_2^{[2]}[f]$	$0.0091\|f''\|_\infty$	$0.0113\|f''\|_2$	$0.0252\|f''\|_1$

Since the corrected quadrature formulas have degree of exactness higher than the original rule, this allowed us to obtain new error estimates in corrected formulas.

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