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Composition of Activation Functions and the Reduction to Finite Domain at Fractional and Fuzzy Framework

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Abstract: This work takes up the aim of the determination of the rate of fractional and fuzzy pointwise and uniform convergences to the unit operator of the ”normalized cusp fuzzy neural network operators”. The cusp is a compact support activation function, which derives by the composition of two general activation functions having as domain the whole real line. These convergences are given via the moduli of continuity of the engaged fuzzy right and left fractional derivatives or just the modulus of continuity of the function under approximation, in the form of Jackson type inequalities. The composition of activation functions aims to more flexible and powerful neural networks, introducing for the first time the reduction of infinite domains to the one domain of compact support.

Keywords: Neural network fuzzy fractional approximation, cusp activation function, fuzzy modulus of continuity, reduction of domain, fuzzy fractional derivatives.

1 Introduction

From AI and computer science we have the following: In essence, composing activation functions in neural networks offers the advantage of potentially tailoring the network's ability to learn and model complex, non-linear relationships in data. Here's a breakdown of the potential benefits:

1. Enhanced Capacity for Complex Modeling:

– Diversification of Non-linearity: Different activation functions have different characteristics. For example, ReLU introduces sparsity, while Sigmoid squashes values into a range. By composing them, the network potentially can learn a wider variety of non-linear transformations and capture more intricate patterns in the data.

2. Improved Training Dynamics:

– Mitigating Gradient Problems: Activation functions influence gradient flow during training. Using different activation functions can potentially help address issues like vanishing or exploding gradients, which hinder learning in deep networks.
– Faster Convergence: Certain activation functions, like ReLU, can accelerate the convergence of the training process compared to others like Sigmoid or Tanh. Combining different functions can potentially lead to faster training and competitive performance.

3. Enhanced Generalization and Robustness:

– Better Generalization: By learning richer representations of the data through diverse activation functions, the network's ability to generalize well to unseen data improves, reducing the risk of overfitting.

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–Increased Robustness: Networks with carefully chosen activation functions can handle variations in input data more effectively, adapting to noise , missing data, or unexpected perturbations.

4. Adaptation to Input Characteristics:

–Handling Diverse Data: Different activation functions can be suited to different data characteristics. For instance, tanh can be useful when dealing with data containing both positive and negative valued.

5. Potential for Architectural Interpretability:

–Insight into Learning: By using distinct activation functions, different parts of the network might become responsible for capturing specific features, which can potentially offer insights into how the model learns.

In summary, composing activation functions potentially allows for a more flexible and powerful neural network capable of:

–Learning more complex patterns.
–Faster and more stable training.
–Better generalization to new data.
–Greater adaptability to diverse data.

Attention: While composing activation functions can offer benefits, it's important to choose them judiciously and with consideration for the specific problem at hand, as some combinations might not be beneficial or could even lead to unwanted behaviors like exploding gradients. Empirical testing and validation are crucial when exploring different activation function compositions.

The author was greatly inspired and motivated by [14] and was the pioneer of quantitative neural network approximation, see [1], and since then he has published numerous of papers and books, e.g. see [12].

In this article we continue this trend at the fuzzy fractional level.

In mathematical neural network approximation AMS Mathscinet lists no articles related to composition of activation functions. So this is a seminal article.

By using composition of activation functions we achieve Section 3 with most notably this composition to lead to an activation function of compact support, though the initial activation functions had an infinite domain, the whole real line.

Now the resulting activation function is an open cusp of compact support $[-1,1]$. Our involved activation functions are very general, and the constructed fuzzy neural network operators resemble the squashing operators in [1], [12], and so do the produced quantitative results.

As a result our produced fuzzy-fractional convergence inequalities look much simpler and nicer.

2 Avout Fuzzy and Fractional Mathematical Analysis

We need the following basic background from [12], Ch. 11.

Definition 1.*(see [23]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathscr{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathscr{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathscr{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([23]).

For $u, v \in \mathbb{R}_{\mathscr{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall\, r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [23]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \;\; \lambda \odot u = u \odot \lambda.$$

If $0 \le r_1 \le r_2 \le 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \le u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define

$$D : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \;\; u, v \in \mathbb{R}_{\mathscr{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathscr{F}}$.

Then $(\mathbb{R}_{\mathscr{F}}, D)$ is a complete metric space, see [23], [24].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{0} : \chi_{\{0\}} \in \mathbb{R}_{\mathscr{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \;\; \forall\, u \in \mathbb{R}_{\mathscr{F}}.$$

Denote

$$D^*(f,g) := \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathscr{F}}$.

We mention

Definition 2. *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathscr{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathscr{F})}(f,\delta)_X = \sup_{x,y \in X,\; |x-y| \le \delta} D(f(x), f(y)), \;\; \delta > 0.$$

We define by $C_{\mathscr{F}}^U(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathscr{F}}$, also $C_{\mathscr{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathscr{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 1. *([4]) Let $f \in C_{\mathscr{F}}^U(X)$. Then $\omega_1^{(\mathscr{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.*

Proposition 2. *([4]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathscr{F})}(f,\delta)_X = \omega_1^{(\mathscr{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathscr{F}}^U(X)$.

Proposition 3. *([4]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathscr{F}}(\mathbb{R})$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 1. It is clear by Propositions 2, 3, that if $f \in C_{\mathscr{F}}^U(\mathbb{R})$, then $f_\pm^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$).

Proposition 4. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathscr{F}}$. Assume that $\omega_1^{\mathscr{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)}, \delta\right)_X$, $\omega_1\left(f_+^{(r)}, \delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathscr{F})}(f,\delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right)_X, \omega_1\left(f_+^{(r)}, \delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [8]. ■

We need

Remark.([2]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. Suppose that

$$\sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max \left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)} \right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right). \tag{1}$$

We need

Definition 3. *Let $x, y \in \mathbb{R}_{\mathscr{F}}$. If there exists $z \in \mathbb{R}_{\mathscr{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 4. *([22]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathscr{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathscr{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathscr{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathscr{F}}^N(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathscr{F}}$.

We mention

Theorem 1. *([20]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathscr{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_\pm^{(r)} = \left(f_\pm^{(r)} \right)', \quad \forall\, r \in [0,1].$$

Remark.([3]) Let $f \in C_{\mathscr{F}}^N(\mathbb{R})$, $N \geq 1$. Then by Theorem 1 we obtain

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)_\pm^{(r)} = \left(f_\pm^{(r)} \right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 2.([3]) Let $f \in C_{\mathscr{F}}^N(\mathbb{R})$, $N \geq 1$. Then by Theorem 1 we have $f_\pm^{(r)} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [23].

Definition 5.(*[18], p. 644) Let $f:[a,b]\to\mathbb{R}_{\mathscr{F}}$. We say that f is Fuzzy-Riemann integrable to $I\in\mathbb{R}_{\mathscr{F}}$ if for any $\varepsilon>0$, there exists $\delta>0$ such that for any division $P=\{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P)<\delta$, we have*

$$D\left(\sum_{P}^{*}(v-u)\odot f(\xi),I\right)<\varepsilon.$$

We write

$$I:=(FR)\int_a^b f(x)\,dx. \tag{3}$$

We mention

Theorem 2.(*[19]) Let $f:[a,b]\to\mathbb{R}_{\mathscr{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^b f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathscr{F}}$, furthermore it holds

$$\left[(FR)\int_a^b f(x)\,dx\right]^r=\left[\int_a^b (f)_-^{(r)}(x)\,dx,\int_a^b (f)_+^{(r)}(x)\,dx\right],$$

$\forall\, r\in[0,1]$.

For the definition of general fuzzy integral we follow [21] next.

Definition 6.*Let (Ω,Σ,μ) be a complete σ-finite measure space. We call $F:\Omega\to R_{\mathscr{F}}$ measurable iff $\forall$ closed $B\subseteq\mathbb{R}$ the function $F^{-1}(B):\Omega\to[0,1]$ defined by*

$$F^{-1}(B)(w):=\sup_{x\in B}F(w)(x),\ all\ w\in\Omega$$

is measurable, see [21].

Theorem 3.(*[21]) For $F:\Omega\to\mathbb{R}_{\mathscr{F}}$,*

$$F(w)=\left\{\left(F_-^{(r)}(w),F_+^{(r)}(w)\right)|0\le r\le 1\right\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\, r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [21], given that for each $r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_-^{(r)}d\mu,\int_A F_+^{(r)}d\mu\right)|0\le r\le 1\right\} \tag{4}$$

is a fuzzy real number for each $A\in\Sigma$.

The last fact leads to

Definition 7.(*[21]) A measurable function $F:\Omega\to\mathbb{R}_{\mathscr{F}}$,*

$$F(w)=\left\{\left(F_-^{(r)}(w),F_+^{(r)}(w)\right)|0\le r\le 1\right\}$$

is integrable if for each $r\in[0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) |0 \leq r \leq 1 \right\}.$$

By [21], F is integrable iff $w \rightarrow \|F(w)\|_{\mathscr{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathscr{F}} := D\left(u, \widetilde{0}\right), \ \forall\, u \in \mathbb{R}_{\mathscr{F}}.$$

We need also

Theorem 4._([21]) Let $F, G : \Omega \rightarrow \mathbb{R}_{\mathscr{F}}$ be integrable. Then_

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathscr{F}} \leq \int_A \|F\|_{\mathscr{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu \right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu \right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \ \ \forall\, r \in [0, 1].$$

We need

Definition 8._Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions), $\forall\, [a, b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [15], pp. 49-52) the function_

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x (x - t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall\, x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$, $\forall\, b > a$. We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a, \infty)$.

Lemma 1._([10]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall\, a \in \mathbb{R}$._

Definition 9._(see also [7], [16], [17]) Let $f : \mathbb{R} \rightarrow \mathbb{R}$, such that $f \in AC^m([a, b])$, $\forall\, [a, b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by_

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (J - x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{7}$$

$\forall\, x \leq b$. We set $D_{b-}^{0} f(x) = f(x)$, $\forall\, x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a, b])$ and $D_{b-}^{\alpha} f$ exists a.e. on $[a, b]$, $\forall\, a < b$.

Lemma 2._([10]) Let $f \in C^{m-1}(R)$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$, $\forall\, b \in \mathbb{R}$._

Convention 5 *We assume that*

$$D^{\alpha}_{*x_0} f(x) = 0, \; \text{for } x < x_0, \tag{8}$$

and

$$D^{\alpha}_{x_0-} f(x) = 0, \; \text{for } x > x_0, \tag{9}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 6. *(by [5]) Let* $f \in C^n(\mathbb{R})$, *where* $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D^{\nu}_{*a} f(x)$ *is continuous in* $x \in [a, \infty)$.

Also we have

Proposition 7. *(by [5]) Let* $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D^{\alpha}_{b-} f(x)$ *is continuous in* $x \in (-\infty, b]$.

We further mention

Proposition 8. *(by [5]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^{\alpha}_{*x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{10}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D^{\alpha}_{*x_0} f(x)$ *is continuous in* x_0.

Proposition 9. *(by [5]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^{\alpha}_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{11}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D^{\alpha}_{x_0-} f(x)$ *is continuous in* x_0.

Corollary 1. *([10]) Let* $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. *Then* $D^{\alpha}_{*x_0} f(x)$, $D^{\alpha}_{x_0-} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $\mathbb{R}^2$ *into* $\mathbb{R}$.

We need

Proposition 10. *([10]) Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be jointly continous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x,+\infty)}, \; \delta > 0, \, x \in \mathbb{R}. \tag{12}$$

(Here ω_1 *is defined over* $[x, +\infty)$ *instead of* $\mathbb{R}$*).*

Then G *is continuous on* $\mathbb{R}$.

Proposition 11. *([10]) Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be jointly continous. Consider*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{(-\infty,x]}, \; \delta > 0, \, x \in \mathbb{R}. \tag{13}$$

(Here ω_1 *is defined over* $(-\infty, x]$ *instead of* $\mathbb{R}$*).*

Then H *is continuous on* $\mathbb{R}$.

By Propositions 10, 11 and Corollary 1 we derive

Proposition 12. *([10]) Let* $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x \in \mathbb{R}$. *Then* $\omega_1 (D^{\alpha}_{*x} f, h)_{[x,+\infty)}$, $\omega_1 \left(D^{\alpha}_{x-} f, h \right)_{(-\infty,x]}$ *are continuous functions of* $x \in \mathbb{R}$, $h > 0$ *fixed.*

We make

*Remark.*Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then

$$\omega_1(g,t) \leq 2\|g\|_\infty < \infty. \tag{14}$$

Assuming that $(D^\alpha_{*x}f)(t)$, $(D^\alpha_{x-}f)(t)$, are both continuous and bounded in $(x,t) \in \mathbb{R}^2$, i.e.

$$\|D^\alpha_{*x}f\|_\infty \leq K_1, \ \forall\, x \in \mathbb{R}; \tag{15}$$

$$\left\|D^\alpha_{x-}f\right\|_\infty \leq K_2, \ \forall\, x \in \mathbb{R}; \tag{16}$$

where $K_1, K_2 > 0$, we get

$$\begin{gathered}\omega_1(D^\alpha_{*x}f,\xi)_{[x,+\infty)} \leq 2K_1; \\ \omega_1\left(D^\alpha_{x-}f,\xi\right)_{(-\infty,x]} \leq 2K_2, \ \forall\, \xi \geq 0,\end{gathered} \tag{17}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \geq 0$,

$$\sup_{x\in\mathbb{R}}\left[\max\left(\omega_1(D^\alpha_{*x}f,\xi)_{[x,+\infty)}, \omega_1\left(D^\alpha_{x-}f,\xi\right)_{(-\infty,x]}\right)\right] \leq 2\max(K_1,K_2) < \infty. \tag{18}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty < \infty$, $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, by Corollary 1 both $(D^\alpha_{*x}f)(t)$, $\left(D^\alpha_{x-}f\right)(t)$ are jointly continuous in (t,x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (18) valid. In particular, each of $\omega_1(D^\alpha_{*x}f,\xi)_{[x,+\infty)}$, $\omega_1\left(D^\alpha_{x-}f,\xi\right)_{(-\infty,x]}$ is finite for any $\xi \geq 0$.

Let us now assume only that $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x \in \mathbb{R}$. Then, by Proposition 15.114, p. 388 of [6], we find that $D^\alpha_{*x}f \in C([x,+\infty))$, and by [9] we obtain that $D^\alpha_{x-}f \in C((-\infty,x])$.

We make

*Remark.*Again let $f \in C^m(\mathbb{R})$, $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1$, $\forall$, $x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. Notice $0 < m-\alpha < 1$. Then

$$D^\alpha_{*x_0}f(x) = \frac{(x-x_0)^{m-\alpha}}{\Gamma(m-\alpha+1)}, \ \forall\, x \geq x_0. \tag{19}$$

Let us consider $x, y \geq x_0$, then

$$\begin{aligned}\left|D^\alpha_{*x_0}f(x) - D^\alpha_{*x_0}f(y)\right| &= \frac{1}{\Gamma(m-\alpha+1)}\left|(x-x_0)^{m-\alpha} - (y-x_0)^{m-\alpha}\right| \\ &\leq \frac{|x-y|^{m-\alpha}}{\Gamma(m-\alpha+1)}.\end{aligned} \tag{20}$$

So it is not strange to assume that

$$\left|D^\alpha_{*x_0}f(x_1) - D^\alpha_{*x_0}f(x_2)\right| \leq K|x_1-x_2|^\beta, \tag{21}$$

$K > 0$, $0 < \beta \leq 1$, $\forall\, x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, where more generally it is $\left\|f^{(m)}\right\|_\infty < \infty$. Thus, one may assume

$$\begin{gathered}\omega_1\left(D^\alpha_{x-}f,\xi\right)_{(-\infty,x]} \leq M_1\xi^{\beta_1}, \text{ and} \\ \omega_1(D^\alpha_{*x}f,\xi)_{[x,+\infty)} \leq M_2\xi^{\beta_2},\end{gathered} \tag{22}$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall\, \xi > 0$, $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\beta = \min(\beta_1,\beta_2)$ and $M = \max(M_1,M_2)$, in that case we obtain

$$\sup_{x\in\mathbb{R}}\left\{\max\left(\omega_1\left(D^\alpha_{x-}f,\xi\right)_{(-\infty,x]}, \omega_1(D^\alpha_{*x}f,\xi)_{[x,+\infty)}\right)\right\} \leq M\xi^\beta \to 0, \text{ as } \xi \to 0+. \tag{23}$$

We need

Definition 10.([11]) *Let $f \in C_{\mathscr{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.*
We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)\,dt, \ \ x \in [a,b], \tag{24}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\,dt, \ \ x \in [a,b], \tag{25}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 11.([11]) *Let $f : [a,b] \to \mathbb{R}_{\mathscr{F}}$ is called fuzzy absolutely continuous iff $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ for every finite, pairwise disjoint, family*

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \ \ \text{with} \ \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \varepsilon. \tag{26}$$

We denote the related space of functions by $AC_{\mathscr{F}}([a,b])$.
If $f \in AC_{\mathscr{F}}([a,b])$, then $f \in C_{\mathscr{F}}([a,b])$.

It holds

Proposition 13.([11]) *$f \in AC_{\mathscr{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 12.([11]) *We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*
Let $f \in C_{\mathscr{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu \mathscr{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\,dt$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\,dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\,dt \right) |0 \le r \le 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\,dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\,dt \right) |0 \le r \le 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu \mathscr{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\,dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\,dt \right) \right], \ \ 0 \le r \le 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathscr{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)} (t)\, dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [15], [6].

I.e. we get that

$$\left(D_{*a}^{\nu\mathscr{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{*a}^{\nu\mathscr{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \ \forall\, r \in [0,1]. \tag{30}$$

We need

Lemma 3.(*[11]*) *$D_{*a}^{\nu\mathscr{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 13.(*[11]*) *We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathscr{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$\begin{aligned} D_{b-}^{\nu\mathscr{F}} f(x) &:= \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt \\ &= \Bigg\{ \Bigg(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{-}^{(r)} (t)\, dt, \\ &\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{+}^{(r)} (t)\, dt \Bigg) \Big| 0 \le r \le 1 \Bigg\} \\ &= \Bigg\{ \Bigg(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)} (t)\, dt, \\ &\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)} (t)\, dt \Bigg) \Big| 0 \le r \le 1 \Bigg\}. \end{aligned} \tag{31}$$

We get

$$\begin{aligned} \left[D_{b-}^{\nu\mathscr{F}} f(x)\right]^r = \Bigg[\Bigg(&\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)} (t)\, dt, \\ &\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)} (t)\, dt \Bigg) \Bigg], \quad 0 \le r \le 1. \end{aligned}$$

That is

$$\left(D_{b-}^{\nu\mathscr{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)} (t)\, dt = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7].

I.e. we get that

$$\left(D_{b-}^{\nu\mathscr{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{32}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{b-}^{\nu\mathscr{F}} f\right)_{\pm}^{(r)} = D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right), \ \forall\, r \in [0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right) \le D_{b-}^{\nu}\left(f_{+}^{(r)}\right), \ \forall\, r \in [0,1].$$

We need

Lemma 4.(*[11]*) *$D_{b-}^{\nu\mathscr{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

3 About reduction of domain in AI ([13])

Let $i = 1,2$, and $h_i : \mathbb{R} \to [-1,1]$ be general sigmoid activation functions, such that they are strictly increasing, $h_i(0) = 0$, $h_i(-x) = -h_i(x)$, $x \in \mathbb{R}$, $h_i(+\infty) = 1$, $h_i(-\infty) = -1$. Also h_i is strictly convex over $(-\infty, 0]$ and stritly concave over $[0,+\infty)$, with $h_i^{(2)} \in C(\mathbb{R})$.

Clearly, $h_1 \circ h_2 = h_1|_{(-1,1)} \circ h_2$ is strictly increasing and $(h_1 \circ h_2)(0) = 0$, and

$$h_1 \circ h_2(-x) = h_1(h_2(-x)) = h_1(-h_2(x)) = -h_1(h_2(x)) = -(h_1 \circ h_2)(x),$$

that is

$$(h_1 \circ h_2)(-x) = -(h_1 \circ h_2)(x), \ \ \forall\, x \in \mathbb{R}.$$

Furthermore

$$(h_1 \circ h_2)(+\infty) = h_1(h_2(+\infty)) = h_1(1),$$
$$(h_1 \circ h_2)(-\infty) = h_1(h_2(-\infty)) = h_1(-1).$$

Next acting over $(-\infty, 0]$: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by convexity of h_2 there we have

$$h_2(\lambda x + \mu y) \leq \lambda h_2(x) + \mu h_2(y), \ \ x, y \in \mathbb{R};$$

and

$$h_1(h_2(\lambda x + \mu y)) \leq h_1(\lambda h_2(x) + \mu h_2(y)) \leq \lambda h_1(h_2(x)) + \mu h_1(h_2(y)),$$

i.e.

$$(h_1 \circ h_2)(\lambda x + \mu y) \leq \lambda (h_1 \circ h_2)(x) + \mu (h_1 \circ h_2)(y),$$

$x, y \in \mathbb{R}$.

So that $h_1 \circ h_2$ is convex over $(-\infty, 0]$.

Similarly, over $[0,+\infty)$ we get: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by concavity of h_2 there we have

$$h_2(\lambda x + \mu y) \geq \lambda h_2(x) + \mu h_2(y), \ \ x, y \in \mathbb{R};$$

and

$$h_1(h_2(\lambda x + \mu y)) \geq h_1(\lambda h_2(x) + \mu h_2(y)) \geq \lambda h_1(h_2(x)) + \mu h_1(h_2(y)).$$

Therefore $h_1 \circ h_2$ is concave over $[0,+\infty)$.

Also, it is

$$h_1(h_2(x))'' = h_1''(h_2(x))\left(h_2'(x)\right)^2 + h_1'(h_2(x))\, h_2''(x) \in C(\mathbb{R}), \ \ x \in \mathbb{R}.$$

So $h_1 \circ h_2$ is a sigmoid activation function generally speaking.

Next we consider the function

$$\psi_{1,2}(x) := \frac{1}{4}(h_1 \circ h_2(x+1) - h_1 \circ h_2(x-1)) > 0, \ \ \forall\, x \in \mathbb{R}.$$

We observe that

$$\psi_{1,2}(-x) = \frac{1}{4}(h_1(h_2(-x+1)) - h_1(h_2(-x-1))) =$$
$$\frac{1}{4}(h_1(h_2(-(x-1))) - h_1(h_2(-(x+1)))) =$$
$$\frac{1}{4}(h_1(-h_2(x-1)) - h_1(-h_2(x+1))) =$$
$$\frac{1}{4}(-h_1(h_2(x-1)) + h_1(h_2(x+1))) =$$
$$\frac{1}{4}(h_1 h_2(x+1) - h_1 h_2(x-1)) = \psi_{1,2}(x),$$

that is

$$\psi_{1,2}(-x) = \psi_{1,2}(x), \ \ \forall\, x \in \mathbb{R}.$$

So $\psi_{1,2}$ can serve as a density function in general.

So we have $h_2 : \mathbb{R} \to (-1,1)$, $h_1|_{(-1,1)} : (-1,1) \to (-1,1)$, and the strictly increasing function $H := h_1|_{(-1,1)} \circ h_2 : \mathbb{R} \to (-1,1)$, with the graph of H containing an arc of finite length, such that $H(0) = 0$, starting at $(-1, h_1(h_2(-1)))$ and terminating at $(1, h_1(h_2(1)))$.We call this arc also H. In particular H is negative and convex over $(-1,0]$, and it is positive and concave over $[0,1)$.

So it has compact support $[-1,1]$ and it is like a squashing function, see [12], Ch. 1, p. 8.

We will work from now on with $|H|$, which has as a graph a cusp joining the points $(-1, |h_1(h_2(-1))|)$, $(0,0)$, $(1, h_1(h_2(1)))$ and with compact support, again, $[-1,1]$. The points $(-1, |h_1(h_2(-1))|)$, $(1, h_1(h_2(1)))$ belong to the graph of $|H|$ and $(0,0)$ too.

Typically H has a steeper slope than of h_2, but it is flatter and closer to the x-axis than h_2 is, e.g. $\tanh(\tanh x)$ has asymptotes ± 0.76, while $\tanh x$ has asymptotes ± 1, notice that $\tanh(1) = 0.76$. Clearly H has applications in spiking neural networks.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ to be either continuous and bounded, or uniformly continuous.

The first modulus of continuity is given by

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0.$$

Here we have that $\omega_1(f,\delta) < +\infty$, $\delta > 0$.

In this article we study the pointwise and uniform convergences, with rates over the real line, to the unit operator, of the "normalized cusp neural network operators",

$$(A_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) \left|H\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|}{\sum_{k=-n^2}^{n^2} \left|H\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|}, \tag{34}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Notice A_n is a positive linear operator with $A_n(1) = 1$.

The terms in the ratio of sums (34) can be nonnegative and make sense, iff $0 < \left|n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \leq 1$, i.e. $0 < \left|x - \frac{k}{n}\right| \leq \frac{1}{n^{1-\alpha}}$, iff

$$nx - n^{\alpha} \leq x \leq nx + n^{\alpha}, \quad x \neq \frac{k}{n}. \tag{35}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - n^{\alpha} \leq nx + n^{\alpha} \leq n^2, \quad x \neq \frac{k}{n}, \tag{36}$$

it is sufficient to assume that

$$n \geq 1 + |x|, \quad x \neq \frac{k}{n}. \tag{37}$$

When $x \in [-1,1]$, $x \neq \frac{k}{n}$, it is enough to assume $n \geq 2$, which implies (36), and $x \neq \frac{k}{n}$.

But the unique case $x = \frac{k}{n}$ contributes nothing and can be ignored.

Thus, without loss of generality we can take always that $x \neq \frac{k}{n}$.

Proposition 14.([1]) *Let $a,b \in \mathbb{R}$, $a \leq b$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a,b]$. Then*

$$\max(0, (b-a) - 1) \leq card(k) \leq (b-a) + 1.$$

*Note 3.*We would like to establish a lower bound on $card(k)$ over the interval $[nx - n^{\alpha}, nx + n^{\alpha}]$. By Proposition 14 we get that

$$card(k) \geq \max(2n^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if $2n^{\alpha} - 1 \geq 1$ iff $n \geq 1$, which is always true.

So to have the desired order (36) and $card(k) \geq 1$ over $[nx - n^{\alpha}, nx + n^{\alpha}]$, it is enough to consider

$$n \geq \max(1 + |x|, 1) = 1 + |x|. \tag{38}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and $\lceil \cdot \rceil$ its ceiling.
Thus, it is clear that

$$(A_n(f))(x) := \frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} f\left(\frac{k}{n}\right) \left|H\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \left|H\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|}, \tag{39}$$

$0 < \alpha < 1$, and $n \in \mathbb{N} : n \geq 1 + |x|$, $x \in \mathbb{R}$.

We present the following fractional approximation results from [13].

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, for any $[a,b] \subset \mathbb{R}$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$. Let also $x \in \mathbb{R}$, $n \in \mathbb{N} : n \geq 1 + |x|$. We further assume that $D^{\beta}_{*x} f$, $D^{\beta}_{x-} f$ are uniformly continuous functions or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively. Then*

1)

$$|A_n(f)(x) - f(x)| \leq \left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j! n^{(1-\alpha)j}}\right) + \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{40}$$

$$\left\{\omega_1\left(D^{\beta}_{*x} f, \frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D^{\beta}_{x-} f, \frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above it is $\sum_{j=1}^{0} \cdot = 0$,

2)

$$\left|A_n(f)(x) - \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(A_n\left((\cdot - x)^j\right)\right)(x)\right| \leq \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{41}$$

$$\left\{\omega_1\left(D^{\beta}_{*x} f, \frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D^{\beta}_{x-} f, \frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\} =: \rho_n^*(x),$$

3) assume further that $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we obtain

$$|A_n(f)(x) - f(x)| \leq \rho_n^*(x), \tag{42}$$

4) in case of $N = 1$, we derive that

$$|A_n(f)(x) - f(x)| \leq \rho_n^*(x). \tag{43}$$

Here we get fractionally with rates the pointwise convergence of $A_n(f)(x) \to f(x)$, as $n \to \infty$, $x \in \mathbb{R}$.

Theorem 6. *([13]) Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$, $n \in \mathbb{N} : n \geq 2$. We further assume that $D^{\beta}_{*x} f(t)$, $D^{\beta}_{x-} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|A(f) - f\|_{\infty,[-1,1]} \leq \tag{44}$$

$$\left(\sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_{\infty,[-1,1]}}{j! n^{(1-\alpha)j}}\right) + \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{\sup_{x\in[-1,1]} \omega_1\left(D^{\beta}_{*x} f, \frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-1,1]} \omega_1\left(D^{\beta}_{x-} f, \frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) in case of $N = 1$, we obtain

$$\|A_n(f) - f\|_{\infty,[-1,1]} \leq \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{45}$$

$$\left\{ \sup_{x\in[-1,1]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-1,1]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Here we get fractionally with rates the uniform convergence of $A_n(f) \to f$, as $n \to \infty$.

Corollary 2.([13]) *Let $x \in [-\phi,\phi]$, $\phi > 0$, and $n \in \mathbb{N} : n \geq 1+\phi$, $0 < \alpha < 1$. Let also $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, $f^{(N)} \in L_\infty(\mathbb{R})$. Here both $D_{*x}^{\alpha} f$, $D_{x-}^{\alpha} f$ are bounded in $(x,t) \in \mathbb{R}^2$. Then*

$$|A_n(f)(x) - f(x)| \leq \left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j! n^{(1-\alpha)j}} \right) + \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{46}$$

$$\left\{ \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

Corollary 3.([13]) *Here $p \geq 1$ and all as in Corollary 2. Then*

1)

$$\|A_n f - f\|_{p,[-\phi,\phi]} \leq \left(\sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_{p,[-\phi,\phi]}}{j! n^{(1-\alpha)j}} \right) + 2^{\frac{1}{p}} \phi^{\frac{1}{p}} \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}, \tag{47}$$

2) when $N = 1$, we derive

$$\|A_n f - f\|_{p,[-\phi,\phi]} \leq 2^{\frac{1}{p}} \phi^{\frac{1}{p}} \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-\phi,\phi]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}. \tag{48}$$

By (47) and (48) we obtain the fractional L_p, $p \geq 1$, convergence with rates of $A_n f$ to f.

4 Fractional Approximation with rates of Fuzzy normalized cusp neural network operators

Based on (39) for $f \in C_{\mathscr{F}}(\mathbb{R})$ we define the corresponding fuzzy operators:

$$\left(A_n^{\mathscr{F}}(f)\right)(x) := \frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]*} f\left(\frac{k}{n}\right) \odot \left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}, \tag{49}$$

$0 < \alpha < 1$, $x \in \mathbb{R}$, and $n \geq 1+|x|$.

We notice that ($r \in [0,1]$)

$$\left[\left(A_n^{\mathscr{F}}(f)\right)(x)\right]^r = \frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \left[f\left(\frac{k}{n}\right)\right]^r \left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|} = \tag{50}$$

$$\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left[f_{-}^{(r)}\left(\frac{k}{n}\right),f_{+}^{(r)}\left(\frac{k}{n}\right)\right]\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}=$$

$$\left[\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}f_{-}^{(r)}\left(\frac{k}{n}\right)\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|},\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}f_{+}^{(r)}\left(\frac{k}{n}\right)\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}\right]= \tag{51}$$

$$\left[\left(A_n\left(f_{-}^{(r)}\right)\right)(x),\left(A_n\left(f_{+}^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(A_n^{\mathscr{F}}(f)\right)_{\pm}^{(r)}=A_n\left(f_{\pm}^{(r)}\right),\ \ \forall\, r\in[0,1], \tag{52}$$

respectively.

We give the following basic fuzzy result:

Theorem 7. *Let $x\in\mathbb{R}$, $n\in\mathbb{N}:n\geq 1+|x|$, $0<\alpha<1$, and $f\in C_{\mathscr{F}}^{b}(\mathbb{R})$ (fuzzy continuous and bounded functions). Then*

$$D\left(\left(A_n^{\mathscr{F}}(f)\right)(x),f(x)\right)\leq\omega_1^{(\mathscr{F})}\left(f,\frac{1}{n^{1-\alpha}}\right)_{\mathbb{R}}. \tag{53}$$

Notice that (53) gives $A_n^{\mathscr{F}}(f)\overset{D}{\to}f$, pointwise, as $n\to\infty$, when $f\in C_{\mathscr{F}}^{U}(\mathbb{R})$.

When $n\geq 2$, we obtain

$$D^{*}\left(A_n^{\mathscr{F}}(f),f\right)_{[-1,1]}\leq\omega_1^{(\mathscr{F})}\left(f,\frac{1}{n^{1-\alpha}}\right)_{\mathbb{R}}. \tag{54}$$

By (54), when $f\in C_{\mathscr{F}}^{U}(\mathbb{R})$, we get $A_n^{\mathscr{F}}(f)\overset{D^{*}}{\to}f$, as $n\to\infty$, uniformly over $[-1,1]$.

Proof. We observe that

$$D\left(\left(A_n^{\mathscr{F}}(f)\right)(x),f(x)\right)=\sup_{r\in[0,1]}\max\left\{\left|\left(A_n^{\mathscr{F}}(f)\right)_{-}^{(r)}(x)-f_{-}^{(r)}(x)\right|,\right.$$

$$\left.\left|\left(A_n^{\mathscr{F}}(f)\right)_{+}^{(r)}(x)-f_{+}^{(r)}(x)\right|\right\}\overset{(52)}{=} \tag{55}$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(A_n\left(f_{-}^{(r)}\right)\right)(x)-f_{-}^{(r)}(x)\right|,\left|\left(A_n\left(f_{+}^{(r)}\right)\right)(x)-f_{+}^{(r)}(x)\right|\right\}\overset{\text{(by [25])}}{\leq} \tag{56}$$

$$\sup_{r\in[0,1]}\max\left\{\omega_1\left(f_{-}^{(r)},\frac{1}{n^{1-\alpha}}\right)_{\mathbb{R}},\omega_1\left(f_{+}^{(r)},\frac{1}{n^{1-\alpha}}\right)_{\mathbb{R}}\right\}$$

(by Proposition 4)

$$=\omega_1^{(\mathscr{F})}\left(f,\frac{1}{n^{1-\alpha}}\right)_{\mathbb{R}}.$$

■

We continue with the following general result:

Theorem 8. *We consider $f:\mathbb{R}\to\mathbb{R}_{\mathscr{F}}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C_{\mathscr{F}}^{N}(\mathbb{R})$, with $D\left(f^{(N)}(\cdot),\widetilde{o}\right)\in L_{\infty}(\mathbb{R})$. Let also $x\in\mathbb{R}$, $n\in\mathbb{N}:n\geq 1+|x|$. We further assume that $D_{*x}^{\beta\mathscr{F}}f$, $D_{x-}^{\beta\mathscr{F}}f$ are fuzzy uniformly continuous functions or fuzzy continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$D\left(\left(A_n^{\mathscr{F}}(f)\right)(x), f(x)\right) \leq \sum_{j=1}^{N-1} \frac{1}{j! n^{(1-\alpha)j}} D\left(f^{(j)}(x), \widetilde{o}\right) + \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{57}$$

$$\left\{\omega_1^{(\mathscr{F})}\left(D_{*x}^{\beta\mathscr{F}} f, \frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1^{(\mathscr{F})}\left(D_{x-}^{\beta\mathscr{F}} f, \frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0} \cdot = 0,$

2) in case of $N=1$ *or* $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, *all* $j = 1, \ldots, N-1$, $N \geq 2$, *we get that*

$$D\left(\left(A_n^{\mathscr{F}}(f)\right)(x), f(x)\right) \leq \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{58}$$

$$\left\{\omega_1^{(\mathscr{F})}\left(D_{*x}^{\beta\mathscr{F}} f, \frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1^{(\mathscr{F})}\left(D_{x-}^{\beta\mathscr{F}} f, \frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with high rates the fuzzy pointwise convergence of $\left(A_n^{\mathscr{F}}(f)\right)(x) \xrightarrow{D} f(x)$, *as* $n \to \infty$, $x \in \mathbb{R}$.

Proof. We have

$$D\left(f^{(N)}(x), \widetilde{o}\right) = \sup_{r\in[0,1]} \max\left\{\left|(f_-^{(r)})^{(N)}(x)\right|, \left|(f_+^{(r)})^{(N)}(x)\right|\right\},$$

so that

$$\left|(f_\pm^{(r)})^{(N)}(x)\right| \leq D\left(f^{(N)}(x), \widetilde{o}\right), \ \forall\, r \in [0,1], \text{ any } x \in \mathbb{R}. \tag{59}$$

Thus

$$(f_\pm^{(r)})^{(N)} \in L_\infty(\mathbb{R}), \ \ \forall\, r \in [0,1].$$

Also we have $f_\pm^{(r)} \in C^N(\mathbb{R})$, hence $f_\pm^{(r)} \in AC^N([a,b])$, $\forall\, [a,b] \subset \mathbb{R}$; $\forall\, r \in [0,1]$.

By assumptions we get that $\left(D_{*x}^{\beta\mathscr{F}} f\right)_\pm^{(r)} \in C_U([x,+\infty))$ or in $C_b([x,+\infty))$ (bounded and continuous on $[x,+\infty)$ functions), also it holds $\left(D_{x-}^{\beta\mathscr{F}} f\right)_\pm^{(r)} \in C_U((-\infty,x])$ or in $C_b((-\infty,x])$, $\forall\, r \in [0,1]$.

By (30) we have

$$\left(D_{*x}^{\beta\mathscr{F}} f\right)_\pm^{(r)} = D_{*x}^{\beta}\left(f_\pm^{(r)}\right), \ \ \forall r \in [0,1]. \tag{60}$$

And by (33) we have that

$$\left(D_{x-}^{\beta\mathscr{F}} f\right)_\pm^{(r)} = D_{x-}^{\beta}\left(f_\pm^{(r)}\right), \ \ \forall r \in [0,1]. \tag{61}$$

Therefore all asumptions of Theorem 5 are fulfilled by each of $f_\pm^{(r)}$, $\forall r \in [0,1]$; thus (40) is valid for these functions.

We observe that

$$D\left(\left(A_n^{\mathscr{F}}(f)\right)(x), f(x)\right) = \sup_{r\in[0,1]} \max\left\{\left|\left(A_n^{\mathscr{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|,\right.$$

$$\left.\left|\left(A_n^{\mathscr{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} \overset{(52)}{=}$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(A_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(A_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \overset{(40)}{\leq} \tag{62}$$

$$\sup_{r\in[0,1]} \max\left\{\sum_{j=1}^{N-1} \frac{\left|\left(f_-^{(r)}\right)^{(j)}(x)\right|}{j! n^{(1-\alpha)j}} + \frac{1}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot\right.$$

$$\left\{\omega_1\left(D^{\beta}_{*x}\left(f^{(r)}_{-}\right),\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D^{\beta}_{x-}\left(f^{(r)}_{-}\right),\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

$$\sum_{j=1}^{N-1}\frac{\left|\left(f^{(r)}_{+}\right)^{(j)}(x)\right|}{j!n^{(1-\alpha)j}}+\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{63}$$

$$\left\{\omega_1\left(D^{\beta}_{*x}\left(f^{(r)}_{+}\right),\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D^{\beta}_{x-}\left(f^{(r)}_{+}\right),\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}=$$

(by Remark 2, (60), (61))

$$\sup_{r\in[0,1]}\max\left\{\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}\right)^{(r)}_{-}(x)\right|}{j!n^{(1-\alpha)j}}+\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot\right.$$

$$\left\{\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

$$\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}\right)^{(r)}_{+}(x)\right|}{j!n^{(1-\alpha)j}}+\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{64}$$

$$\left.\left\{\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}\overset{(1)}{\leq}$$

$$\sum_{j=1}^{N-1}\frac{1}{j!n^{(1-\alpha)j}}\sup_{r\in[0,1]}\max\left\{\left|\left(f^{(j)}\right)^{(r)}_{-}(x)\right|,\left|\left(f^{(j)}\right)^{(r)}_{+}(x)\right|\right\}+ \tag{65}$$

$$\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\right.$$

$$\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]},\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+$$

$$\left.\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\overset{\text{(by Definition 1, (1))}}{\leq}$$

$$\sum_{j=1}^{N-1}\frac{1}{j!n^{(1-\alpha)j}}D\left(f^{(j)}(x),\widetilde{o}\right)+\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)},\omega_1\left(\left(D^{\beta\mathscr{F}}_{*x}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right\}+\right.$$

$$\left.\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{-},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]},\omega_1\left(\left(D^{\beta\mathscr{F}}_{x-}f\right)^{(r)}_{+},\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}$$

$$\overset{\text{(by Proposition 4)}}{=}\sum_{j=1}^{N-1}\frac{1}{j!n^{(1-\alpha)j}}D\left(f^{(j)}(x),\widetilde{o}\right)+\frac{1}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1^{(\mathscr{F})}\left(\left(D^{\beta\mathscr{F}}_{*x}f\right),\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1^{(\mathscr{F})}\left(\left(D^{\beta\mathscr{F}}_{x-}f\right),\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}, \tag{66}$$

proving the claim. ■

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