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Characterization for Gompertz Distribution Based on Progressive First Failure Censoring

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Abstract: In this article, I establish recurrence relations for moment generating function and moments based on progressive first failure censoring. Characterization for Gompertz distribution using recurrence relations of moment generating function and single and product moments of progressive first failure censoring are obtained. Further, the results are specialized to the progressively type-II right censored (PTIIRCOS).

Keywords: Characterization; Gompertz Distribution; progressive First Failure Censoring; Recurrence Relations.

1 Introduction

Suppose that n independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}$ has occurred and finally when the m^{th} failure $X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ is observed, the remaining groups R_m are removed from the test. Then $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)} < \cdots < X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ are called progressively first-failure censored order statistics with the progressive censored scheme, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with cdf and pdf, the joint pdf for $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ is defined as follows:

$$f_{X_{1:m:n},\dots,X_{m:m:n}}(x_1,x_2,x_3,\dots,x_{m-1},x_m) = K_{(n,m-1)}k^m \prod_{i=1}^{m} f(x_i)[\bar{F}(x_i)]^{kR_i+k-1}, 0 < x_1 < x_2 < x_3 < \dots < x_{m-1} < x_m < \infty, \quad (1)$$

where,
$K_{(n,m-1)} = n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-R_2-R_3\cdots-R_{m-1}-R_m-m+1)$.

Aggarwala and Balakrishnan [1] derived recurrence relations (RR) for single and product moments of PTIIRCOS from exponential distribution. Mohie El-Din et al. [6,7,8,9,?] derived RR of moments of the extended power Lindley, generalized Pareto, Gompertz, linear failure rate and Marshall-Olkin extended Burr XII distributions based on general PTIIRCOS and characterization. Sadek et al. [10] derived characterization for generalized power function distribution using RR based on general PTIIRCOS. Marwa Mohie El-Din and Sharawy [?,5] derived RR for the extension of exponential and generalized exponential distributions based on general PTIIRCOS. Kotb et al. [4] derived E-Bayesian estimation for Kumaraswamy distribution using progressive first failure censoring (PFFC).

Through out this paper, I introduce RR of moment generating function (MGF) based on PFFC. Also characterization for exponential distribution using RR of MGF based on PFFC, are obtained.

I introduce the RR of MGF for Gompertz distribution (GD) based on PFFC. The pdf is

$$f(x,\alpha,\beta) = \alpha e^{\beta x - \frac{\alpha}{\beta}(e^{\beta x}-1)}, \quad \alpha > 0, \beta > 0 \quad x \geq 0 \quad (2)$$

and the cdf given by

$$f(x,\alpha,\beta) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}, \quad \alpha > 0, \beta > 0 \quad x \geq 0. \quad (3)$$

The GD was introduced by Benjamin Gompertz [2].

It may be noticed that from (2) and (3) the relation between pdf and cdf is given by,

$$f(x) = \alpha e^{\beta x}\bar{F}(x). \quad (4)$$

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For any continuous distribution, we shall denote the i^{th} single moment of the PFFC in view of (1) as

$$\mu_{q:m:n,k}^{(kR_1+k-1,kR_2+k-1,\ldots,kR_m+k-1)^{(i)}} = E[X_{q:m:n,k}^{(kR_1+k-1,kR_2+k-1,\ldots,kR_m+k-1)}]^i = K_{(n,m-1)}\int\int\cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i k^m f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\times f(x_2) [\bar{F}(x_2)]^{kR_2+k-1}\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_m, \quad (5)$$

the i^{th} and j^{th} product moments as

$$\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} = E[X_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^i}X_{s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^r}] = K_{(n,m-1)}\int\int\cdots\int_{0<x_1<\cdots<x_m<\infty} x_q^i x_s^j k^m f(x_1) [\bar{F}(x_1)]^{kR_1+k-1}\times f(x_2)[\bar{F}(x_2)]^{kR_2+k-1}\ldots f(x_m) [\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_m, \quad (6)$$

the single MGF of the PFFC in view of (1) as

$$M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = E[e^{tx_q}X_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}] = K_{(n,m-1)}\int\int\cdots \int_{0<x_1<\cdots<x_m<\infty} e^{tx_q}k^m f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\times f(x_2) [\bar{F}(x_2)]^{kR_2+k-1}\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_m, \quad (7)$$

and product MGF

$$M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = K_{(n,m-1)}\int\int\cdots\int_{0<x_1<\cdots<x_m<\infty} e^{(t_1x_q+t_2x_s)}k^m\times f(x_1) [\bar{F}(x_1)]^{kR_1+k-1}\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_m. \quad (8)$$

2 RR of Moments and MGF

In this section, we introduce the RR for single and product moments based on PFFC.

In the next theorem we introduce the RR for single moments based on PFFC.

Theorem 1. *For $2\le q\le m-1$, $m\le n$ then*

$$\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i)}} = \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!(i+j+1)}\{k(R_q+1) \mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+j+1)}} - (n-R_1-\cdots-R_{q-1}-q+1) \mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+j+1)}} +(n-R_1-\cdots-R_q-q) \mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+j+1)}}\}. \quad (9)$$

Proof. From Eq. (4) and Eq. (5), we get

$$\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i)}} = \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!}K_{(n,m-1)}\int\int\cdots \int_c Q_1(x_{q-1},x_{q+1})k^m\times f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\ldots f(x_{q-1}) [\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\times\ldots f(x_m) [\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_{q-1}dx_{q+1}\ldots dx_m, \quad (10)$$

where $c = 0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty$, and

$$Q_1(x_{q-1},x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+j}[\bar{F}(x_q)]^{kR_q+k}dx_q. \quad (11)$$

Now, integrating by parts gives

$$Q_1(x_{q-1},x_{q+1}) = \frac{x_{q+1}^{i+j+1}[\bar{F}(x_{q+1})]^{kR_q+k} - x_{q-1}^{i+j+1}[\bar{F}(x_{q-1})]^{kR_q+k}}{i+j+1} +(\frac{kR_q+k}{i+j+1})\int_{x_{q-1}}^{x_{q+1}} x_q^{i+j+1}f(x_q)[\bar{F}(x_q)]^{kR_q+k-1}dx_q. \quad (12)$$

Substituting by Eq. (12) in Eq. (10) and simplifying, yields Eq. (9).

This completes the proof.

Special case

1- Theorem 1 will be valid for the PTIIRCOS when k=1,

$$\mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}} = \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!(i+j+1)}\{(R_q+1) \mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+j+1)}} - (n-R_1-\cdots-R_{q-1}-q+1) \mu_{q-1:m-1:n}^{(R_1,R_2,\ldots,R_{q-1}+R_q+1,R_{q+1},\ldots,R_m)^{(i+j+1)}} +(n-R_1-\cdots-R_q-q) \mu_{q:m-1:n}^{(R_1,R_2,\ldots,R_q+R_{q+1}+1,R_{q+2},\ldots,R_m)^{(i+j+1)}}\}.$$

In the next two theorems, I introduce RR for product moments based on PFFC.

Theorem 2. *For $1 \le q < s \le m-1$, $m \le n$ then*

$$\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{k(R_q+1)\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}} - (n-R_1-\cdots-R_{q-1}-q+1)\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}} + (n-R_1-\cdots-R_q-q)\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}}\}. \tag{13}$$

Proof. Similarly as proved in theorem 1.

Special case

This theorem will be valid for the PTIICOS as a special case from the PFFC when k=1,

$$\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,r)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_q+1)\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+j+1,r)}} - (n-R_1-\cdots-R_{q-1}-q+1)\mu_{q-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{q-1}+R_q+1,R_{q+1},\ldots,R_m)^{(i+j+1,r)}} + (n-R_1-\cdots-R_q-q)\mu_{q,s-1:m-1:n}^{(R_1,R_2,\ldots,R_q+R_{q+1}+1,R_{q+2},\ldots,R_m)^{(i+j+1,r)}}\}.$$

Theorem 3. *For $1 \le q < s \le m-1$, $m \le n$ then*

$$\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{k(R_s+1)\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}} - (n-R_1-\cdots-R_{s-1}-s+1)\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}} + (n-R_1-\cdots-R_s-s)\mu_{q,s:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}}\}. \tag{14}$$

Proof. Similarly as proved in theorem 1.

Special cases

For k=1, we obtain the recurrence relation of PTIICOS.

$$\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,r)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{(R_s+1)\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+r+1)}} - (n-R_1-\cdots-R_{s-1}-s+1)\mu_{q,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+r+1)}} + (n-R_1-\cdots-R_s-s)\mu_{q,s:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(i,j+r+1)}}\},$$

and for i=0

$$\mu_{s:m:n}^{(R_1,R_2,\ldots,R_m)^{(r)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(j+1)} \{(R_s+1)\mu_{s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j+r+1)}} - (n-R_1-\cdots-R_{s-1}-s+1)\mu_{s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(j+r+1)}} + (n-R_1-\cdots-R_s-s)\mu_{s:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(j+r+1)}}\},$$

and for s=m

$$\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(r)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j+r+1)}} - (n-R_1-\cdots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-1}+R_m+1)^{(j+r+1)}}\}.$$

In the next theorem we introduce the RR for single MGF based on PFFC.

Theorem 4. *For $2 \le q \le m-1$, $m \le n$ then*

$$\frac{t+\beta}{\alpha} M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \{k(R_q+1)M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(j)}} - (n-R_1-\cdots-R_{q-1}-q+1)M_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(j)}} + (n-R_1-\cdots-R_q-q)M_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(j)}}\}. \tag{15}$$

Proof. From Eq. (4) and Eq. (7), we get

$$M_{q:m:n,k}^{kR_1+k-1,\ldots,kR_m+k-1} = K_{(n,m-1)} \int\int\cdots\int_c Q_2(x_{q-1},x_{q+1}) k^m \times f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\ldots f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1} f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1} \times \ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1\ldots dx_{q-1}dx_{q+1}\ldots dx_m, \tag{16}$$

where

$$Q_2(x_{q-1},x_{q+1}) = \alpha \int_{x_{q-1}}^{x_{q+1}} e^{(t+\beta)x_q}[\bar{F}(x_q)]^{kR_q+k}dx_q. \tag{17}$$

Now, integrating by parts gives

$$Q_2(x_{q-1},x_{q+1}) = \frac{\alpha e^{(t+\beta)x_{q+1}}[\bar{F}(x_{q+1})]^{kR_q+k} - \alpha e^{(t+\beta)x_{q-1}}[\bar{F}(x_{q-1})]^{kR_q+k}}{t+\beta} + \alpha\left(\frac{kR_q+k}{t+\beta}\right) \int_{x_{q-1}}^{x_{q+1}} e^{(t+\beta)x_q} f(x_q)[\bar{F}(x_q)]^{kR_q+k-1}dx_q. \tag{18}$$

Substituting by Eq. (18) in Eq. (16) and simplifying, yields Eq. (15).

This completes the proof.

In the next two theorems, I introduce RR for product MGF based on PFFC.

Theorem 5. *For $1 \le q < s \le m-1$, $m \le n$ then*

$$\frac{t+\beta}{\alpha} M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \{k(R_q+1) M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)(j)} - (n-R_1-\cdots-R_{q-1}-q+1) M_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)(j)} + (n-R_1-\cdots-R_q-q) M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)(j)}\}. \tag{19}$$

Proof. Similarly as proved in theorem 4.

Theorem 6. *For $1 \le q < s \le m-1$, $m \le n$ then*

$$\frac{t+\beta}{\alpha} M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \{k(R_q+1) M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)(j)} - (n-R_1-\cdots-R_{s-1}-s+1) M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\dots,kR_m+k-1)(j)} + (n-R_1-\cdots-R_s-s) M_{q,s:m-1:n,k}^{(kR_1+k-1,\dots,kR_s+kR_{s+1}+2k-1,kR_{s+2}+k-1,\dots,kR_m+k-1)(j)}\}. \tag{20}$$

Proof. Similarly as proved in theorem 4.

3 The Characterization

In this section, we introduce the characterization of the GD using the relation between pdf and cdf and using RR for single and product moments and MGF based on PFFC.

3.1 Characterization via differential equation for the GD

In the next theorem, we introduce the characterization of the GD using relation between pdf and cdf.

Theorem 7. *Let X be a continuous random variable with pdf $f(.)$, cdf $F(.)$ and survival function $\bar{F}(.)$. Then X has GD iff*

$$f(x) = \alpha e^{\beta x}[\bar{F}(x)]. \tag{21}$$

Proof. **Necessity:** From Eq. (2) and Eq. (3) we can easily obtain Eq. (21).

Sufficiency:

Suppose that Eq. (21) is true. Then we have:

$$\frac{-d[\bar{F}(x)]}{\bar{F}(x)} = \alpha e^{\beta x} dx.$$

By integrating, we get

$$-ln|\bar{F}(x)| = \frac{\alpha}{\beta} e^{\beta x} + C \tag{22}$$

where C is an arbitrary constant.

Now, since $[\bar{F}(\mu)] = 1$, then putting $x = 0$ in Eq. (22), we get $C = \frac{-\alpha}{\beta}$.

Therefore,

$$ln|\bar{F}(x)| = -\frac{\alpha}{\beta} e^{\beta x} + \frac{\alpha}{\beta}$$

or,

$$\bar{F}(x) = exp\{-\frac{\alpha}{\beta} e^{\beta x} + \frac{\alpha}{\beta}\}$$

Hence,

$$F(x) = 1 - exp\{\frac{\alpha}{\beta} - \frac{\alpha}{\beta} e^{\beta x}\}$$

That is the distribution function of GD.

This completes the proof.

3.2 Characterization via RR for single moment

In the next theorem, we will introduce the characterization of the GD using RR for single moments based on PFFC.

Theorem 8. *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GD iff, for $2 \le q \le m-1$, $m \le n$ and $i \ge 0$,*

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)(i)} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{k(R_q+1) \mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)(i+j+1)} - (n-R_1-\cdots-R_{q-1}-q+1) \mu_{q-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)(i+j+1)} + (n-R_1-\cdots-R_q-q) \mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)(i+j+1)}\}. \tag{23}$$

Proof. **Necessity** 1 proved the necessary part of this theorem. Theorem **Sufficiency:** Assuming that Eq. (23)

holds, then we have:

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!(i+j+1)} \{k(R_q+1) \mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+j+1)}} - (n-R_1-\dots-R_{q-1}-q+1) \mu_{q-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)^{(i+j+1)}} + (n-R_1-\dots-R_q-q) \mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)^{(i+j+1)}}\}. \tag{24}$$

where,

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+j+1)}} = K_{(n,m-1)} \int\int\cdots\int_c Q_2(x_{q-1},x_{q+1}) \times k^m f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\dots f(x_{q-1}) [\bar{F}(x_{q-1})]^{kR_{q-1}+k-1} \times f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m) [\bar{F}(x_m)]^{kR_m+k-1} dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m, \tag{25}$$

where

$$Q_2(x_{q-1},x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+j+1} f(x_q)[\bar{F}(x_q)]^{kR_q+k-1} dx_q. \tag{26}$$

Integrating by parts, we obtain

$$Q_2(x_{q-1},x_{q+1}) = \frac{-1}{kR_q+k} x_{q+1}^{i+j+1}[\bar{F}(x_{q+1})]^{kR_{q+1}+k} + \frac{1}{kR_q+k} x_{q-1}^{i+j+1} [\bar{F}(x_{q-1})]^{kR_{q-1}+k} + \frac{i+j+1}{kR_q+k} \int_{x_{q-1}}^{x_{q+1}} x_q^{i+j}[\bar{F}(x_q)]^{kR_q+k} dx_q. \tag{27}$$

Substituting in Eq. (25), we get

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+j+1)}} = \frac{i+j+1}{kR_q+k} K_{(n,m-1)} \int\int\cdots \int_c k^m \times f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\dots f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1} \int_{x_{q-1}}^{x_{q+1}} x_q^{i+j}[\bar{F}(x_q)]^{kR_q+k} dx_q f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m + \frac{K_{(n,m-1)}}{kR_q+k} \int\int\cdots\int_c x_{q-1}^{i+j+1} k^m f(x_1)\times[\bar{F}(x_1)]^{kR_1+k-1}\dots f(x_{q-1}) [\bar{F}(x_{q-1})]^{kR_{q-1}+kR_q+k} f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots\times f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m - \frac{K_{(n,m-1)}}{kR_q+k} \int\int\cdots\int_c x_{q+1}^{i+j+1} k^m \times f(x_1)[\bar{F}(x_{r+1})]^{kR_1+k-1}\dots f(x_{q+1}) [\bar{F}(x_{q+1})]^{kR_q+kR_{q+1}+k}\dots\times f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_{r+1}\dots dx_{q-1}dx_{q+1}\dots dx_m = K_{(n,m-1)} \frac{i+j+1}{kR_q+k} \int\int\cdots\int_c k^m\dots\times \int_{x_{q-1}}^{x_{q+1}} x_q^{i+j}[\bar{F}(x_q)]^{kR_q+k} dx_q f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\dots f(x_{q-1}) [\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}\times f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m) [\bar{F}(x_m)]^{kR_m+k-1} dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m - \frac{(n-R_1-\dots-R_{q-1}-q+1)}{kR_q+k} \mu_{q-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)^{(i+j+1)}} + \frac{(n-R_{r+1}-\dots-R_q-q)}{kR_q+k} \mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)^{(i+j+1)}}. \tag{28}$$

Substituting for $\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+j+1)}}$ from (28) in (24), we get

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i)}} = \sum_{j=0}^{\infty} \frac{\alpha\beta^j}{j!} K_{(n,m-1)} \int\int\cdots \int_{0<x_1<\dots<x_m<\infty} x_q^{i+j}\times k^m[\bar{F}(x_q)]^{kR_q+k} f(x_1)[\bar{F}(x_1)]^{kR_1+k-1} \dots f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}\times f(x_{q+1}) [\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1\dots dx_m, \tag{29}$$

we get

$$K_{(n,m-1)}\int\int\cdots\int_{0<x_1<\cdots<x_m<\infty} x_q^i k^m[\bar{F}(x_q)]^{kR_q+k-1}\times f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\ldots f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1} f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1\ldots dx_m = K_{(n,m-1)}\int\int\cdots\int_{0<x_1<\cdots<x_m<\infty} \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!}x_q^{i+j}k^m[\bar{F}(x_q)]^{kR_q+k}\cdots\times f(x_1)[\bar{F}(x_1)]^{kR_1+k-1} f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\cdots\times f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\ldots dx_m. \quad (30)$$

We get

$$K_{(n,m-1)}\int\int\cdots\int_{0<x_1<\cdots<x_m<\infty} x_q^i[\bar{F}(x_q)]^{kR_q+k-1} f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\cdots\times f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1} f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} [f(x_q)-\alpha e^{\beta x_q}[\bar{F}(x_q)]]k^m dx_1\ldots dx_m = 0. \quad (31)$$

Using Muntz-Szasz theorem, [See, Hwang and Lin [2]], we get

$$f(x_q) = \alpha e^{\beta x_q}[\bar{F}(x_q)].$$

Using Theorem 7, we get

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}.$$

That is the distribution function of GD.
This completes the proof.

3.3 Characterization via RR for product moments

In the next two theorems, we will introduce the characterization of GD using RR for product moments based on PFFC.

Theorem 9. *Let $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GD iff, for $1\le q<s\le m-1$, $m\le n$ and $i,j\ge 0$,*

$$\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} = \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!(i+j+1)}\{k(R_q+1) \mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}} - (n-R_1-\cdots-R_{q-1}-q+1) \mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}} + (n-R_1-\cdots-R_q-q) \mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+j+1,r)}}\}. \quad (32)$$

Proof. Similarly as proved in theorem 8.

Theorem 10. *Let $X_{1:n}\le\cdots\le X_{n:n}$ be the order stasistics of a random sample of size (n-r). Then X has GD iff, for $1\le q<s\le m-1$, $m\le n$ and $i,j\ge 0$,*

$$\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} = \sum_{j=0}^{\infty}\frac{\alpha\beta^j}{j!(i+j+1)}\{k(R_s+1) \mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}} - (n-R_1-\cdots-R_{s-1}-s+1) \mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}} + (n-R_1-\cdots-R_s-s) \mu_{q,s:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1)^{(i,j+r+1)}}\}. \quad (33)$$

Proof. Similarly as proved in theorem 8.

3.4 Characterization via RR for single MGF

In the next theorem, we will introduce the characterization of the GD using RR for single MGF based on PFFC.

Theorem 11. *Let $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GD iff, for $2\le q\le m-1$, $m\le n$ and $i\ge 0$,*

$$\frac{t+\beta}{\alpha}M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = \sum_{j=0}^{\infty}\frac{\beta^j}{j!}\{k(R_q+1) M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(j)}} - (n-R_1-\cdots-R_{q-1}-q+1) M_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(j)}} + (n-R_1-\cdots-R_q-q) M_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(j)}}\}. \quad (34)$$

Proof. Similarly as proved in theorem 8.

3.5 Characterization via RR for product MGF

In the next two theorems, we will introduce the characterization of GD using RR for product MGF based on GPTIICOS.

Theorem 12. *Let $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GD* *iff, for* $1 \le q \le m-1$, m

$$\frac{t+\beta}{\alpha} M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \{k(R_q+1) M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(j)}} - (n-R_1-\cdots-R_{q-1}-q+1) M_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(j)}} + (n-R_1-\cdots-R_q-q) M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(j)}}\}. \quad (35)$$

Proof. Similarly as proved in theorem 8.

Theorem 13. *Let* $X_{1:n} \le \cdots \le X_{n:n}$ *be the order stasistics of a random sample of size n. Then X has GD iff, for* $1 \le q < s \le m-1$, $m \le n$ *and* $i, j \ge 0$,

$$\frac{t+\beta}{\alpha} M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \{k(R_q+1) M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(j)}} - (n-R_1-\cdots-R_{s-1}-s+1) M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)^{(j)}} + (n-R_1-\cdots-R_s-s) M_{q,s:m-1:n,k}^{(kR_1+k-1,\ldots,kR_s+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1)^{(j)}}\}. \quad (36)$$

Proof. Similarly as proved in theorem 8.

Abbreviations

Abbreviation	Defination
PTIIRCOS	Progressively type-II right censored
MGF	Moment generating function
GD	Gompertz distribution
PFFC	Progressive first failure censoring
RR	Recurrence relations

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