Some linear isometric operators on the Dirichlet space

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1. Introduction

Let \(D\) be the unit disk in the complex plane \(C\), and \(S()\) be the set of holomorphic self-maps of \(D\). The algebra of all holomorphic functions on \(D\) will be denoted by \(H()\) and the set of all holomorphic automorphisms on \(D\) will be denoted by \(Aut()\). Let \(n\) be the unit poly-disk in \(C^n\), and \(S()\) be the set of all holomorphic self-maps of \(n\).

At the beginning, we introduce a generalized Dirichlet space with respect to \(\mu\) on the open unit disk. If \(\mu\) is a positive measure on \(D\), let the notation \(D_2(\mu)\) denote the space of holomorphic functions on \(D\) with

\[
\|f\|_\mu = \left( \int_D |f(z)|^2 d\mu(z) \right)^{1/2} < \infty,
\]

and we call it a Dirichlet space with respect to \(\mu\). It is clear that \(D_2(\mu)\) is a Hilbert space, and its inner product induced by its norm has the following formula:

\[
\langle f, g \rangle_\mu = f(0)g(0) + \int_D f(z)\overline{g(z)}d\mu(z).
\]

The Sobolev space of order 1 on the unit poly-disk is denoted by \(W^1_2\), and it is a Hilbert space which is the completed space of smooth functions \(f\) on \(D^n\) for which

\[
\|f\|_2^2 = \left( \int_{D^n} |f(z)|^2 dv \right)^2 + \sum_{i=1}^n \int_{D^n} \left( \left| \frac{\partial f}{\partial z_i} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}_i} \right|^2 \right) dv < \infty.
\]

Here, \(\frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right)\) and \(\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right)\).

Let \(\langle \cdot, \cdot \rangle\) denote the inner product of \(W^1_2\) induced by its norm as follow

\[
\langle f, g \rangle = \int_{D^n} f(z)g(z) dv + \sum_{i=1}^n \int_{D^n} \left( \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_i} + \frac{\partial f}{\partial \bar{z}_i} \frac{\partial g}{\partial \bar{z}_i} \right) dv.
\]

Recall that the Dirichlet space on the poly-disk \(D\) is a subspace of \(W^1_2\). Each point evaluation is easily verified to be a bounded linear functional on \(D\).

In this paper, the notation \(D_2\) is applied to denote the Dirichlet space in the unit disk. And the norm on \(D_2\) is as the following:

\[
\|f\| = \left( \int_D |f(z)|^2 dA(z) \right)^{1/2}.
\]

Then the inner product on \(D_2\) is defined as:

\[
\langle f, g \rangle = f(0)g(0) + \int f(z)\overline{g(z)} dA(z).
\]

For each \(z \in D\), there exists an unique reproducing kernel \(R_z \in D_2\) which has the reproducing property

\[
f(z) = \langle f, R_z \rangle
\]

for each \(f \in D_2\). It is well known that the formula of the reproducing kernel \(R_z\) is given by

\[
R_z(w) = 1 + \log \left( \frac{1}{1 - \bar{z}w} \right) \quad (w \in D).
\]

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Readers interested in the topic of holomorphic function spaces can refer to the books [16] and [17].

Let $P$ be the orthogonal projection from $W^2_2$ onto $D$, and

$$\Omega = \left\{ u \in C^1(D^n) : u, \frac{\partial u}{\partial z_i}, \frac{\partial u}{\partial \bar{z}_i} \in L^\infty(D^n, dv) \right\},$$

where $i = 1, \ldots, n$. Given $u \in \Omega$, the Toeplitz operator $T_u$ with symbol $u$ is a linear operator on $D$ which is defined by

$$T_u f = P(u f)$$

for functions $f \in D$. It is easy to show that the Toeplitz operator $T_u : D \to D$ is always bounded whenever $u \in \Omega$.

Let $\psi \in H(D)$ and $\varphi \in S(D)$, the composition operator $C_{\psi \varphi}$ induced by $\varphi$ is defined as $C_{\psi \varphi} f = f \circ \varphi$ for $f \in H(D)$; the multiplication operator induced by $\psi$ is defined by $M_{\psi} f(z) = \psi(z) f(z)$. The Volterra operator $J_\psi$ is defined by

$$J_\psi f(z) = \int_0^z f(\zeta) \psi'(\zeta) d\zeta,$$

and another integral operator, named co-Volterra operator, $I_\psi$ is defined by

$$I_\psi f(z) = \int_0^z f'(\zeta) \psi(\zeta) d\zeta$$

where $z \in D$ and $f \in H(D)$. The operator $J_\psi$ is actually a generalization of the integral operator (when $\psi(z) = z$). The operators $J_\psi$ and $I_\psi$ are close companions as a consequence of their relations to the multiplication operator $M_{\psi} f(z) = \psi(z) f(z)$. To see this, note that integration by parts gives

$$M_{\psi} f = f(0) \psi(0) + J_\psi f + I_\psi f.$$

A linear operator $T$ on a normed space $X$ is said to be an isometric linear operator if $\|T f\|_X = \|f\|_X$ for any $f \in X$. Let $X$ and $Y$ be two Banach spaces and $T_1$ and $T_2$ are bounded linear operators on $X$ and $Y$ respectively. We say that $T_1$ and $T_2$ are isometrically equivalent if there exists surjective isometries $U_X$ and $U_Y$ on $X$ and $Y$ respectively such that $U_X T_1 = T_2 U_Y$. For $Y = X$, two operators $T_1$ and $T_2$ are said to be similar if there is a bounded invertible operator $S$ on $X$ such that $S T_2 = T_1 S$.

If $S$ could be chosen to be an isometry as well, then $T_1$ and $T_2$ are said to be isometrically isomorphic.

The isometric composition operator on analytic function spaces has been studied by many authors. In [12,7], the authors characterized the isometric composition operator on several classical reflexive spaces just like Hardy space, weighted Bergman space and analytic Besov spaces, respectively. The isometric composition operators on the Bloch spaces both in the unit disk, unit poly-disk and unit ball were discussed in [14,3,1,2,11,4,9,10]. Recently, isometric problems on the spaces of BMOA and Dirichlet spaces are concerned in [8,13].

Isometric equivalence of composition operators on Hardy space and several Banach spaces of analytic functions spaces are investigated in [15,5] and [6], resp.

Base on these foundations, we firstly characterize isometric composition operators on generalized Dirichlet spaces in the unit disk. Secondly, we study isometric composition and Volterra type operators on classic Dirichlet space in the unit disk. Then, we investigate the isometric equivalence of Toeplitz operators on classic Dirichlet space. At the end of the paper, isometric composition operators on Dirichlet space in polydisk are also studied to some degree.

2. Isometry on generalized Dirichlet space

Let $\nu_0$ be a positive Borel measure on the unit interval $[0,1]$, another positive measure $\nu$ on $\mathcal{L}$ is called a radial measure with respect to $\nu_0$ if $d\nu = d\nu_0(r) dr/2\pi$ where $d\theta$ represents the Lebesgue measure of $\partial$. Suppose that $\nu_0$ is a positive Borel measure on $[0,1]$, a real sequence $\{a_n\}$ is called a moment sequence of $\nu_0$ if

$$a_n = \int_0^1 r^n d\nu_0$$

for $n = 0, 1, 2, \ldots$

Let $\nu_0$ be a positive Borel measure on $[0,1]$ and $\varphi \in S(1)$ with $\|\varphi\|_\infty = 1$. If positive measure $\nu$ is a radial measure with respect to $\nu_0$, and $\mu$ is another positive measure on $\mathcal{L}$, then the composition operator $C_\varphi$ is isometric from $D_2(\nu)$ into $D_2(\mu)$ if and only if

$$\int_D |\phi^n|^2 \frac{\phi'^m}{\nu} d\mu = 0 \quad \forall n \neq m,$$

and the sequence $\{\int_D |\phi|^n |\phi'|^2 d\mu\}$ is a moment sequence of $\nu_0$.

Proof. Firstly, assume that $C_\varphi$ is an isometric composition operator from $D_2(\nu_0)$ into $D_2(\mu)$, then

$$(z^n, z^m)_\nu = (C_\varphi z^n, C_\varphi z^m)_\mu.$$  (2)

Putting $n = 1$ and $m = 0$ in equation 1, we have $\phi(0) = 0$. Hence

$$\int_D (z^n)(z^m)' d\nu = \int_D (\phi^n)(\phi^m)' d\mu.$$

By the polarization formula, we have the following two items:

1. $\int_D |\phi'|^2 d\mu = \int_0^1 \nu_0(r)$ when $n = m = 1$
2. $\int_D \phi^n d\mu = \int_0^1 \nu_0(r)$ when $n = m = 1$

$$nm\delta_{(n-1)(m-1)}\int_0^1 r^{(n-1)} r^{(m-1)} d\nu_0(r)$$

$$= \int_D (z^n)(z^m)' d\nu = \int_D (\phi^n)(\phi^m)' d\mu$$

when $n > 1$ and $m > 1$. 

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Thus, for any $n = 1, 2, 3, \ldots$,

$$\int_D (\phi^n)'^2 d\mu = n^2 \int_0^1 r^{2(n-1)} dv_0 \quad (n = 1, 2, 3, \ldots),$$

that is,

$$\int_D |\phi^n| |\phi'|^2 d\mu = \int_0^1 r^{2(n-1)} dv_0 \quad (n = 1, 2, 3, \ldots).$$

For $0 \leq x \leq 1$, we have

$$x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n$$

and

$$\sum_{n=0}^{\infty} |a_n| (1 - x^2)^n < \infty \quad (0 \leq x \leq 1).$$

Since

$$\left| \sum_{n=0}^{k} a_n (1 - |\phi|^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n|$$

and

$$\left| \sum_{n=0}^{k} a_n (1 - r^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n| < \infty,$$

by the dominated convergence theorem, we have

$$\int_D |\phi| |\phi'|^2 d\mu = \int_D \sum_{n=0}^{\infty} a_n (1 - |\phi|^2) |\phi'|^2 d\mu$$

$$= \sum_{n=0}^{\infty} a_n \int_D (1 - |\phi|^2) |\phi'|^2 d\mu$$

$$= \sum_{n=0}^{\infty} a_n \int_D (1 - r^2)^n dv_0$$

$$= \int_0^1 \sum_{n=0}^{\infty} a_n (1 - r^2)^n dv_0 = \int_0^1 r dv_0.$$ 

Similarly, because $x^{2l+1} = \sqrt{1 - (1 - x^{2l+2})}$, we can get

$$\int_D |\phi^{2l+1}| |\phi'|^2 d\mu = \int_0^1 r^{2n+1} dv_0 \quad (n = 0, 1, 2, \ldots).$$

Thus

$$\left\{ \int_D |\phi^n| |\phi'|^2 d\mu \right\}$$

is a moment sequence of $r_0$. Conversely, if $\int_D \phi^n \bar{\phi}_m |\phi'|^2 d\mu = 0$ for all $m$, and

$$\left\{ \int_D |\phi^n| |\phi'|^2 d\mu \right\}$$

is a moment sequence of $r_0$, then

$$\int_D \left| \sum_{n=0}^{k} a_n \phi^n \right|^2 |\phi'|^2 d\mu$$

$$= \sum_{n=0}^{k} |a_n|^2 \int_D |\phi^n|^2 |\phi'|^2 d\mu$$

$$= \sum_{n=0}^{k} |a_n|^2 \int_0^1 r^{2n} dv_0 = \int_D \sum_{n=0}^{k} a_n z^n d\nu,$$

that is, the operator $C_\phi$ is an isometric operator.

### 3. Isometric operators on $D_2$

In this section, our discussion starts from the isometric property for the product of composition and Volterra type operators.

Let $\phi \in S()$ and $\psi \in H()$. Then neither the products of composition and Volterra type operators $C_\phi I_\psi$ and $C_\psi I_\phi$, nor the products of Volterra type and composition operators $I_\phi C_\psi$ and $I_\psi C_\phi$ can be isometric on Dirichlet space $D_2$.

**Proof.** We here just prove the front half of the theorem, and it is similar for the back half. Note that the Dirichlet norm of $f \in D_2$ has the following formula:

$$\|f\|_{D_2}^2 = |f(0)|^2 + \int f'(z)|^2 dA(z).$$

(3)

We assume the operator $C_\phi I_\psi$ to be isometric on $D_2$. And making $C_\phi I_\psi$ acting on a holomorphic function $f \in D_2$, we have

$$C_\phi I_\psi (f)(z) = \int_0^\phi (\zeta) \psi(\zeta) d\zeta.$$ 

Next we take the Dirichlet norm of the operator $C_\phi I_\psi (f)$.

$$\|C_\phi I_\psi f\|_{D_2}^2 = \left| \int_0^\phi (\zeta) \psi(\zeta) d\zeta \right|^2$$

$$+ \int |f'(\zeta)| |\psi'(\zeta)|^2 d\zeta.$$ 

(4)

The formulas V1 and V2 are equal by the assumption on isometry of $C_\phi I_\psi$. By putting $f \equiv 1$ in both V1 and V2, we have $V1 = 1$ and $V2 = 0$. That is a contradiction, so the operator $C_\phi I_\psi$ cannot be isometric.

Now assume that $C_\phi J_\psi$ is isometric on $D_2$. By making $C_\phi J_\psi$ acting on $f \in D_2$, we obtain

$$C_\phi J_\psi (f)(z) = \int_0^\phi (\zeta) \psi'(\zeta) d\zeta,$$

and then take its Dirichlet norm,

$$\|C_\phi J_\psi f\|_{D_2}^2 = \left| \int_0^\phi (\zeta) \psi'(\zeta) d\zeta \right|^2$$

$$+ \int |f'(\zeta)| |\psi'(\zeta)|^2 d\zeta.$$ 

(5)

Similarly, formulas V1 and V3 are equal from the isometric assumption of $C_\phi J_\psi$. Putting $f \equiv 1$ in both V3 and V1, we have

$$\int_0^\phi \psi'(\zeta) d\zeta$$

$$+ \int |\psi'(\zeta)|^2 d\zeta = 1.$$ 

(6)
Let \( f(z) = z^n \) for \( n = 1, 2, \ldots \) and \( z \in \) both V3 and V1, then
\[
\begin{align*}
\int_0^\infty (\zeta^n |\zeta'|^2) d\zeta \\
+ \int |\zeta^n| |\zeta'|^2 dA(z)
\end{align*}
\]
\( = 0 + \int_{n^2}^{n^2-1} dA(z) \)
\( = n^2 \int_0^{2\pi} d\theta \int_0^1 r^{2(n-1)} r dr = n\pi \to \infty, \)
as \( n \to \infty \). But from the formula of V5, it is obvious that
\[
\int_0^\infty (\zeta^n |\zeta'|^2) d\zeta + \int |\zeta^n| |\zeta'|^2 dA(z)
\leq \left( \int_0^\infty (\zeta^n |\zeta'|^2) d\zeta - \int_0^\infty (\zeta^n |\zeta'|^2) d\zeta \right) \\
+ \int_0^\infty |\zeta^n| |\zeta'|^2 dA(z)
\leq 1
\rightarrow \infty,
\]
Hence \( \int_0^\infty (\zeta^n |\zeta'|^2) d\zeta \to \infty \) as \( n \to \infty \). Considering that the line integral is independent on the integration path, we assume that the line integral between 0 and \( \varphi(0) \) is taken along the straight complex line from 0 to \( \varphi(0) \), and here we denote this straight line by \( [0, \varphi(0)] \) and the notation \( L[0, \varphi(0)] \) as its Lebesque length. So
\[
\int_0^\infty (\zeta^n |\zeta'|^2) d\zeta
\leq \sup_{z \in D_\varphi(0)} |\zeta^n| |\varphi(z)| L[0, \varphi(0)] < \infty,
\]
where \( D_\varphi \) represents the closed disc with the origin as its center and \( r \) as its radius. Hence this contradiction implies that the isometric assumption on \( C_\varphi J_\psi \) on \( D_2 \) must be false.

The proof of last two negative isometric assertion is left to the interested readers.

In the remaining text of this section, we will look into the properties of quasi-isometric composition operator. A bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \) is called a quasi-isometry if \( T^*T = T^*T \). We will characterize the quasi-isometric composition operator on \( D_2 \) in the following theorem. Let \( \varphi \in S() \), then \( C_\varphi \) is a quasi-isometry on \( D_2 \) if and only if \( \varphi \) is a rotation.

**Proof:** Since the composition operator induced by a rotation is clearly a quasi-isometry, it suffices to show that if \( C_\varphi \) is a quasi-isometry, then \( \varphi \) is a rotation.

Suppose that the composition operator \( C_\varphi \) is quasi-isometry. By the definition of quasi-isometry, we have
\[
\langle C_\varphi ^2 f, g \rangle_{D_2} = \langle C_\varphi f, C_\varphi g \rangle_{D_2}
\]
for \( \forall f, g \in D_2 \). That is
\[
\langle C_\varphi ^2 f, C_\varphi ^2 g \rangle_{D_2} = \langle C_\varphi f, C_\varphi g \rangle_{D_2}
\]
(7)

If we put \( f = id, g \equiv 1 \), then \( \varphi(\varphi(0)) = \varphi(0) \), that is \( \varphi(0) = 0 \).

Put \( f(z) = g(z) = z^n \) for \( \forall z \) in \( D_2 \), then
\[
\langle \varphi^n \circ \varphi, \varphi^n \circ \varphi \rangle_{D_2} = \langle \varphi^n, \varphi^n \rangle_{D_2},
\]
From the definition of inner product on \( D_2 \), we get
\[
\int_D |\varphi^n \circ \varphi|^2 |\varphi'|^2 dA(z) = \int_D |\varphi|^2 |\varphi'|^2 dA(z)
\]
that is
\[
\int_D (|\varphi^n \circ \varphi|^2 - |\varphi^n|^2) |\varphi'|^2 dA(z) = 0.
\]
From Schwarz lemma, we obtain that
\[
|\varphi^n \circ \varphi|^2 - |\varphi^n|^2 = 0.
\]
(8)

Noting that \( \varphi \in S() \) and \( \varphi(0) = 0 \), suppose \( \sum_{n=1}^{\infty} a_n z^n \) is the Taylor expansion of \( \varphi(z) \), and we put it in equation 2, then \( a_1 = \lambda, |\lambda| = 1, a_n = 0 \) for \( n \geq 2 \), that is \( \varphi \) is a rotation.

If \( \varphi \) is a rotation of the open unit disk, then \( C_\varphi \) is an isometry. Hence if \( C_\varphi \) is a quasi-isometry on \( D_2 \), then \( C_\varphi \) is an isometry on \( D_2 \).

### 4. Isometric equivalence of Toeplitz operators

In this section, we will study the isometric equivalence of two Toeplitz operators on the Dirichlet space of the disk. Let \( f, g \in W^1_0() \), if two Toeplitz operators \( T_f \) and \( T_g \) are isometric equivalence on \( D_2 \), then there exists a constant \( \gamma \) with modular 1 and an automorphism \( \sigma \) of the unit disk such that
\[
f(\sigma(z)) - f(\sigma(0)) = \gamma(g(z) - g(0)).
\]
Suppose $\lambda$ and $\kappa$ are two complex numbers with modular 1 and $\sigma$ is an automorphism of the disk, we define an integral operator $Q$ on $h \in W_1^2()$ as the following:

$$Qh(z) = \lambda h(0) + \kappa \int_0^z h'(\sigma(\xi))\sigma'(\xi) d\xi$$

for any $h \in W_1^2()$. It is easy to see that $Q$ is a surjective isometry on $W_1^2()$ and $QD_2 = D_2$. Suppose $P$ is the Bergman projection from $W_1^2()$ to $D_2$, then we have

$$P(h - Ph) = Ph - PPh = Ph - Ph = 0$$

for any $h \in W_1^2()$. Hence

$$PQh = PQ(h - Ph) + PQh = PQPh = PQh.$$ Then it follows from the above equation that $PQ = PQ$.

For $h \in W_1^2()$, we have

$$QT_1h = PQ(fh) = P(Qf(0) + \kappa Z\int_0^z (f'h + h'f)(\sigma(\xi))\sigma'(\xi) d\xi / \Sigma_i \sum_{n} \int_{D^n} |\partial f / \partial w_i|^2 d\nu.$$ \section{5. Isometric composition operators on $\mathcal{D}$}

Recall that a self-map $\varphi$ of is said to be a univalent full map if it is one-to-one and $A[\varphi()] = 0$. In the following, we will characterize the isometric composition operator on Dirichlet space on the polydisk. Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{D}^n$ and $\tau$ be a permutation of $\{1, 2, \ldots, n\}$. For $j = 1, 2, \ldots, n$, suppose that $\varphi_j \in S()$, $\varphi_j(0) = 0$ and all $\varphi_j$’s are univalent full maps on $\mathbb{R}$. Denote by $\varphi(z) = (\varphi_1(z_{\tau(1)}), \ldots, \varphi_n(z_{\tau(n)})$. Then $C_\varphi$ is isometric composition operator on $\mathcal{D}$.

**Proof.** For simplicity, we shall assume that $\varphi_j(0) = 0$ for each $j = 1, 2, \ldots, n$. Let us set aside the trivial case for each $\varphi_j \in Aut()$. Then

$$f(\varphi) = f(\varphi_1(z_1), \ldots, \varphi_n(z_n))$$

for each $f \in \mathcal{D}$, where $z_i \in D$ with $i = 1, 2, \ldots, n$.

From the definition of the Dirichlet norm and the fact that $\varphi_j(0) = 0$ for $j = 1, 2, \ldots, n$, we have:

$$\|C_\varphi f\|^2 = |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial \varphi_i} \right|^2 d\nu.$$ \begin{equation}
\|C_\varphi f\|^2 = |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial \varphi_i} \right|^2 n_{\varphi}, \nu d
\end{equation}

However $\varphi_j$ is univalent self map, $n_{\varphi_j} = 1$ for each $j = 1, 2, \ldots, n$, so

$$\|C_\varphi f\|^2 = |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial \varphi_i} \right|^2 d\nu.$$ that is $\|C_\varphi f\| = \|f\|$, then $C_\varphi$ is an isometry on $\mathcal{D}$.

From the expression of inner product on $\mathcal{D}$, we can easily get the following proposition: For $\varphi \in S(\mathbb{D})$, if the composition operator $C_\varphi$ is isometric on $\mathcal{D}$, then $\varphi(0) = 0$.

The converse of Theorem 5 is still an open question.

**References**


