

# $(\in, \in \forall q)$ -Fuzzy Prime Ideals of $BCK$ -Algebras

S. R. Barbhuiya\*

Department of Mathematics, Srikishan Sarda College, Hailakandi, Hailakandi-788151, Assam, India

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**Abstract:** The aim of this paper is  $(\in, \in \forall q)$ -fuzzification of the concept of prime ideals in commutative  $BCK$ -algebras. We state and prove some theorems in  $(\in, \in \forall q)$ -fuzzy prime ideals in commutative  $BCK$ -algebras also we study relation between an  $(\in, \in \forall q)$ -fuzzy prime ideal and an  $(\in \forall q)$ -fuzzy level prime ideals.

**Keywords:**  $BCK$ -algebra, Prime ideal, Fuzzy prime ideal,  $(\in, \in \forall q)$ -fuzzy prime ideal.

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## 1 Introduction

The concept of fuzzy sets was first initiated by Zadeh([21]) in 1965. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, vector spaces etc. Imai and Iseki ([11]) introduced two classes of abstract algebras:  $BCK$ -algebras and  $BCI$ -algebras. It is known that the class of  $BCK$ -algebra is a proper sub class of the class of  $BCI$ -algebras. Iseki([12]) introduced the concept of prime ideal in commutative  $BCK$ -algebras. In ([2]) Ahsan, Deeba and Thaheem have studied the theory of ideals, in particular, prime ideals of a commutative  $BCK$ -algebras. In([18, 19]) Jun and Xin have studied fuzzy prime ideals and invertible fuzzy ideals in  $BCK$ -algebras. Abdullah ([1]) introduced the notion of intuitionistic fuzzy prime ideals of commutative  $BCK$ -algebras. Murali ([16]) proposed a definition of a fuzzy point belonging to fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das ([5, 6]) used the relation of "belongs to" and "quasi-coincident" between fuzzy point and fuzzy set to introduced the concept of  $(\in, \in \forall q)$ -fuzzy subgroup,  $(\in, \in \forall q)$ -fuzzy subring and  $(\in \forall q)$ -level subset. Basnet and Singh ([7]) introduced  $(\in, \in \forall q)$ -fuzzy ideals of  $BG$ -algebra in 2011. Dhanani and Pawar([10]) introduced the concept of  $(\in, \in \forall q)$ -fuzzy ideals(prime) of lattice. It is now natural to investigate similar type of generalisation of the existing fuzzy subsystem with other algebraic structure. In this paper, we introduced the notion of  $(\in, \in \forall q)$ - fuzzy prime ideals of commutative  $BCK$ -algebras and got some interesting result.

## 2 Preliminaries

**Definition 2.1**([1, 18, 19]) An algebra  $(X, *, 0)$  of type  $(2, 0)$  is called a  $BCK$ -algebra if it satisfies the following axioms:

- (i)  $((x * y) * (x * z)) * (z * y) = 0$
  - (ii)  $(x * (x * y)) * y = 0$
  - (iii)  $x * x = 0$
  - (iv)  $0 * x = 0$
  - (v)  $x * y = 0$  and  $y * x = 0 \Rightarrow x = y$  for all  $x, y, z \in X$ .
- We can define a partial ordering " $\leq$ " on  $X$  by  $x \leq y$  iff  $x * y = 0$ .

**Definition 2.2**([1, 18, 19]) A  $BCK$ -algebra  $X$  is said to be commutative if it satisfies the identity  $x \wedge y = y \wedge x$  where  $x \wedge y = y * (y * x) \forall x, y \in X$ . In a commutative  $BCK$ -algebra, it is known that  $x \wedge y$  is the greatest lower bound of  $x$  and  $y$ .

In a  $BCK$ -algebra  $X$ , the following hold:

- (i)  $x * 0 = x$
- (ii)  $(x * y) * z = (x * z) * y$
- (iii)  $x * y \leq x$
- (iv)  $(x * y) * z \leq (x * z) * (y * z)$
- (v)  $x \leq y$  implies  $x * z \leq y * z$  and  $z * y \leq z * x$ .

**Definition 2.3**([20]) A nonempty subset  $I$  of a  $BCK$ -algebra  $X$  is called an ideal of  $X$  if

- (i)  $0 \in I$
- (ii)  $x * y \in I$  and  $y \in I \Rightarrow x \in I$  for all  $x, y \in X$ .

\* Corresponding author e-mail: [saidurbarbhuiya@gmail.com](mailto:saidurbarbhuiya@gmail.com)

**Definition 2.4**[20] A fuzzy set  $\mu$  in BCK-algebra  $X$  is called a fuzzy ideal of  $X$  if it satisfies

- (i)  $\mu(0) \geq \mu(x)$
- (ii)  $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$  for all  $x, y \in X$ .

**Definition 2.5**[1] An ideal  $I$  of a commutative BCK-algebra  $X$  is said to be prime if  $x \wedge y \in I \Rightarrow x \in I$  or  $y \in I$

**Definition 2.6**[19] A non constant fuzzy ideal  $\mu$  of a commutative BCK-algebra  $X$  is said to be fuzzy prime if  $\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$  for all  $x, y \in X$ . Since  $x \wedge y \leq x, y$  and  $\mu$  is order reversing, it follows that  $\mu(x) \leq \mu(x \wedge y)$  and  $\mu(y) \leq \mu(x \wedge y)$  therefore a non constant fuzzy ideal  $\mu$  of a commutative BCK-algebra  $X$  is fuzzy prime iff  $\mu(x \wedge y) = \max\{\mu(x), \mu(y)\}$  for all  $x, y \in X$  or equivalently  $\mu(x \wedge y) = \mu(x)$  or  $\mu(y)$  for all  $x, y \in X$ .

**Definition 2.7** Let  $\lambda$  and  $\mu$  be two fuzzy sets, then their union  $\lambda \cup \mu$  and their intersection  $\lambda \cap \mu$  are defined by  $(\lambda \cup \mu)(x) = \max\{\lambda(x), \mu(x)\}$  and  $(\lambda \cap \mu)(x) = \min\{\lambda(x), \mu(x)\}$ .

### 3 $(\in, \in \vee q)$ -fuzzy prime ideals of BCK-algebra

In what follows, let  $X$  denote a commutative BCK-algebra unless otherwise stated.

**Definition 3.1**[5] A fuzzy set  $\mu$  of the form

$$\mu(y) = \begin{cases} t, & \text{if } y = x, t \in (0, 1], \\ 0, & \text{if } y \neq x, \end{cases}$$

is called a fuzzy point with support  $x$  and value  $t$  and it is denoted by  $x_t$ .

**Definition 3.2**[15] Let  $\mu$  be a fuzzy set in  $X$  and  $x_t$  be a fuzzy point then

- (i) If  $\mu(x) \geq t$  then we say  $x_t$  belongs to  $\mu$  and write  $x_t \in \mu$
- (ii) If  $\mu(x) + t > 1$  then we say  $x_t$  quasi coincident with  $\mu$  and write  $x_t q \mu$
- (iii) If  $x_t \in \vee q \mu \Leftrightarrow x_t \in \mu$  or  $x_t q \mu$
- (iv) If  $x_t \in \wedge q \mu \Leftrightarrow x_t \in \mu$  and  $x_t q \mu$

The symbol  $x_t \overline{\alpha} \mu$  means  $x_t \alpha \mu$  does not hold and  $\overline{\in \wedge q}$  means  $\overline{\in \vee q}$

For a fuzzy point  $x_t$ . and a fuzzy set  $\mu$  in set  $X$ , Pu and Liu gave meaning to the symbol  $x_t \alpha \mu$  where  $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

**Definition 3.3**[3] Let  $\mu$  be a fuzzy set in BCK-algebra  $X$  and  $t \in (0, 1]$ , let

$$\mu_t = \{x \in X | x_t \in \mu\} = \{x \in X | \mu(x) \geq t\}$$

$$\begin{aligned} \langle \mu \rangle_t &= \{x \in X | x_t q \mu\} = \{x \in X | \mu(x) + t > 1\} \\ [\mu]_t &= \{x \in X | x_t \in \vee q \mu\} = \\ &= \{x \in X | \mu(x) \geq t \text{ or } \mu(x) + t > 1\} \end{aligned}$$

Here  $\mu_t$  is called  $t$  level set of  $\mu$ ,  $\langle \mu \rangle_t$  is called  $q$  level set of  $\mu$  and  $[\mu]_t$  is called  $(\in \vee q)$  level set of  $\mu$ . Clearly  $[\mu]_t = \langle \mu \rangle_t \cup \mu_t$

**Definition 3.4**[20] A fuzzy set  $\mu$  of a BCK-algebra  $X$  is said to be  $(\alpha, \beta)$ -fuzzy ideal of  $X$ , where  $\alpha \neq \in \wedge q$  if

- (i)  $x_t \alpha \mu \Rightarrow 0_t \beta \mu$
- (ii)  $(x * y)_t \alpha \mu, y_s \alpha \mu \Rightarrow x_{m(t,s)} \beta \mu$  for all  $x, y \in X$  where  $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}; t, s \in (0, 1]$  and  $m(t, s) = \min(t, s)$ .

**Example 3.5** Consider BCK-algebra  $X = \{0, x, y, z\}$  with the following cayley table.

*	0	x	y	z	w
0	0	0	0	0	0
x	x	0	x	0	x
y	y	y	0	y	0
z	z	x	z	0	z
w	w	w	y	w	0

Define a map  $\mu : X \rightarrow [0, 1]$  by  $\mu(0) = 0.7, \mu(x) = \mu(z) = 0.3, \mu(y) = \mu(w) = 0.2$ , then  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Definition 3.6** An  $(\alpha, \beta)$ -fuzzy ideal  $\mu$  of a BCK-algebra  $X$  is said to be an  $(\alpha, \beta)$ -fuzzy prime ideal of  $X$  if  $(x \wedge y)_t \alpha \mu \Rightarrow x_t \beta \mu$  or  $y_t \beta \mu$ , for all  $x, y \in X$ . where  $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}; \alpha \neq \in \wedge q$  and  $t \in (0, 1]$ .

**Theorem 3.7** A fuzzy set  $\mu$  of a BCK-algebra  $X$  is a fuzzy prime ideal if and only if  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal.

*Proof.* Let  $\mu$  be a fuzzy prime ideal, therefore  $\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$

$$\begin{aligned} \text{Let } (x \wedge y)_t \in \mu &\Rightarrow \mu(x \wedge y) \geq t \\ &\Rightarrow \max\{\mu(x), \mu(y)\} \geq \mu(x \wedge y) \geq t \\ &\Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t \\ &\Rightarrow x_t \in \mu \text{ or } y_t \in \mu \end{aligned}$$

Therefore  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal.

Conversely, Let  $\mu$  be an  $(\in, \in)$ -fuzzy prime ideal.

Let  $x, y \in X$  and  $\mu(x \wedge y) = t$  where  $t \in [0, 1]$

$$\begin{aligned} \text{then } \mu(x \wedge y) \geq t &\Rightarrow (x \wedge y)_t \in \mu \\ &\Rightarrow x_t \in \mu \text{ or } y_t \in \mu \\ &\Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t \\ &\Rightarrow \max\{\mu(x), \mu(y)\} \geq t = \mu(x \wedge y) \end{aligned}$$

Therefore  $\mu$  is a fuzzy prime ideal.

**Theorem 3.8**  $\mu$  is a  $(q, q)$ -fuzzy prime ideal if and only if  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal.

*Proof.* Let  $\mu$  be a  $(q, q)$ -fuzzy ideal of a BCK-algebra X.

Let  $x, y \in X$  such that  $(x \wedge y)_t \in \mu$

$$\Rightarrow \mu(x \wedge y) \geq t$$

$$\Rightarrow \mu(x \wedge y) + \delta > t, \text{ for some } \delta > 0$$

$$\Rightarrow \mu(x \wedge y) + \delta - t + 1 > 1$$

$$\Rightarrow (x \wedge y)_{\delta-t+1} q\mu$$

Since  $\mu$  is a  $(q, q)$  fuzzy prime ideal X.

Therefore we have  $x_{\delta-t+1} q\mu$  or  $y_{\delta-t+1} q\mu$

$$\Rightarrow \mu(x) + \delta - t + 1 > 1 \text{ or } \mu(y) + \delta - t + 1 > 1$$

$$\Rightarrow \mu(x) + \delta > t \text{ or } \mu(y) + \delta > t$$

$$\Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t$$

$$\Rightarrow x_t \in \mu \text{ or } y_t \in \mu$$

therefore  $(x \wedge y)_t \in \mu \Rightarrow x_t \in \mu \text{ or } y_t \in \mu$

Hence  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal of X.

Conversely,

Assume  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal of X.

Let  $(x \wedge y)_t q\mu$

$$\Rightarrow \mu(x \wedge y) + t > 1$$

$$\Rightarrow \mu(x \wedge y) > 1 - t$$

$$\Rightarrow \mu(x \wedge y) \geq \delta - t + 1 > 1 - t \text{ for some } \delta > 0$$

$$\Rightarrow (x \wedge y)_{\delta-t+1} \in \mu$$

Since  $\mu$  is an  $(\in, \in)$  fuzzy prime ideal X.

Therefore we have  $x_{\delta-t+1} \in \mu$  or  $y_{\delta-t+1} \in \mu$

$$\Rightarrow \mu(x) \geq \delta - t + 1 > 1 - t \text{ or } \mu(y) \geq \delta - t + 1 > 1 - t$$

$$\Rightarrow \mu(x) > 1 - t \text{ or } \mu(y) > 1 - t$$

$$\Rightarrow \mu(x) + t > 1 \text{ or } \mu(y) + t > 1$$

$$\Rightarrow x_t q\mu \text{ or } y_t q\mu$$

therefore  $(x \wedge y)_t q\mu \Rightarrow x_t q\mu \text{ or } y_t q\mu$

Hence  $\mu$  is a  $(q, q)$ -fuzzy prime ideal of X.

**Theorem 3.9** An  $(\in, \in \vee q)$ -fuzzy ideal  $\mu$  of a BCK-algebra X is an  $(\in, \in \vee q)$ -fuzzy prime ideal of X iff

$$\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\} \quad \forall x, y \in X$$

*Proof.* First let  $\mu$  be an  $(\in, \in \vee q)$ -fuzzy prime ideal of X.

To prove

$$\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\} \quad (3.1)$$

Assume that (3.1) is not valid, then there exists  $x, y \in X$  such that

$$\max\{\mu(x), \mu(y)\} < \min\{\mu(x \wedge y), 0.5\}$$

choose a real number t such that

$$\max\{\mu(x), \mu(y)\} < t < \min\{\mu(x \wedge y), 0.5\}$$

then  $t \in (0, 0.5]$  and  $\mu(x \wedge y) > t \Rightarrow (x \wedge y)_t \in \mu$

and also  $\mu(x) < t$  and  $\mu(y) < t$  i.e.  $x_t \notin \mu, y_t \notin \mu$

also  $\mu(x) + t < 2t < 2 \times 0.5 = 1$  i.e.  $x_t \notin \vee q\mu, y_t \notin \vee q\mu$  which is a contradiction, since  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal. Hence we must have

$$\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\}$$

**Conversely,** Suppose

$$\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\} \quad (3.2)$$

Let  $x, y \in X$  such that  $(x \wedge y)_t \in \mu$  where  $t \in (0, 1]$

$$\text{i.e. } \mu(x \wedge y) \geq t.$$

Now

$$(3.2) \Rightarrow \max\{\mu(x), \mu(y)\} \geq \min\{t, 0.5\}.$$

Now we have

Case I:  $t \leq 0.5$

$$\text{then } \max\{\mu(x), \mu(y)\} \geq t \Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t \Rightarrow x_t \in \mu \text{ or } y_t \in \mu$$

Case II:  $t > 0.5$

$$\text{then } \max\{\mu(x), \mu(y)\} \geq 0.5 \Rightarrow \mu(x) \geq 0.5 \text{ or } \mu(y) \geq 0.5$$

$$\Rightarrow \mu(x) + t \geq 0.5 + t > 0.5 + 0.5 = 1 \text{ or } \mu(y) + t \geq 0.5 + t > 0.5 + 0.5 = 1$$

$$\Rightarrow x_t q\mu \text{ or } y_t q\mu \text{ combining case I and case II } x_t \in \vee q\mu \text{ or } y_t \in \vee q\mu$$

$$\Rightarrow x_t q\mu \text{ or } y_t q\mu \text{ combining case I and case II } x_t \in \vee q\mu \text{ or } y_t \in \vee q\mu$$

Hence  $(x \wedge y)_t \in \mu \Rightarrow x_t \in \vee q\mu \text{ or } y_t \in \vee q\mu$

that is  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of X.

**Corollary 3.10** Every fuzzy prime ideal is an  $(\in, \in \vee q)$ -fuzzy prime ideal.

**Note 1.** Converse of above is not true as seen from the following Example.

**Example 3.11** Consider a BCK-algebra  $X = \{0, x, y, z\}$  with the following cayley tables.

*	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

Table 1

$\wedge$	0	x	y	z
0	0	0	0	0
x	0	x	x	0
y	0	x	y	0
z	0	0	0	z

Table 2

Define a map  $\mu : X \rightarrow [0, 1]$  by  $\mu(0) = 0.6, \mu(x) = 0.5, \mu(y) = 0.7, \mu(z) = 0.56$  then  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal X, but not a fuzzy prime ideal of X because  $\mu(x \wedge z) = \mu(0) = 0.6 > \max\{\mu(x), \mu(z)\} = \max\{0.5, 0.56\} = 0.56$ .

**Theorem 3.12** If a fuzzy set  $\mu$  of a BCK-algebra X is an  $(\in, \in \vee q)$ -fuzzy prime ideal of X and  $\mu(x) < 0.5 \forall x \in X$ , then  $\mu$  is also an  $(\in, \in)$ -fuzzy prime ideal of X.

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q)$ -fuzzy prime ideal of X and  $\mu(x) < 0.5 \forall x \in X$  Let  $(x \wedge y)_t \in \mu \Rightarrow \mu(x \wedge y) \geq t$  therefore  $t \leq \mu(x \wedge y) < 0.5$  and also  $\mu(x) < 0.5, \mu(y) < 0.5$  therefore  $t < 0.5$  and also  $\mu(x) + t < 0.5 + 0.5 = 1$  and  $\mu(y) + t < 0.5 + 0.5 = 1 \Rightarrow x_t \bar{q}\mu$  and  $y_t \bar{q}\mu$  Since  $\mu$  is  $(\in, \in \vee q)$ -fuzzy prime ideal, so we must have  $x_t \in \mu$  or  $y_t \in \mu$  i.e.  $\mu$  is an  $(\in, \in)$ -fuzzy prime ideal of X.

**Theorem 3.13A** *fuzzy set  $\mu$  in  $X$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$  if and only if the set  $\mu_t = \{x \in X | \mu(x) \geq t\}$  is a prime ideal of  $X$ , where  $\mu_t \neq \emptyset$  for all  $t \in (0, 0.5]$*

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ . Let  $t \in (0, 0.5]$  and  $x \wedge y \in \mu_t$ . Therefore  $\mu(x \wedge y) \geq t$ . It follows that  $\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\} \geq \min\{t, 0.5\} = t$  therefore  $\mu(x) \geq t$  or  $\mu(y) \geq t$ , that is  $x \in \mu_t$  or  $y \in \mu_t$  therefore  $\mu_t$  is a prime ideal of  $X$ .

Conversely  
Suppose that  $\mu_t$  is a prime ideal of  $X$  for all  $t \in (0, 0.5]$  and assume (3.1) is not valid, then there exists some  $a, b \in X$  such that  $\max\{\mu(a), \mu(b)\} < \min\{\mu(a \wedge b), 0.5\}$  hence we can take  $t \in (0, 0.5)$  such that

$$\max\{\mu(a), \mu(b)\} < t < \min\{\mu(a \wedge b), 0.5\} \quad (3.3)$$

(3.3)  $\Rightarrow a \wedge b \in \mu_t$ . Since  $\mu_t$  is a prime ideal of  $X$ , it follows that  $a \in \mu_t$  or  $b \in \mu_t$ , so that  $\mu(a) \geq t$  or  $\mu(b) \geq t$ , which contradicts (3.3), therefore we must have  $\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\}$  consequently  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

**Theorem 3.14** *Let  $A$  be a non empty subset of a BCK-algebra  $X$ . Consider the fuzzy set  $\mu_A$  in  $X$  defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

*Then  $A$  is a prime ideal of  $X$  iff  $\mu_A$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .*

*Proof.* Let  $A$  be an ideal of  $X$ . Then  $(\mu_A)_t = \{x \in X | \mu_A(x) \geq t\} = A, \forall t \in (0, 0.5]$  which is a prime ideal. Hence by above theorem  $\mu_A$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

Conversely, assume that  $\mu_A$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

Let  $x \wedge y \in A$ , then  $\mu_A(x \wedge y) = 1$   
 $\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\} = \min\{1, 0.5\} = 0.5$   
 $\Rightarrow \max\{\mu_A(x), \mu_A(y)\} \geq 0.5$   
 $\Rightarrow \mu_A(x) \geq 0.5$  or  $\mu_A(y) \geq 0.5$   
 $\Rightarrow \mu_A(x) = 1$  or  $\mu_A(y) = 1$   
 $\Rightarrow x \in A$  or  $y \in A$

Therefore  $x \wedge y \in A \Rightarrow x \in A$  or  $y \in A$   
Hence  $A$  is a prime ideal of  $X$ .

**Theorem 3.15** *Let  $A$  be a prime ideal of  $X$ , then for every  $t \in (0, 0.5]$ , there exists an  $(\in, \in \vee q)$ -fuzzy prime ideal  $\mu$  of  $X$ , such that  $\mu_t = A$ .*

*Proof.* Let  $\mu$  be a fuzzy set in  $X$  defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A, \\ s, & \text{otherwise,} \end{cases}$$

for all  $x \in X$   
where  $s < t \in (0, 0.5]$ ,  $\mu_t = \{x \in X | \mu(x) \geq t > s\} = A$   
hence  $\mu_t$  is a prime ideal. Now if  $\mu$  is not an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ , then there exists some  $a, b \in X$  such that  $\max\{\mu(a), \mu(b)\} < \min\{\mu(a \wedge b), 0.5\}$  hence we can take  $t \in (0, 0.5)$  such that

$$\max\{\mu(a), \mu(b)\} < t < \min\{\mu(a \wedge b), 0.5\} \quad (3.4)$$

therefore  $\mu(a \wedge b) \geq t$  for some  $a, b \in X \Rightarrow a \wedge b \in \mu_t = A$ , a prime ideal therefore  $a \in A$  or  $b \in A \Rightarrow \mu(a) = 1$  or  $\mu(b) = 1$  which contradicts (3.4). Hence  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

**Theorem 3.16** *Let  $\mu$  be a fuzzy set in BCK-algebra  $X$ . Then  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$  iff  $[\mu]_t$  is a prime ideal of  $X$  for all  $t \in (0, 1]$ . We call  $[\mu]_t$  an  $(\in \vee q)$ -level prime ideal of  $\mu$ .*

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ , to prove  $[\mu]_t$  is a prime ideal of  $X$ . Let  $x \wedge y \in [\mu]_t$  for  $t \in (0, 1]$  then  $(x \wedge y)_t \in \vee q \mu$  then  $\mu(x \wedge y) \geq t$  or  $\mu(x \wedge y) + t > 1$  since  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ . Therefore  $\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\}$ . Now we have the following cases:

Case I:  $\mu(x \wedge y) > t$

$$\max\{\mu(x), \mu(y)\} \geq \min\{t, 0.5\}$$

Subcase I:  $t > 0.5$

$$\begin{aligned} \max\{\mu(x), \mu(y)\} &\geq 0.5 \\ \Rightarrow \mu(x) \geq 0.5 \text{ or } \mu(y) &\geq 0.5 \\ \Rightarrow \mu(x) + t > 0.5 + 0.5 = 1 \text{ or } \mu(y) + t > 0.5 + 0.5 = 1 \\ \Rightarrow x_t q \mu \text{ or } y_t q \mu \end{aligned}$$

Subcase II:  $t \leq 0.5$

$$\begin{aligned} \max\{\mu(x), \mu(y)\} &\geq t \\ \Rightarrow \mu(x) \geq t \text{ or } \mu(y) &\geq t \\ \Rightarrow x_t \in \mu \text{ or } y_t \in \mu \end{aligned}$$

Hence  $(x \wedge y)_t \in \vee q \mu \Rightarrow x_t \in \vee q \mu$  or  $y_t \in \vee q \mu$

i.e.  $(x \wedge y)_t \in [\mu]_t \Rightarrow x_t \in [\mu]_t$  or  $y_t \in [\mu]_t$

Case II:  $\mu(x \wedge y) + t > 1$

$$\max\{\mu(x), \mu(y)\} \geq \min\{1 - t, 0.5\}$$

Subcase I:  $t \leq 0.5$

$$\begin{aligned} \max\{\mu(x), \mu(y)\} &\geq 0.5 \geq t \\ \Rightarrow \mu(x) \geq t \text{ or } \mu(y) &\geq t \\ \Rightarrow x_t \in \mu \text{ or } y_t \in \mu \end{aligned}$$

Subcase II:  $t > 0.5$

$$\begin{aligned} \max\{\mu(x), \mu(y)\} &\geq 1 - t \\ \Rightarrow \mu(x) \geq 1 - t \text{ or } \mu(y) &\geq 1 - t \\ \Rightarrow \mu(x) + t \geq 1 \text{ or } \mu(y) + t &\geq 1 \\ \Rightarrow x_t q \mu \text{ or } y_t q \mu \end{aligned}$$

Hence  $(x \wedge y)_t \in \vee q \mu \Rightarrow x_t \in \vee q \mu$  or  $y_t \in \vee q \mu$

i.e.  $(x \wedge y)_t \in [\mu]_t \Rightarrow x_t \in [\mu]_t$  or  $y_t \in [\mu]_t$

Conversely, let  $\mu$  be a fuzzy set in  $X$  and  $t \in (0, 1]$  such that  $[\mu]_t$  is a prime ideal of  $X$ . To prove  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ . If  $\mu$  is not an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ , then there exists  $a, b \in X$  such that  $\max\{\mu(a), \mu(b)\} < \min\{\mu(a \wedge b), 0.5\}$  holds. Choose  $t$  such that

$$\max\{\mu(a), \mu(b)\} < t < \min\{\mu(a \wedge b), 0.5\} \quad (3.5)$$

then  $\mu(a \wedge b) > t \Rightarrow a \wedge b \in \mu_t \subseteq [\mu]_t$  which is a prime ideal  $\Rightarrow a \in [\mu]_t$  or  $b \in [\mu]_t$   
 $\Rightarrow \mu(a) \geq t$  or  $\mu(a) + t > 1$  or  $\mu(b) \geq t$  or  $\mu(b) + t > 1$  which contradicts (3.5). Hence we must have  $\max\{\mu(x), \mu(y)\} \geq \min\{\mu(x \wedge y), 0.5\}$

**Theorem 3.17** Let  $\lambda$  and  $\mu$  be two  $(\in, \in \vee q)$ -fuzzy prime ideals of  $X$  then  $\lambda \cup \mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

*Proof.* Here  $\lambda$  and  $\mu$  be  $(\in, \in \vee q)$ -fuzzy prime ideals of  $X$ . Therefore

$$\begin{aligned} \max\{\lambda(x), \lambda(y)\} &\geq \min\{\lambda(x \wedge y), 0.5\}, \text{ and} \\ \max\{\mu(x), \mu(y)\} &\geq \min\{\mu(x \wedge y), 0.5\}, \quad \forall x, y \in X. \end{aligned} \quad (3.6)$$

To prove  $\lambda \cup \mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ . It is enough to show that

$$\max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} \geq \min\{(\lambda \cup \mu)(x \wedge y), 0.5\} \quad \forall x, y \in X. \quad (3.7)$$

$$\begin{aligned} &\text{Now } \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} \\ &= \max\{\max\{\lambda(x), \mu(x)\}, \max\{\lambda(y), \mu(y)\}\} \\ &= \max\{\max\{\lambda(x), \lambda(y)\}, \max\{\mu(x), \mu(y)\}\} \\ &\geq \max\{\min\{\lambda(x \wedge y), 0.5\}, \min\{\mu(x \wedge y), 0.5\}\} \text{ by (3.6)} \\ &\Rightarrow \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} \geq \max\{\min\{(\lambda(x \wedge y), 0.5), \min\{\mu(x \wedge y), 0.5\}\}\} \end{aligned} \quad (3.8)$$

Now we have the following cases:

Case I:  $\lambda(x \wedge y) \leq 0.5$  and  $\mu(x \wedge y) \leq 0.5$ , then

$$\begin{aligned} (3.8) \Rightarrow \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} &\geq \max\{\lambda(x \wedge y), \mu(x \wedge y)\} \\ &\geq (\lambda \cup \mu)(x \wedge y) \\ &= \min\{(\lambda \cup \mu)(x \wedge y), 0.5\} \end{aligned}$$

Case II:  $\lambda(x \wedge y) \leq 0.5$  and  $\mu(x \wedge y) > 0.5$ , then

$$\begin{aligned} (3.8) \Rightarrow \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} &\geq \max\{\lambda(x \wedge y), 0.5\} = 0.5 \\ &= \min\{\max\{\lambda(x \wedge y), \mu(x \wedge y)\}, 0.5\} \\ &= \min\{(\lambda \cup \mu)(x \wedge y), 0.5\} \end{aligned}$$

Case III:  $\lambda(x \wedge y) > 0.5$  and  $\mu(x \wedge y) \leq 0.5$

$$\begin{aligned} (3.8) \Rightarrow \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} &\geq \max\{0.5, \mu(x \wedge y)\} = 0.5 \\ &= \min\{\max\{\lambda(x \wedge y), \mu(x \wedge y)\}, 0.5\} \\ &= \min\{(\lambda \cup \mu)(x \wedge y), 0.5\} \end{aligned}$$

Case IV:  $\lambda(x \wedge y) > 0.5$  and  $\mu(x \wedge y) > 0.5$ , then

$$\begin{aligned} (3.8) \Rightarrow \max\{(\lambda \cup \mu)(x), (\lambda \cup \mu)(y)\} &\geq \max\{0.5, 0.5\} = 0.5 \\ &= \min\{\max\{\lambda(x \wedge y), \mu(x \wedge y)\}, 0.5\} \\ &= \min\{(\lambda \cup \mu)(x \wedge y), 0.5\} \end{aligned}$$

Hence from above (3.7) hold  $\forall x, y \in X$ .

Therefore  $(\lambda \cup \mu)$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

**Theorem 3.18** Let  $\{\mu_i : i \in \Lambda\}$  be a family of  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ , then  $\mu = \cup\{\mu_i : i \in \Lambda\}$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

**Theorem 3.19** Let  $\lambda$  and  $\mu$  be two  $(\in, \in \vee q)$ -fuzzy prime ideals of  $X$ , then  $\lambda \cap \mu$  may not be an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$  as shown in Example below.

**Example 3.20** Consider a BCK-algebra  $X$  as in Example 3.11 and fuzzy sets  $\lambda$  and  $\mu$  defined by  $\lambda(0) = 0.45, \lambda(x) = \lambda(y) = 0.46, \lambda(z) = 0.2$  and  $\mu(0) = 0.4, \mu(x) = 0.32, \mu(y) = 0.35, \mu(z) = 0.45$ . Then by routine calculations it can be verified that  $\lambda$  and  $\mu$  both are  $(\in, \in \vee q)$ -fuzzy prime ideals  $X$  and also  $(\lambda \cap \mu)(0) = 0.4, (\lambda \cap \mu)(x) = 0.32, (\lambda \cap \mu)(y) = 0.35, (\lambda \cap \mu)(z) = 0.2$ . But  $\lambda \cap \mu$  is not an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$  because  $\max\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(z)\} = \max\{0.32, 0.2\} = 0.32 \not\geq \min\{(\lambda \cap \mu)(x \wedge z), 0.5\} = \min\{(\lambda \cap \mu)(0), 0.5\} = \min\{0.4, 0.5\} = 0.4$ .

### 4 Cartesian product of BCK-algebras and their $(\in, \in \vee q)$ -fuzzy prime ideals

**Theorem 4.1** Let  $X, Y$  be two BCK-algebras, then their cartesian product  $X \times Y = \{(x, y) | x \in X, y \in Y\}$  is also a BCK-algebra under the binary operation  $*$  defined in  $X \times Y$  by  $(x, y) * (p, q) = (x * p, y * q)$  for all  $(x, y), (p, q) \in X \times Y$ .

*Proof.* Straightforward.

**Definition 4.2** Let  $\mu_1$  and  $\mu_2$  be two  $(\in, \in \vee q)$ -fuzzy prime ideals of a BCK-algebra  $X$ . Then their cartesian product  $\mu_1 \times \mu_2$  is defined by  $(\mu_1 \times \mu_2)(x, y) = \max\{\mu_1(x), \mu_2(x)\}$  Where  $(\mu_1 \times \mu_2) : X \times X \rightarrow [0, 1] \quad \forall x, y \in X$ .

**Theorem 4.3** Let  $\mu_1$  and  $\mu_2$  be two  $(\in, \in \vee q)$ -fuzzy prime ideals of a BCK-algebra  $X$ . Then  $\mu_1 \times \mu_2$  is also an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X \times X$ .

*Proof.* Similar to theorem 3.17, just replace  $\cup$  by  $\times$ ,  $\lambda$  by  $\mu_1$  and  $\mu$  by  $\mu_2$ .

### 5 Homomorphism of BCK-algebras and $(\in, \in \vee q)$ -fuzzy prime ideals

**Definition 5.1** Let  $X$  and  $X'$  be two commutative BCK-algebras, then a mapping  $f : X \rightarrow X'$  is said to be a lattice homomorphism if  $f(x \wedge y) = f(x) \wedge f(y) \quad \forall x, y \in X$ .

**Theorem 5.2** Let  $X$  and  $X'$  be two commutative BCK-algebras and  $f : X \rightarrow X'$  be a lattice homomorphism. If  $\mu$  be an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X'$ , then  $f^{-1}(\mu)$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .

*Proof.*  $f^{-1}(\mu)$  is defined as  $f^{-1}(\mu)(x) = \mu(f(x)) \quad \forall x \in X$ . Let  $\mu$  be an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X'$  and  $x, y \in X$  such that  $(x \wedge y)_t \in f^{-1}(\mu)$  then  $f^{-1}(\mu)(x \wedge y) \geq t$   
 $\mu(f(x \wedge y)) \geq t$

$\Rightarrow (f(x \wedge y))_t \in \mu$   
 $\Rightarrow (f(x) \wedge f(y))_t \in \mu$  [ Since  $f$  is a lattice homomorphism. ]  
 $\Rightarrow (f(x))_t \in \vee q\mu$  or  $(f(y))_t \in \vee q\mu$  because  $\mu$  be an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X'$   
 $\Rightarrow \mu(f(x)) \geq t$  or  $\mu(f(x)) + t > 1$  or  $\mu(f(y)) \geq t$  or  $\mu(f(y)) + t > 1$   
 $\Rightarrow f^{-1}(\mu)(x) \geq t$  or  $f^{-1}(\mu)(x) + t > 1$  or  $f^{-1}(\mu)(y) \geq t$  or  $f^{-1}(\mu)(y) + t > 1$   
 $\Rightarrow x_t \in f^{-1}(\mu)$  or  $x_t q f^{-1}(\mu)$  or  $y_t \in f^{-1}(\mu)$  or  $y_t q f^{-1}(\mu)$   
 $\Rightarrow x_t \in \vee q f^{-1}(\mu)$  or  $y_t \in \vee q f^{-1}(\mu)$   
 Therefore  $(x \wedge y)_t \in f^{-1}(\mu) \Rightarrow x_t \in \vee q f^{-1}(\mu)$  or  $y_t \in \vee q f^{-1}(\mu)$   
 Hence  $f^{-1}(\mu)$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$

**Theorem 5.3** Let  $X$  and  $X'$  be two commutative BCK-algebras and  $f : X \rightarrow X'$  be an onto lattice homomorphism. If  $\mu$  be a fuzzy subset of  $X'$  such that  $f^{-1}(\mu)$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ , then  $\mu$  is also an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X'$ .

*Proof.* Let  $x', y' \in X'$  such that  $(x' \wedge y')_t \in \mu$  where  $t \in [0, 1]$  then  $\mu(x' \wedge y') \geq t$  since  $f$  is onto so there exists  $x, y \in X$  such that  $f(x) = x', f(y) = y'$  also  $f$  is lattice homomorphism so

$f(x \wedge y) = f(x) \wedge f(y) = x' \wedge y'$  Now  
 $\mu(x' \wedge y') \geq t \Rightarrow \mu(f(x \wedge y)) \geq t$   
 $\Rightarrow f^{-1}(\mu)(x \wedge y) \geq t$   
 $\Rightarrow (x \wedge y)_t \in f^{-1}(\mu)$   
 $\Rightarrow x_t \in \vee q f^{-1}(\mu)$  or  $y_t \in \vee q f^{-1}(\mu)$

[since  $f^{-1}(\mu)$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X$ .]

$\Rightarrow f^{-1}(\mu)(x) \geq t$  or  $f^{-1}(\mu)(x) + t > 1$  or  $f^{-1}(\mu)(y) \geq t$  or  $f^{-1}(\mu)(y) + t > 1$   
 $\Rightarrow \mu(x') \geq t$  or  $\mu(x') + t > 1$  or  $\mu(y') \geq t$  or  $\mu(y') + t > 1$   
 $\Rightarrow x'_t \in \mu$  or  $x'_t q \mu$  or  $y'_t \in \mu$  or  $y'_t q \mu$   
 $\Rightarrow x'_t \in \vee q \mu$  or  $y'_t \in \vee q \mu$   
 therefore  $(x' \wedge y')_t \in \mu \Rightarrow x'_t \in \vee q \mu$  or  $y'_t \in \vee q \mu$   
 Hence  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy prime ideal of  $X'$ .

## 6 Conclusion

In this paper, we have studied  $(\in, \in \vee q)$ -fuzzy prime ideals in commutative BCK-algebras and investigated some of their useful properties. In my opinion, these definitions and results can be extended to other algebraic systems also. In the notions of  $(\alpha, \beta)$ -fuzzy ideals we can define twelve different types of ideals by three choices of  $\alpha$  and four choices of  $\beta$ . we can apply  $(\in, \in \vee q)$ -fuzzy prime ideals is in the field of fuzzy medical diagnosis, artificial intelligence, information science, agriculture etc.

In future, the following studies may be carried out:

(1)  $(\in, \in \vee q)$ -weakly prime and weakly semiprime fuzzy ideals.

(2)  $(\in, \in \vee q)$ -fuzzy prime ideals of lattice.

(3)  $(\in, \in \vee q)$ -fuzzy prime and irreducible ideals in BCK-algebras.

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**Saidur Rahman Barbhuiya** is an Assistant Professor at Department of Mathematics, Srikishan Sarda College, Hailakandi, Assam, India. He received his M. Sc. degree from Assam university, Silchar and submitted Ph.D. thesis in Fuzzy algebra to Assam

university, Silchar. His research interests include Fuzzy sets, Fuzzy algebraic structures and their applications.