

A New Extension of Lomax Distribution: Properties, Inference, Simulation, and Applications

Hazar A. Khoeger¹, Mustafa Kamal², Yousef Alharbi³, Eslam Hussam^{4,5}, Mohamed Said Mohamed⁶, Ahmad Abubakar Suleiman⁷, Maryam Ibrahim Habadi⁸, and Mohammed Elgarhy^{9,10,*}

¹ Department of Mathematics, College of Science, Umm Al-Qura University, Makkah, Saudi Arabia

² Department of Basic Sciences, College of Science and Theoretical Studies, Prince Musaid bin Abdulrahman University, Dammam 32256, Saudi Arabia

³ Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

⁴ Department of Accounting, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim, Saudi Arabia

⁵ Department of Mathematics, Faculty of Science, Capital University, Cairo, Egypt

⁶ Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim 16511, Saudi Arabia

⁷ Department of Statistics, Faculty of Computing and Mathematical Sciences, Aliko Dangote University of Science and Technology, Wudil 713281, Nigeria

⁸ Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

⁹ Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt

¹⁰ Department of Computer Engineering, Biruni University, 34010 Istanbul, Turkey

Received: 15 Apr. 2026, Revised: 12 Jul. 2026, Accepted: 12 Oct. 2026

Published online: 1 Nov. 2026

Abstract: Lifetime data arising in reliability engineering, survival analysis and risk modeling are often skewed and heavy-tailed, and their tails are not always well described by classical lifetime models. We propose a two-parameter model, the Entropy-Transformed Lomax (ETLx) distribution, obtained by applying an entropy-based transformation to the Lomax distribution. Most Lomax generalizations gain flexibility by adding shape parameters; the ETLx model keeps the two original parameters and can still accommodate positively skewed, heavy-tailed data. We derive explicit expressions for the density, distribution, survival and hazard rate functions, the quantile function (written with the Lambert W function), the ordinary and incomplete moments, the moment generating function, skewness, kurtosis, and the Rényi and Tsallis entropies. The parameters are estimated by maximum likelihood, and a Monte Carlo study is used to check how the estimators behave in finite samples. Three lifetime datasets are then fitted with the ETLx model and with the Lomax, Weibull, Gamma, Lognormal and Exponential distributions. The fits are compared through the log-likelihood, the Akaike (AIC), Bayesian (BIC), Hannan–Quinn (HQIC) and consistent Akaike (CAIC) information criteria, and the Kolmogorov–Smirnov (KS), Cramér–von Mises (CVM) and Anderson–Darling (AD) statistics. For all three datasets the ETLx model gave the smallest information criteria and the best goodness-of-fit values, which suggests that it is a sensible choice for heavy-tailed lifetime data.

Keywords: Entropy transformation; Lomax distribution; Lifetime distribution; Lambert W function; Hazard rate function; Model selection; Goodness-of-fit

1 Introduction

Statistical models for lifetime and reliability data are needed in engineering, medicine, economics and the environmental sciences. Data of this kind are frequently skewed and have heavier tails than the classical

distributions allow [16], which has led to a large body of work on more flexible lifetime models [11, 20, 24]. Recent examples include odds-based Weibull models, extended Lindley models and one-parameter models with a decreasing hazard rate [4, 28, 44].

* Corresponding author e-mail: Mohammed.Elgarhy@ecu.edu.eg

One way to make a probability model more flexible is to borrow ideas from information theory, in particular entropy. The concept goes back to Clausius [17] in thermodynamics, and Shannon [50] gave it its information-theoretic form. In statistics, entropy measures the uncertainty, or information, carried by a distribution, and it appears in many inferential problems [5, 26, 34, 42, 51]. In reliability work, the Shannon and Rényi entropies of a failure-time distribution tell us how predictable failures are: the larger the entropy, the less predictable the failure pattern. Arimoto and Tsallis entropies are used for non-extensive systems and in signal processing, for instance to bound the bit rate needed for lossless compression. Entropy measures also act as diversity and complexity indices in ecology and physics, where they help separate the stages of a process according to how random the observations are [32]. Apart from measuring uncertainty, entropy can be used as an operator that produces new families of distributions: building it into a distribution changes its shape and can make it better suited to complicated data. Isaic-Maniu and Voda [29] proposed a transformation of this type, namely

$$\ell(t) = G(t) + R(t) \ln R(t), \quad (1)$$

where $R(t) = 1 - G(t)$ is the survival function of the baseline distribution $G(t)$. Differentiating (1) with respect to t gives the density

$$f^*(t) = R'(t) \ln R(t). \quad (2)$$

The transformation has several useful features. It keeps the number of parameters of the baseline model, so nothing is added to the parameter vector. It nevertheless makes the baseline distribution more flexible, widening the range of shapes and tail behaviors it can produce. It changes the hazard structure, so that different failure-rate shapes can be modeled. Finally, because the survival function enters through the factor $R(t) \ln R(t)$, the new model has a log-weighted survival behavior, which is useful when survival decays logarithmically.

Equation (1) is therefore a simple device for generating a new distribution from an existing one, and the corresponding density (2) can be fitted to many kinds of data. For these reasons the transformation is attractive in applied fields such as engineering, economics and biology.

The Lomax distribution is a standard heavy-tailed model. It has been used in reliability and survival analysis, risk assessment, queueing theory and finance [1, 9, 12, 22, 25, 52, 55, 59]. Inference for related lifetime models under progressive, hybrid and step-stress censoring, and for stress-strength reliability with censored and record data, is studied in [3, 6–8, 21, 45]. For a shape parameter $\alpha > 0$ and a scale parameter $\lambda > 0$, the main functions of the Lomax distribution are as follows.

The cumulative distribution function (CDF) is

$$G(t; \alpha, \lambda) = 1 - \left(1 + \frac{t}{\lambda}\right)^{-\alpha}, \quad t > 0. \quad (3)$$

The probability density function (PDF) is

$$g(t; \alpha, \lambda) = \frac{\alpha}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)}. \quad (4)$$

The survival function is

$$R(t) = \left(1 + \frac{t}{\lambda}\right)^{-\alpha}. \quad (5)$$

The hazard rate function is

$$h(t) = \frac{\alpha}{\lambda + t}. \quad (6)$$

Finally, for $0 < r < \alpha$, the r th moment is

$$E[T^r] = \lambda^r \frac{\Gamma(\alpha - r) \Gamma(1 + r)}{\Gamma(\alpha)}. \quad (7)$$

Because its survival function decays as a power law, the Lomax distribution is a natural candidate for extreme events. It can still be too rigid for the variety of tail behaviors and shapes seen in practice, and many extensions have been suggested to make it more flexible [19, 38, 39, 48, 49]. Among them are the beta Lomax [36], transmuted Lomax [15], Marshall–Olkin alpha power Lomax [10], gamma Lomax [18], Weibull Lomax [57], Lomax exponential [27], Gumbel Lomax [58], transmuted Weibull Lomax [2], weighted Lomax [33], half-logistic Lomax [14], extended odd Weibull power Lomax [13], odd Lomax [19], logistic Lomax [60], odd Lomax generalized exponential [47], truncated exponential Marshall–Olkin Lomax [23], log-Lomax [31] and Kumaraswamy modified Kies power Lomax [56] distributions.

Even with this literature, there is room for models that are flexible and yet easy to handle, both mathematically and numerically. The Lomax distribution has served well in reliability, actuarial science, economics and survival analysis because of its heavy tail, but its simple form cannot always reproduce what modern lifetime data show: tails that change from one dataset to another, several failure mechanisms acting together, and different amounts of positive skewness [40, 54].

Most Lomax extensions gain flexibility by adding shape parameters through beta, Kumaraswamy, Marshall–Olkin, transmuted, exponentiated or other generators. The extra parameters help the fit, but they make inference harder, raise the computational cost and can make numerical estimation unstable [41]. Extensions that add flexibility without adding parameters are therefore worth looking for.

The entropy transformation is one way to obtain such an extension. It does not introduce new parameters; it uses the information in the baseline survival function, through $R(t) \ln R(t)$, to change both the density and the reliability functions. The resulting models stay small but

can represent different degrees of skewness, tail weight and hazard shape.

Our interest in this approach comes from the fact that many lifetime datasets are highly variable and heavy-tailed in ways that classical distributions do not always reproduce. In reliability and survival studies, heavy tails often result from mixed failure mechanisms, changing operating conditions, manufacturing defects or unobserved risk factors [30]. Entropy-based transformations can account for such features while keeping closed-form distribution functions and a tractable likelihood [35, 46, 53].

In this paper we apply the entropy transformation to the Lomax distribution and call the result the Entropy-Transformed Lomax (ETLx) distribution. The aim is a model that is more flexible than the Lomax distribution but still has two parameters and simple formulas. We study its distributional properties, moments, entropy measures, reliability functions and quantile function, estimate its parameters by maximum likelihood, and check the estimators by Monte Carlo simulation. We then fit the model to three generated lifetime datasets and compare it with several standard lifetime distributions.

Generator methods usually change the CDF of the baseline model by means of extra generator parameters. The entropy transformation works differently: it reweights the baseline density by the logarithm of the survival function. The density, survival and hazard functions all change, yet the model keeps only the two Lomax parameters, so the gain in flexibility does not cost tractability or computing effort.

The model has practical advantages as well. Its quantile function has a closed form in terms of the Lambert W function, so random numbers are easy to generate and simulation is straightforward. The moments, incomplete moments and entropy measures can all be written explicitly. Together with the simulation and data results reported below, this makes the ETLx distribution a convenient option for heavy-tailed lifetime data.

In practice, the model is intended for reliability and lifetime data with heavy tails, positive skewness and mixed failure mechanisms. In our simulations the maximum likelihood estimators behaved well in finite samples, and for the three datasets considered the ETLx distribution fitted better than the Weibull, Lomax, Gamma, Lognormal and Exponential distributions. The model may therefore be useful in reliability engineering, survival analysis, actuarial science and risk modeling.

The main contributions of the paper are the following.

- i. A new two-parameter model, the ETLx distribution, is introduced to make the Lomax distribution more flexible.
- ii. Its main mathematical properties are derived: the density, distribution, survival and hazard functions, the quantiles, the moments and the entropy measures.

iii. Maximum likelihood estimators of the parameters are obtained, and their performance is examined by Monte Carlo simulation.

iv. It is shown that the model can describe different hazard-rate shapes.

v. The model is fitted to three datasets and compared with several classical lifetime distributions.

The rest of the paper is organized as follows. Section 2 defines the ETLx distribution and gives its distribution and reliability functions. Section 3 derives its main statistical properties, and Section 4 deals with maximum likelihood estimation. The simulation results are given in Section 5, and Section 6 presents three data applications. Section 7 concludes the paper and suggests topics for further work.

2 Construction of the Entropy-Transformed Lomax Distribution

The ETLx distribution is obtained from the entropy transformation

$$f_{ETLx}(t) = R'(t) \ln R(t), \quad (8)$$

where $R(t)$ denotes the survival function of the Lomax distribution.

From (5), the derivative of $R(t)$ appearing in (8) is

$$R'(t) = -\frac{\alpha}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)}, \quad (9)$$

and the logarithm of the survival function is

$$\ln R(t) = -\alpha \ln \left(1 + \frac{t}{\lambda}\right). \quad (10)$$

Inserting (9) and (10) into (8) gives the PDF of the ETLx distribution:

$$f_{ETLx}(t) = \frac{\alpha^2}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)} \ln \left(1 + \frac{t}{\lambda}\right), \quad (11)$$

$$t > 0, \alpha > 0, \lambda > 0.$$

For (11) to be a proper density, it must integrate to one over the support, that is,

$$\int_{-\infty}^{\infty} f_{ETLx}(t) dt = 1. \quad (12)$$

Substituting (11) into (12) gives

$$\int_0^{\infty} \frac{\alpha^2}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)} \ln \left(1 + \frac{t}{\lambda}\right) dt. \quad (13)$$

Let $u = 1 + \frac{t}{\lambda}$, $dt = \lambda du$. Then (13) becomes

$$\alpha^2 \int_1^{\infty} u^{-(\alpha+1)} \ln(u) du. \quad (14)$$

Next, let $v = \ln u$, $u = e^v$, $du = e^v dv$ so that (14) becomes

$$\alpha^2 \int_0^\infty v e^{-\alpha v} dv. \quad (15)$$

Using the gamma integral $\int_0^\infty v e^{-\alpha v} dv = \frac{1}{\alpha^2}$ in (15), we obtain $\alpha^2 \times \frac{1}{\alpha^2} = 1$.

which proves (12). Hence (11) is a proper probability density function. The CDF of the ETLx distribution is

$$F_{ETLx}(t) = 1 - \left[1 + \alpha \ln \left(1 + \frac{t}{\lambda} \right) \right] \left(1 + \frac{t}{\lambda} \right)^{-\alpha}, \quad (16)$$

$t > 0, \alpha > 0, \lambda > 0$.

Figure 1 shows the PDF for several parameter values. The density is unimodal and right-skewed, and both the degree of asymmetry and the weight of the tail change noticeably with α and λ . This is what makes the model suitable for skewed and heavy-tailed data.

Figure 2 shows the corresponding CDFs. The parameters control how fast probability accumulates: for some combinations the CDF rises steeply and approaches one early, while for others it climbs slowly, so the model can describe both short and long lifetimes. In the second case the tail is heavy, which means that extreme values become rare much more slowly than they would under a normal distribution.

The survival function of the ETLx distribution is

$$S_{ETLx}(t) = \left[1 + \alpha \ln \left(1 + \frac{t}{\lambda} \right) \right] \left(1 + \frac{t}{\lambda} \right)^{-\alpha}. \quad (17)$$

and its hazard rate function is

$$h_{ETLx}(t) = \frac{\alpha^2 \ln(1 + t/\lambda)}{(\lambda + t)[1 + \alpha \ln(1 + t/\lambda)]}. \quad (18)$$

To study the shape of the hazard rate, put

$$u = 1 + \frac{t}{\lambda}, \quad u > 1.$$

so that

$$h(u) = \frac{\alpha^2}{\lambda} \frac{\ln(u)}{u(1 + \alpha \ln(u))}.$$

Differentiating with respect to u gives

$$\frac{dh(u)}{du} = \frac{\alpha^2}{\lambda} \frac{(1 - \ln u)(1 + \alpha \ln u) - \alpha \ln u}{u^2(1 + \alpha \ln u)^2}.$$

and, after simplifying the numerator,

$$\frac{dh(u)}{du} = \frac{\alpha^2}{\lambda} \frac{1 - \ln u - \alpha (\ln u)^2}{u^2(1 + \alpha \ln u)^2}.$$

Because $u > 1$, we have $\ln u > 0$ and the denominator is positive, so the sign of the derivative is the sign of

$$1 - \ln u - \alpha (\ln u)^2.$$

As a function of $\ln u$, this expression equals 1 at $u = 1$ and decreases to $-\infty$, so it changes sign exactly once, at $\ln u^* = (\sqrt{1 + 4\alpha} - 1)/(2\alpha)$. Writing $t^* = \lambda[\exp((\sqrt{1 + 4\alpha} - 1)/(2\alpha)) - 1]$, we therefore have

$$h'(t) > 0 \quad \text{for } t < t^*, \quad h'(t) < 0 \quad \text{for } t > t^*.$$

The ETLx hazard rate is thus unimodal (upside-down bathtub): it rises over a short initial period, reaches its maximum at t^* , and then decreases towards zero. Since t^* is small unless λ is large, the model mainly describes early failures followed by a steadily falling failure intensity.

Figure 3 shows survival functions for several combinations of α and λ . Some curves decrease slowly, so that the probability of long survival remains high, while others drop sharply at the start, which corresponds to short lifetimes. The shape parameter α mainly controls how steep the curve is, and the scale parameter λ controls how far it stretches along the time axis. The model can therefore describe residual lifetimes under quite different conditions, which matters in actuarial and engineering work.

Figure 4 shows the hazard rate function. From this figure we can note that the hazard rate function is up-side-down.

Table 1 gives numerical values of the main functions of the ETLx distribution for several pairs (α, λ) and time points x . The PDF values show how the shape changes: a larger α puts more probability mass at small x . With $\alpha = 3.0$ and $\lambda = 1.0$, for instance, the PDF equals 0.7208 at $x = 0.5$ and falls to 0.0124 at $x = 5.0$, whereas with $\alpha = 0.5$ and $\lambda = 1.0$ it is much smaller and changes little over the same range.

The two parameters play different roles. With λ fixed, a larger α makes the survival function fall faster and raises the hazard at early times, so the distribution becomes more concentrated. A larger λ simply rescales the time axis: the mass spreads over a wider range, the PDF becomes lower and the CDF rises more slowly.

The values in Table 1 were computed for representative combinations of α , λ and t in order to show how each parameter affects these functions.

2.1 Quantile Function

The quantile function $Q(u)$, $0 < u < 1$, is the solution in t of

$$F_{ETLx}(t) = u, \quad 0 < u < 1. \quad (19)$$

Inserting the CDF (16) into (19) gives

ETLx Probability Density Function Shapes for Various Parameter Sets

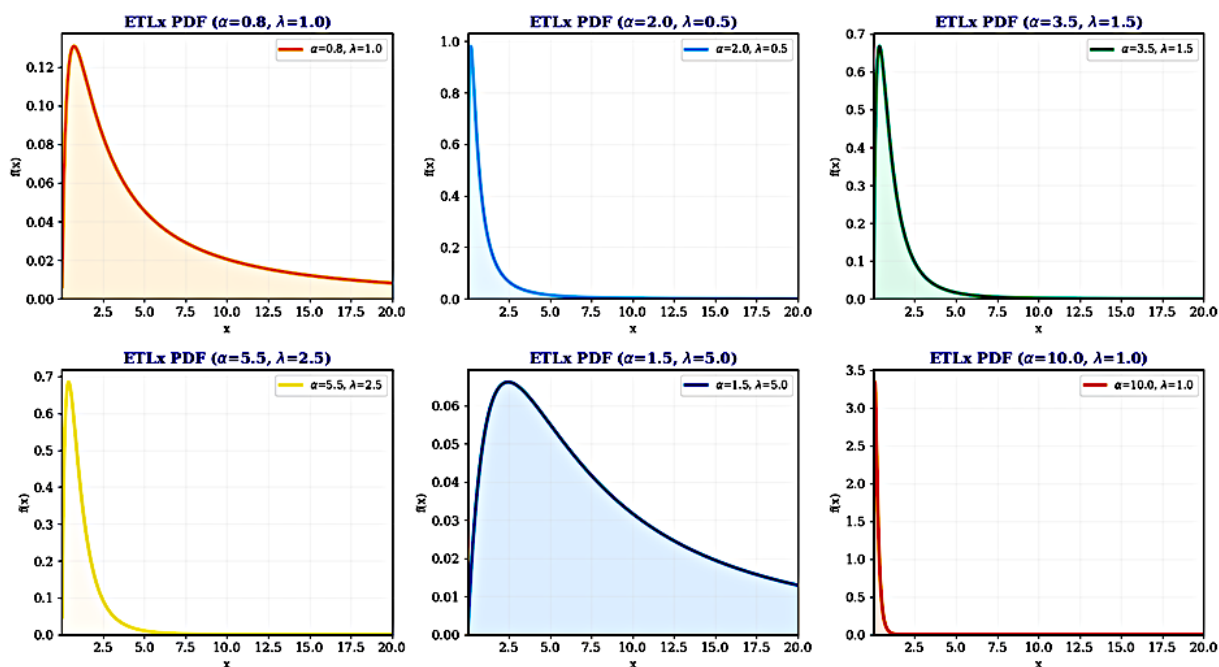


Fig. 1: PDF of the ETLx distribution for selected values of α and λ .

ETLx Cumulative Distribution Function Shapes

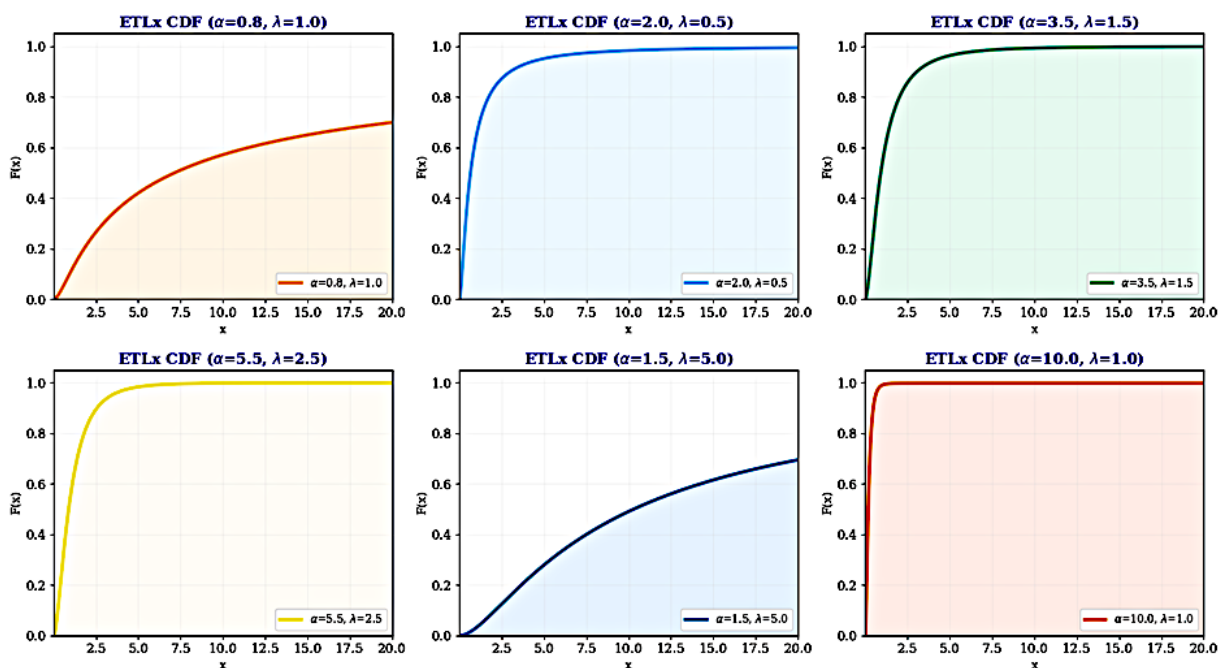


Fig. 2: CDF of the ETLx distribution for selected values of α and λ .

Survival Function Shapes for the ETLx Distribution

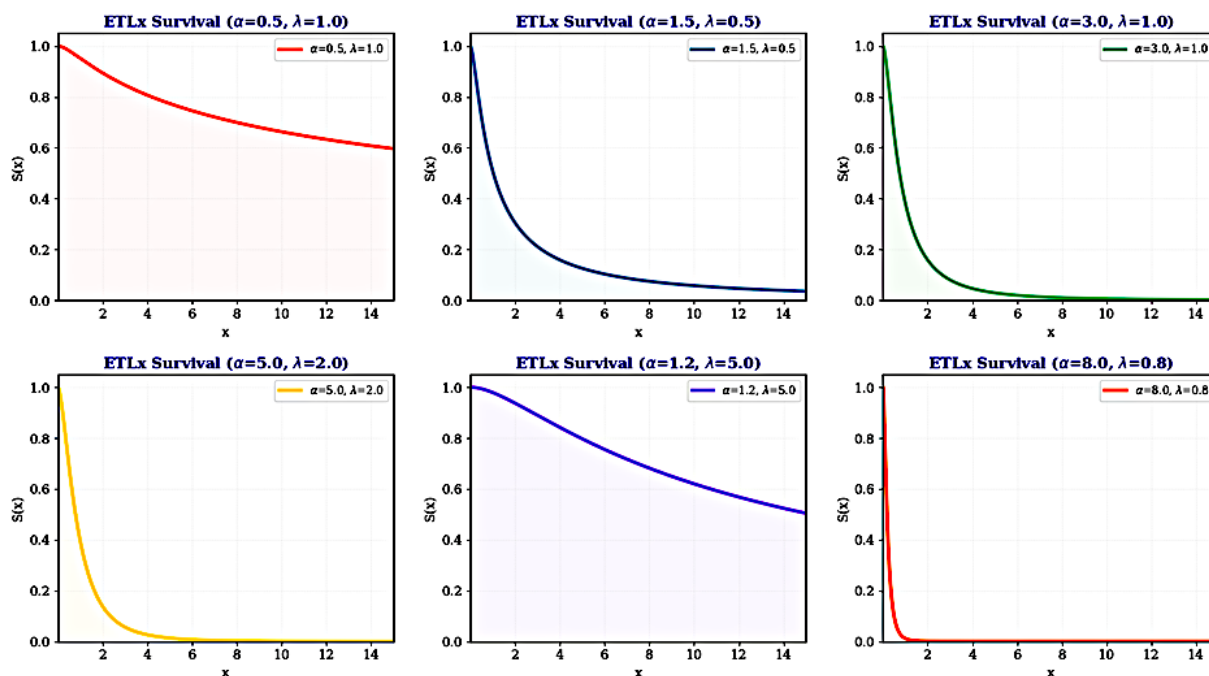


Fig. 3: Survival function of the ETLx distribution for selected values of α and λ .

Hazard Rate Function Shapes for the ETLx Distribution

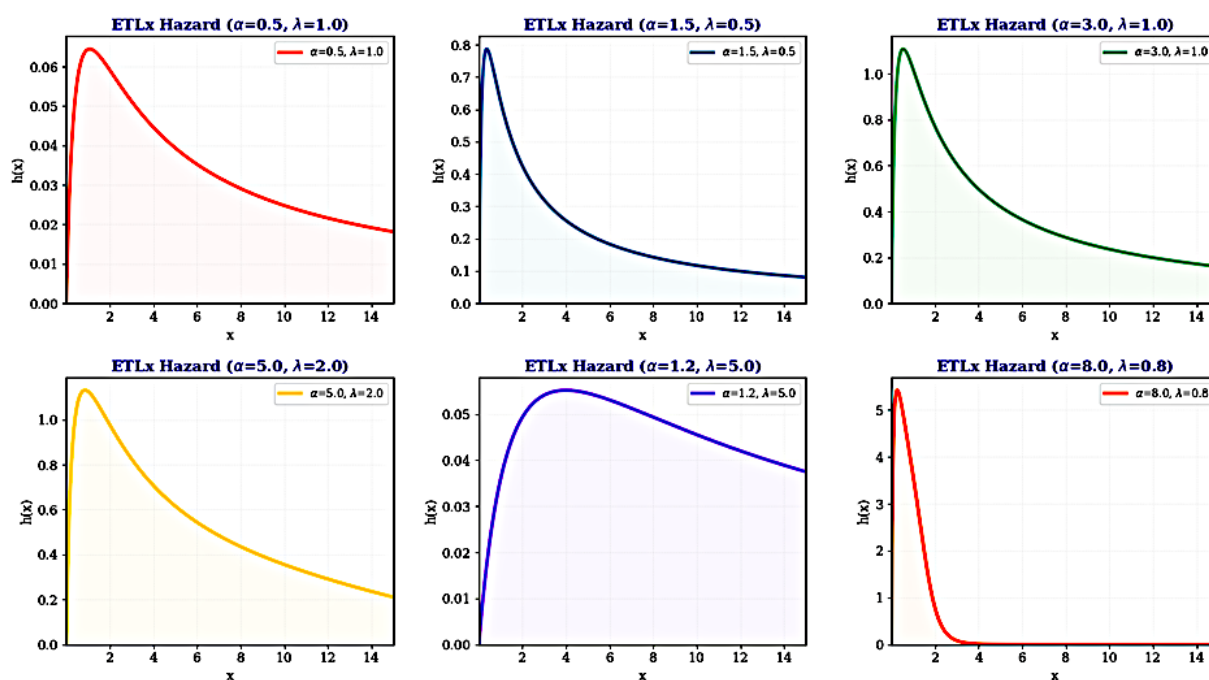


Fig. 4: Hazard rate function of the ETLx distribution for selected values of α and λ .

Table 1: PDF, CDF, survival function and hazard rate function of the ETLx distribution for selected (α, λ) and several values of t .

α	λ	t	PDF	CDF	Survival	Hazard
0.5	1	0.5	0.055177	0.017973	0.982027	0.056187
0.5	1	1	0.061266	0.047829	0.952171	0.064344
0.5	1	1.5	0.057951	0.077788	0.922212	0.062839
0.5	1	2	0.052857	0.105508	0.894492	0.059092
0.5	1	2.5	0.047831	0.130662	0.869338	0.05502
0.5	1	3	0.043322	0.153426	0.846574	0.051173
0.5	1	3.5	0.03939	0.174081	0.825919	0.047693
0.5	1	4	0.035988	0.192905	0.807095	0.04459
0.5	1	4.5	0.033041	0.210145	0.789855	0.041832
0.5	1	5	0.030478	0.22601	0.77399	0.039378
0.5	2	0.5	0.019959	0.00578	0.99422	0.020075
0.5	2	1	0.027588	0.017973	0.982027	0.028093
0.5	2	1.5	0.030216	0.032556	0.967444	0.031233
0.5	2	2	0.030633	0.047829	0.952171	0.032172
0.5	2	2.5	0.030034	0.063023	0.936977	0.032055
0.5	2	3	0.028976	0.077788	0.922212	0.03142
0.5	2	3.5	0.027728	0.091968	0.908032	0.030536
0.5	2	4	0.026429	0.105508	0.894492	0.029546
0.5	2	4.5	0.025146	0.1184	0.8816	0.028523
0.5	2	5	0.023915	0.130662	0.869338	0.02751
1.5	1	0.5	0.331061	0.124608	0.875392	0.378186
1.5	1	1	0.275698	0.27885	0.72115	0.382303
1.5	1	1.5	0.208625	0.399309	0.600691	0.347308
1.5	1	2	0.158571	0.490407	0.509593	0.311172
1.5	1	2.5	0.122993	0.560294	0.439706	0.279717
1.5	1	3	0.097474	0.615068	0.384932	0.253224
1.5	1	3.5	0.078781	0.658898	0.341102	0.23096
1.5	1	4	0.064779	0.694625	0.305375	0.212128
1.5	1	4.5	0.054067	0.724221	0.275779	0.196054
1.5	1	5	0.045718	0.749083	0.250917	0.182203
1.5	2	0.5	0.143702	0.044955	0.955045	0.150466
1.5	2	1	0.16553	0.124608	0.875392	0.189093
1.5	2	1.5	0.155399	0.205444	0.794556	0.195579
1.5	2	2	0.137849	0.27885	0.72115	0.191151
1.5	2	2.5	0.120138	0.34329	0.65671	0.182939
1.5	2	3	0.104312	0.399309	0.600691	0.173654
1.5	2	3.5	0.090747	0.447981	0.552019	0.16439
1.5	2	4	0.079286	0.490407	0.509593	0.155586
1.5	2	4.5	0.069636	0.527568	0.472432	0.147398
1.5	2	5	0.061497	0.560294	0.439706	0.139858
3	1	0.5	0.720827	0.34329	0.65671	1.097634
3	1	1	0.389895	0.615069	0.384931	1.012897
3	1	1.5	0.211113	0.76007	0.23993	0.879897
3	1	2	0.122068	0.840892	0.159108	0.767202

$$\left(1 + \frac{t}{\lambda}\right)^{-\alpha} \left[1 + \alpha \ln \left(1 + \frac{t}{\lambda}\right)\right] = 1 - u. \quad (20)$$

$$y^{-\alpha} (1 + \alpha \ln y) = 1 - u. \quad (22)$$

Now let

To simplify (20), put

$$x = \ln y, \quad y = e^x. \quad (23)$$

$$y = 1 + \frac{t}{\lambda}. \quad (21)$$

so that (22) becomes

Substituting (21) into (20) and simplifying gives

$$e^{-\alpha x} (1 + \alpha x) = 1 - u. \quad (24)$$

To bring (24) into Lambert W form, set $s = 1 + \alpha x$:

$$x = \frac{s-1}{\alpha}. \quad (25)$$

Using (25) in (24) and simplifying, we obtain

$$se^{-s} = \frac{1-u}{e}.$$

Multiplying both sides by -1 gives the standard Lambert form

$$(-s)e^{-s}(-1) = -\frac{1-u}{e}.$$

With $z = -s$ this reads

$$ze^z = -\frac{1-u}{e}. \quad (26)$$

which is the defining equation of the Lambert W function, because

$$W(a)e^{W(a)} = a.$$

Hence $z = W\left(-\frac{1-u}{e}\right)$. Returning to $s = -z$,

$$s = -W\left(-\frac{1-u}{e}\right).$$

Solving (25) for x gives

$$x = \frac{-W\left(-\frac{1-u}{e}\right) - 1}{\alpha}. \quad (27)$$

and, since $y = e^x$ by (23),

$$y = \exp\left(\frac{-W\left(-\frac{1-u}{e}\right) - 1}{\alpha}\right). \quad (28)$$

Finally, using $t = \lambda(y-1)$ from (21) in (28), we obtain

$$Q_{ETLx}(u) = \lambda \left[\exp\left(\frac{-W_{-1}\left(-\frac{1-u}{e}\right) - 1}{\alpha}\right) - 1 \right], \quad (29)$$

$$0 < u < 1.$$

Equation (29) gives the quantile function of the ETLx distribution in closed form. For $0 < u < 1$ the argument $-\frac{1-u}{e}$ lies in $(-\frac{1}{e}, 0)$, where the Lambert function has two real branches. Positive quantiles come from the lower branch $W_{-1}(\cdot)$, and this branch was used for all quantiles in Table 2 and for all random numbers generated in the Monte Carlo study. Setting $u = 0.5$ gives the median:

$$t_{0.5} = \lambda \left[\exp\left(\frac{-W_{-1}\left(-\frac{0.5}{e}\right) - 1}{\alpha}\right) - 1 \right]. \quad (30)$$

Because the quantile function is explicit, random numbers can be generated directly by inversion and percentiles can

be computed exactly, without solving $F(t) = u$ numerically. Table 2 and Figure 5 show how the quantiles depend on the parameters. The tail is governed mainly by α , and small values give very large upper quantiles: for $\alpha = 0.5$ and $\lambda = 1$ the 0.95 quantile is 13,195.79, about 476 times the median (27.69). For larger α the distribution is far more concentrated; at $\alpha = 5.5$ the 0.95 quantile is only 1.369. The scale parameter λ multiplies all quantiles by the same factor and leaves the shape unchanged. For fixed α (e.g. $\alpha = 1.056$), moving from $\lambda = 1$ to $\lambda = 3$ triples every quantile.

Figure 5 plots the quantile function. The curves are close together for $p < 0.5$ but separate quickly above the median, and for small α they rise steeply as $p \rightarrow 1$, as expected for a heavy-tailed model.

3 Mathematical Properties of the ETLx Distribution

3.1 Moments of the ETLx Distribution

The r th raw moment of a continuous random variable T is

$$u'_r = E[T^r] = \int_0^\infty t^r f(t) dt, \quad (31)$$

where $f(t)$ denotes the ETLx density. Using the density (11), this becomes

$$u'_r = \int_0^\infty t^r \frac{\alpha^2}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)} \ln\left(1 + \frac{t}{\lambda}\right) dt. \quad (32)$$

To evaluate the integral, we change variables:

$$y = 1 + \frac{t}{\lambda}. \quad (33)$$

so that

$$t = \lambda(y-1), \quad dt = \lambda dy. \quad (34)$$

and

$$t^r = \lambda^r (y-1)^r. \quad (35)$$

The integral converges only when $\alpha > r$. Substituting (33)–(35) into (32) gives

$$u'_r = \int_{y=1}^\infty \lambda^r (y-1)^r \frac{\alpha^2}{\lambda} y^{-(\alpha+1)} \ln y \lambda dy.$$

After collecting the constants,

$$u'_r = \alpha^2 \lambda^r \int_1^\infty (y-1)^r y^{-(\alpha+1)} \ln y dy. \quad (36)$$

By the binomial theorem,

Table 2: Selected quantiles of the ETLx distribution for several values of α and λ .

α	λ	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.75$	$p = 0.9$	$p = 0.95$
0.5	0.5	0.948424	2.919213	13.84709	108.5844	1194.968	6597.896
0.5	1	1.896848	5.838426	27.69417	217.1688	2389.936	13195.79
0.5	1.5	2.845272	8.757638	41.54126	325.7532	3584.905	19793.69
0.5	2	3.793696	11.67685	55.38834	434.3376	4779.873	26391.58
0.5	2.5	4.74212	14.59606	69.23543	542.922	5974.841	32989.48
0.5	3	5.690544	17.51528	83.08251	651.5064	7169.809	39587.38
1.056	0.5	0.327517	0.743013	1.951906	5.909426	19.42252	44.24771
1.056	1	0.655034	1.486025	3.903812	11.81885	38.84505	88.49542
1.056	1.5	0.982551	2.229038	5.855719	17.72828	58.26757	132.7431
1.056	2	1.310068	2.97205	7.807625	23.63771	77.69009	176.9908
1.056	2.5	1.637585	3.715063	9.759531	29.54713	97.11261	221.2385
1.056	3	1.965102	4.458076	11.71144	35.45656	116.5351	265.4863
1.611	0.5	0.195547	0.408018	0.917062	2.159514	5.091021	9.000272
1.611	1	0.391093	0.816035	1.834123	4.319027	10.18204	18.00054
1.611	1.5	0.58664	1.224053	2.751185	6.478541	15.27306	27.00082
1.611	2	0.782187	1.632071	3.668246	8.638054	20.36408	36.00109
1.611	2.5	0.977733	2.040089	4.585308	10.79757	25.4551	45.00136
1.611	3	1.17328	2.448106	5.50237	12.95708	30.54613	54.00163
2.167	0.5	0.139099	0.279206	0.584886	1.232572	2.510506	3.965266
2.167	1	0.278198	0.558412	1.169771	2.465145	5.021013	7.930532
2.167	1.5	0.417297	0.837617	1.754657	3.697717	7.531519	11.8958
2.167	2	0.556397	1.116823	2.339542	4.930289	10.04203	15.86106
2.167	2.5	0.695496	1.396029	2.924428	6.162862	12.55253	19.82633
2.167	3	0.834595	1.675235	3.509313	7.395434	15.06304	23.7916
2.722	0.5	0.107874	0.211753	0.42625	0.844448	1.587004	2.356214
2.722	1	0.215748	0.423506	0.852499	1.688897	3.174008	4.712428
2.722	1.5	0.323622	0.635259	1.278749	2.533345	4.761012	7.068642
2.722	2	0.431496	0.847012	1.704998	3.377794	6.348017	9.424856
2.722	2.5	0.53937	1.058765	2.131248	4.222242	7.935021	11.78107
2.722	3	0.647243	1.270518	2.557498	5.066691	9.522025	14.13728
3.278	0.5	0.088076	0.170403	0.334344	0.636933	1.138117	1.625769
3.278	1	0.176151	0.340807	0.668689	1.273867	2.276233	3.251538
3.278	1.5	0.264227	0.51121	1.003033	1.9108	3.41435	4.877306
3.278	2	0.352303	0.681614	1.337377	2.547733	4.552466	6.503075
3.278	2.5	0.440379	0.852017	1.671722	3.184666	5.690583	8.128844
3.278	3	0.528454	1.02242	2.006066	3.8216	6.828699	9.754613
3.833	0.5	0.074409	0.142506	0.27467	0.509322	0.879281	1.223544
3.833	1	0.148818	0.285012	0.549341	1.018645	1.758562	2.447088
3.833	1.5	0.223227	0.427519	0.824011	1.527967	2.637843	3.670632
3.833	2	0.297636	0.570025	1.098682	2.03729	3.517124	4.894176
3.833	2.5	0.372045	0.712531	1.373352	2.546612	4.396406	6.11772
3.833	3	0.446454	0.855037	1.648023	3.055934	5.275687	7.341265
4.389	0.5	0.06441	0.122432	0.232905	0.423453	0.713026	0.973637
4.389	1	0.128819	0.244863	0.46581	0.846906	1.426052	1.947274
4.389	1.5	0.193229	0.367295	0.698715	1.270359	2.139078	2.92091

$$(y-1)^r = \sum_{k=0}^r \binom{r}{k} y^k (-1)^{r-k}. \quad (37)$$

Using (37) in (36) gives

$$u'_r = \alpha^2 \lambda^r \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} \int_1^{\infty} y^{k-\alpha-1} \ln y dy. \quad (38)$$

Evaluating the remaining integral and simplifying, we obtain the r th raw moment of the ETLx distribution:

$$u'_r = \alpha^2 \lambda^r \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} \frac{1}{(\alpha-k)^2}. \quad (39)$$

This moment exists only for $\alpha > r$.

Taking $r = 1, 2, 3, 4$ in (39) gives the first four raw moments. For $r = 1$ we obtain the mean,

ETLx Quantile Function Shapes (Percentile points)

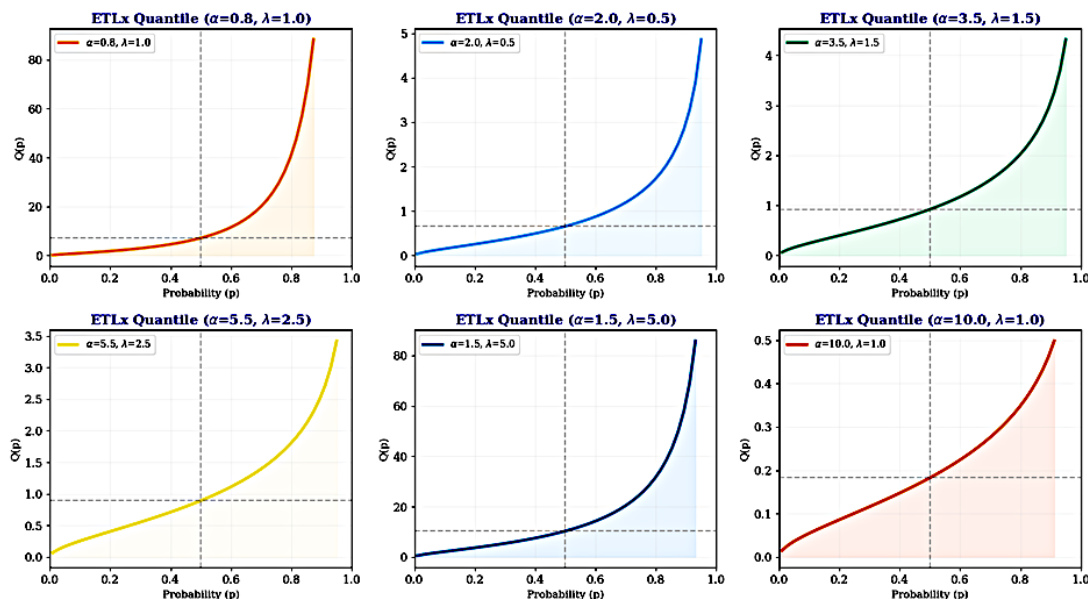


Fig. 5: Quantile function of the ETLx distribution for selected parameter values.

$$u'_1 = \lambda \left(\frac{\alpha^2}{(\alpha-1)^2} - 1 \right). \quad (40)$$

which is finite for $\alpha > 1$. For $r = 2$,

$$u'_2 = \alpha^2 \lambda^2 \left(\frac{1}{\alpha^2} - \frac{2}{(\alpha-1)^2} + \frac{1}{(\alpha-2)^2} \right). \quad (41)$$

For $r = 3$,

$$u'_3 = \alpha^2 \lambda^3 \left(-\frac{1}{\alpha^2} + \frac{3}{(\alpha-1)^2} - \frac{3}{(\alpha-2)^2} + \frac{1}{(\alpha-3)^2} \right), \quad (42)$$

and for $r = 4$,

$$u'_4 = \alpha^2 \lambda^4 \left(\frac{1}{\alpha^2} - \frac{4}{(\alpha-1)^2} + \frac{6}{(\alpha-2)^2} - \frac{4}{(\alpha-3)^2} + \frac{1}{(\alpha-4)^2} \right), \quad \alpha > 4. \quad (43)$$

The r th central moment is

$$u_r = E[(T - \mu)^r], \quad (44)$$

where $\mu = E[T]$ is the mean. In particular, the variance is

$$\sigma^2 = u'_2 - (u'_1)^2. \quad (45)$$

which exists for $\alpha > 2$. The third central moment is

$$u_3 = u'_3 - 3\mu u'_2 + 2\mu^3. \quad (46)$$

and the fourth is

$$u_4 = u'_4 - 4\mu u'_3 + 6\mu^2 u'_2 - 3\mu^4. \quad (47)$$

The coefficients of skewness and kurtosis are

$$\gamma_1 = \frac{\mu_3}{\sigma^3}, \quad (48)$$

and

$$\gamma_2 = \frac{\mu_4}{\sigma^4}. \quad (49)$$

They are defined for $\alpha > 3$ and $\alpha > 4$, respectively.

Table 3 lists the mean, variance, skewness and kurtosis for parameter values at which these quantities exist. For fixed α , the mean increases almost linearly with λ ; for fixed λ , it decreases as α increases. With $\alpha = 5$, for example, the mean rises from 0.2861 to 5.6252 as λ goes from 0.5 to 10.0, while with $\lambda = 10$ it falls from 5.6252 at $\alpha = 5$ to 3.6114 at $\alpha = 7$. Thus α mainly determines how concentrated the distribution is, and λ determines its overall scale.

All entries in Table 3 were obtained directly from the closed-form expressions (40)–(49), without numerical integration. Because the distribution is heavy-tailed, the moments exist only for sufficiently large α : the mean for $\alpha > 1$, the variance for $\alpha > 2$, the skewness for $\alpha > 3$

and the kurtosis for $\alpha > 4$. Only parameter values that meet these conditions were used.

Figures 6–9 show the mean, variance, skewness and kurtosis as surfaces over (α, λ) . They agree with Table 3: the mean and the variance increase with λ and decrease as α increases. The skewness is positive over the whole parameter range shown, so the distribution is always right-skewed, and the kurtosis stays above three, so the distribution is always leptokurtic. The model can therefore describe positively skewed data with different amounts of spread and tail weight.

3.2 Incomplete Moments of the ETLx Distribution

The r th incomplete moment about the origin is

$$M_r(x) = \int_0^x t^r f(t) dt, \quad (50)$$

where $f(t)$ is the ETLx density. Using (11), we have

$$M_r(x) = \int_0^x t^r \frac{\alpha^2}{\lambda} \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)} \ln\left(1 + \frac{t}{\lambda}\right) dt. \quad (51)$$

The integral is simplified by the same change of variables as before. Let $y = 1 + \frac{t}{\lambda}$. Then,

$$M_r(x) = \alpha^2 \lambda^r \int_1^{1+\frac{x}{\lambda}} (y-1)^r y^{-(\alpha+1)} \ln y dy. \quad (52)$$

Expanding $(y-1)^r$ by the binomial theorem (37) and inserting the result in (52), we get

$$M_r(x) = \alpha^2 \lambda^r \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} \int_1^{1+\frac{x}{\lambda}} y^{k-\alpha-1} \ln y dy. \quad (53)$$

Evaluating the integral and simplifying gives the incomplete moment of the ETLx distribution:

$$M_r(x) = \alpha^2 \lambda^r \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} \left[\frac{A^{k-\alpha} \ln A}{k-\alpha} - \frac{A^{k-\alpha} - 1}{(\alpha-k)^2} \right], \quad (54)$$

where $A = 1 + \frac{x}{\lambda}$, $k > \alpha$.

3.3 Entropy Measures

Entropy measures how uncertain, or random, the outcome of a probability distribution is. For lifetime data it

Table 3: Mean, variance, skewness and kurtosis of the ETLx distribution for selected parameter values.

α	λ	Mean	Variance	Skewness	Kurtosis
5	0.5	0.2861	0.0842	4.4328	76.3528
5	1.36	0.7689	0.6256	4.4027	73.2941
5	2.23	1.254	1.6687	4.3955	71.1337
5	3.09	1.7395	3.2136	4.3888	69.1995
5	3.95	2.2251	5.2601	4.3813	67.4069
5	4.82	2.7108	7.8081	4.3728	65.723
5	5.68	3.1965	10.8574	4.3633	64.1288
5	6.55	3.6822	14.4078	4.3528	62.612
5	7.41	4.168	18.4588	4.3415	61.1636
5	8.27	4.6537	23.0102	4.3294	59.7764
5	9.14	5.1395	28.0613	4.3166	58.4452
5	10	5.6252	33.6116	4.3031	57.1652
6	0.5	0.2254	0.044	3.4433	34.443
6	1.36	0.6021	0.3279	3.4074	33.9355
6	2.23	0.9813	0.875	3.4033	33.8434
6	3.09	1.3609	1.6852	3.402	33.7747
6	3.95	1.7407	2.7585	3.4013	33.7025
6	4.82	2.1206	4.095	3.4008	33.6221
6	5.68	2.5005	5.6946	3.4003	33.5329
6	6.55	2.8804	7.5573	3.3997	33.4349
6	7.41	3.2604	9.6832	3.399	33.3285
6	8.27	3.6404	12.0722	3.3982	33.2142
6	9.14	4.0203	14.7242	3.3973	33.0927
6	10	4.4003	17.6394	3.3963	32.9642
7	0.5	0.1865	0.0267	2.965	23.1821
7	1.36	0.4948	0.1995	2.9195	22.7244
7	2.23	0.8057	0.5325	2.9142	22.6722
7	3.09	1.1172	1.0256	2.9126	22.656
7	3.95	1.4288	1.679	2.9119	22.6478
7	4.82	1.7406	2.4925	2.9115	22.642
7	5.68	2.0523	3.4662	2.9113	22.6367
7	6.55	2.3641	4.6001	2.9112	22.6312
7	7.41	2.6759	5.8942	2.9111	22.625
7	8.27	2.9878	7.3484	2.911	22.6179
7	9.14	3.2996	8.9628	2.9109	22.6098
7	10	3.6114	10.7374	2.9108	22.6006
8	0.5	0.1596	0.0178	2.6846	18.261
8	1.36	0.42	0.1333	2.629	17.7908
8	2.23	0.6834	0.356	2.6221	17.7359
8	3.09	0.9474	0.6859	2.6201	17.7196
8	3.95	1.2115	1.1229	2.6192	17.7127
8	4.82	1.4757	1.6671	2.6188	17.709
8	5.68	1.74	2.3184	2.6185	17.7069
8	6.55	2.0043	3.0768	2.6183	17.7054

describes uncertainty in a way that complements the usual summaries such as the mean, variance, skewness and kurtosis [43].

In a heavy-tailed lifetime model, entropy measures the uncertainty associated with failure occurrence: a higher entropy means more variable and less predictable failure times, and a lower entropy means that lifetimes are more concentrated [37]. Entropy therefore carries

Mean of the ETLx Distribution

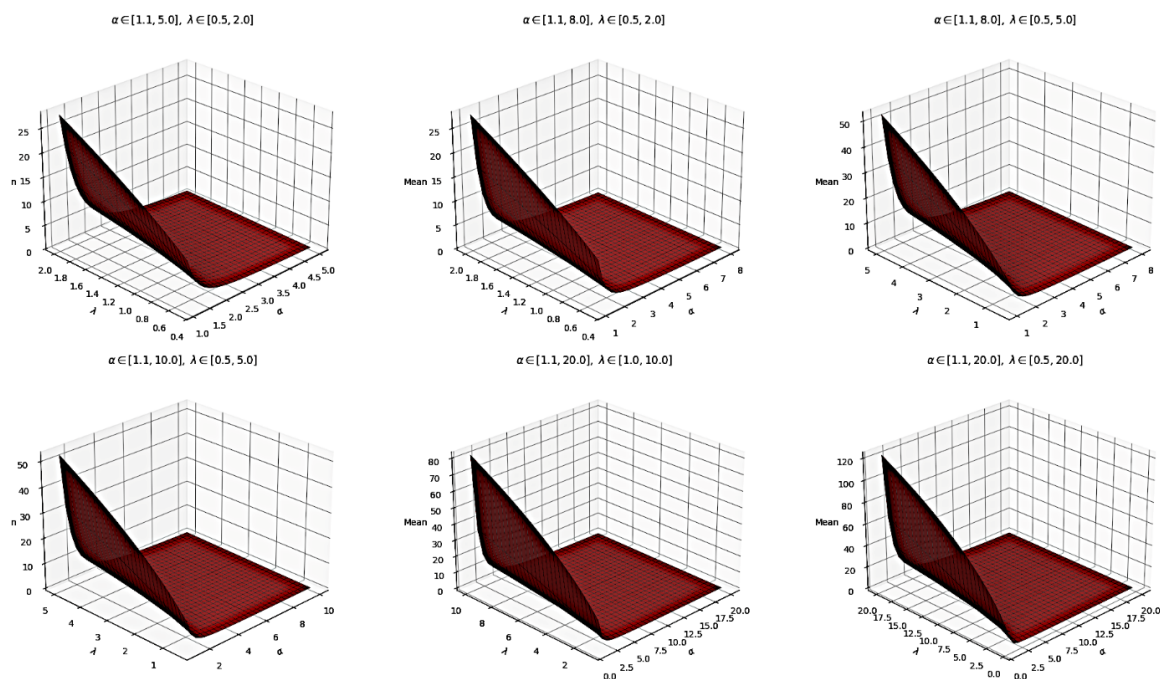


Fig. 6: Mean of the ETLx distribution as a function of α and λ .

Variance of the ETLx Distribution

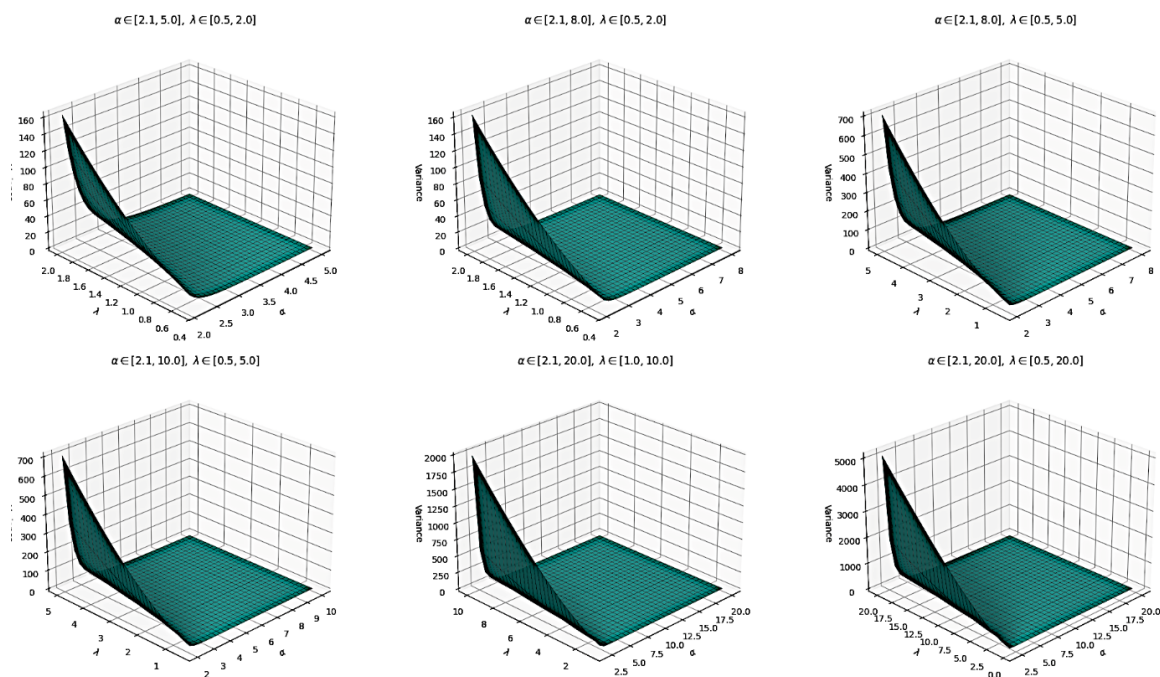


Fig. 7: Variance of the ETLx distribution as a function of α and λ .

Skewness of the ETLx Distribution

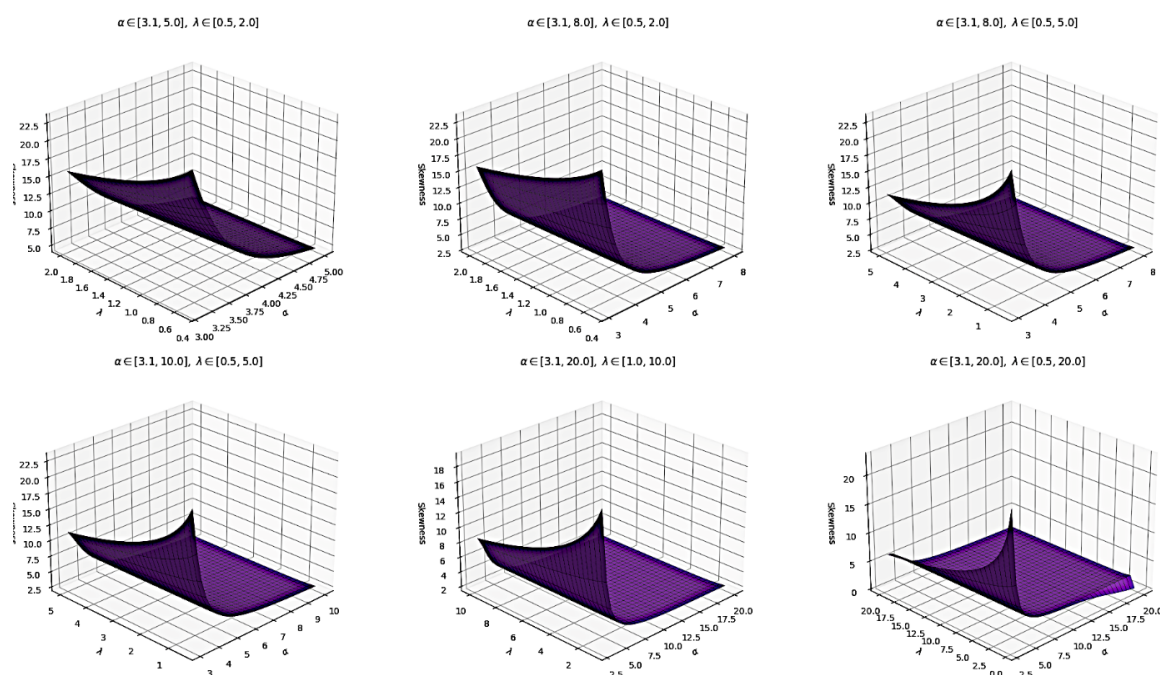


Fig. 8: Skewness of the ETLx distribution as a function of α and λ .

Kurtosis of the ETLx Distribution

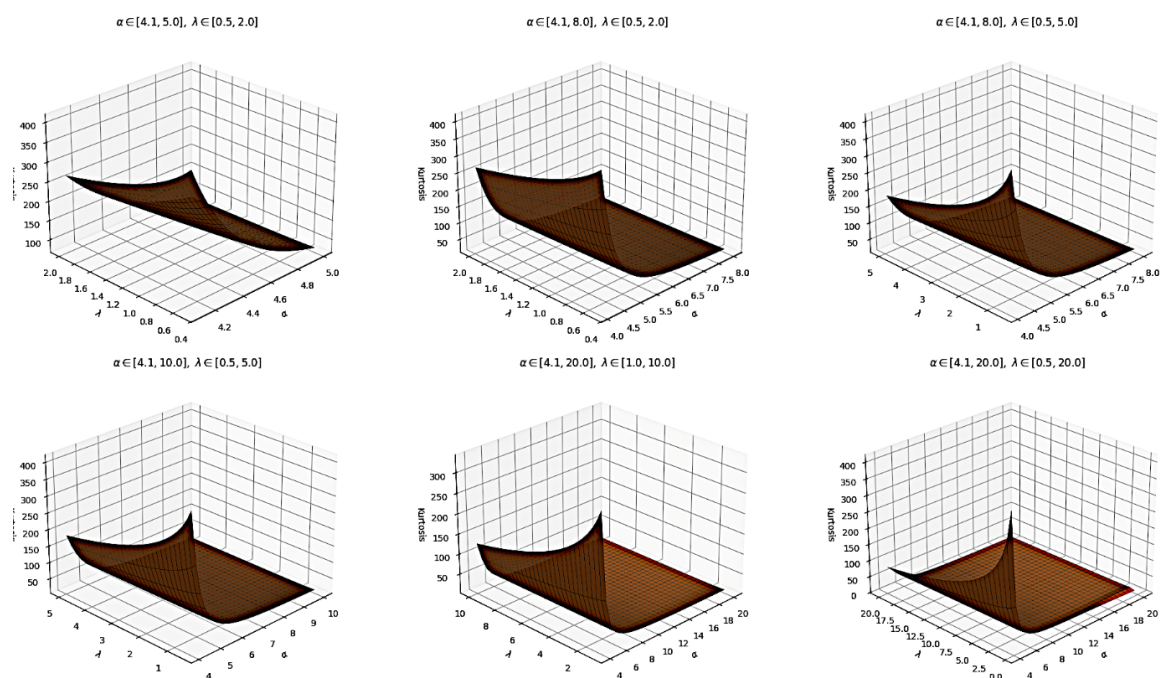


Fig. 9: Kurtosis of the ETLx distribution as a function of α and λ .

information about system reliability that moment-based summaries do not provide.

3.3.1 Rényi Entropy

The Rényi entropy depends on an order parameter $q > 0$, $q \neq 1$, which changes the weight given to different parts of the lifetime distribution. It is defined as

$$H_R(q) = \frac{1}{1-q} \ln \left(\int_0^\infty f(t)^q dt \right), \quad (55)$$

where $f(t)$ is the ETLx density.

We first need $f_{ETLx}(t)^q$, which is

$$f_{ETLx}(t)^q = \left(\frac{\alpha^2}{\lambda} \right)^q \left(1 + \frac{t}{\lambda} \right)^{-q(\alpha+1)} \left[\ln \left(1 + \frac{t}{\lambda} \right) \right]^q. \quad (56)$$

Changing to the variable y gives

$$f_{ETLx}(t)^q = \left(\frac{\alpha^2}{\lambda} \right)^q \lambda \int_1^\infty y^{-q(\alpha+1)} (\ln y)^q dy. \quad (57)$$

With $y = e^u$ this becomes

$$f_{ETLx}(t)^q = \left(\frac{\alpha^2}{\lambda} \right)^q \lambda \int_0^\infty u^q e^{-u(q(\alpha+1)-1)} du. \quad (58)$$

The last integral is a gamma integral, so

$$f_{ETLx}(t)^q = \left(\frac{\alpha^2}{\lambda} \right)^q \lambda \frac{\Gamma(q+1)}{[q(\alpha+1)-1]^{q+1}}. \quad (59)$$

Inserting (59) in (55) gives the Rényi entropy of the ETLx distribution:

$$H_R(q) = \frac{1}{1-q} \ln \left[\left(\frac{\alpha^2}{\lambda} \right)^q \lambda \frac{\Gamma(q+1)}{[q(\alpha+1)-1]^{q+1}} \right], \quad (60)$$

where $q(\alpha+1) > 1$.

3.3.2 Tsallis Entropy

The Tsallis entropy of order q is

$$H_T(q) = \frac{1}{q-1} \left(1 - \int_0^\infty f(t)^q dt \right). \quad (61)$$

Using the integral (59) again, the Tsallis entropy of the ETLx distribution is

$$H_T(q) = \frac{1}{q-1} \left(1 - \left(\frac{\alpha^2}{\lambda} \right)^q \lambda \frac{\Gamma(q+1)}{[q(\alpha+1)-1]^{q+1}} \right), \quad (62)$$

where $q(\alpha+1) > 1$. Table 4 reports both entropies for several parameter combinations. Their values depend clearly on the shape parameter α , the scale parameter λ and the order q , which is to be expected because entropy reflects both the shape and the spread of a distribution.

Figures 10 and 11 show the Rényi and Tsallis entropies as surfaces over α and λ for $q = 0.8$ and $q = 1.5$. Both entropies decrease as α increases, because the distribution becomes more concentrated. For fixed α they increase with λ , since a larger scale spreads the distribution over a wider range. Raising q from 0.8 to 1.5 lowers the whole surface, as the theory predicts for generalized entropies. These results agree with Table 4 and show that the ETLx model can represent data with quite different levels of uncertainty, including strongly skewed and heavy-tailed data.

4 Maximum Likelihood Estimation of the ETLx Distribution

Let $T_1, T_2, \dots, T_n$ be a random sample from the ETLx distribution with PDF (11). The likelihood function is

$$L(\alpha, \lambda) = \prod_{i=1}^n f_{ETLx}(t_i; \alpha, \lambda). \quad (63)$$

Substituting (11) into (63) gives

$$L(\alpha, \lambda) = \left(\frac{\alpha^2}{\lambda} \right)^n \prod_{i=1}^n \left(1 + \frac{t_i}{\lambda} \right)^{-(\alpha+1)} \prod_{i=1}^n \ln \left(1 + \frac{t_i}{\lambda} \right). \quad (64)$$

The log-likelihood function is

$$\begin{aligned} \ell(\alpha, \lambda) &= 2n \ln \alpha - n \ln \lambda - (\alpha+1) \sum_{i=1}^n \ln \left(1 + \frac{t_i}{\lambda} \right) \\ &\quad + \sum_{i=1}^n \ln \ln \left(1 + \frac{t_i}{\lambda} \right). \end{aligned} \quad (65)$$

The estimates are found by differentiating the log-likelihood with respect to α and λ . The derivative with respect to α is

$$\frac{\partial \ell(\alpha, \lambda)}{\partial \alpha} = \frac{2n}{\alpha} - \sum_{i=1}^n \ln \left(1 + \frac{t_i}{\lambda} \right). \quad (66)$$

and setting it equal to zero gives the equation for $\hat{\alpha}$:

$$\alpha = \frac{2n}{\sum_{i=1}^n \ln \left(1 + \frac{t_i}{\lambda} \right)}. \quad (67)$$

Table 4: Rényi and Tsallis entropies of the ETLx distribution for selected parameter values.

α	λ	Rényi				Tsallis			
		$q = 0.5$	$q = 1.0$	$q = 1.5$	$q = 2.0$	$q = 0.5$	$q = 1.0$	$q = 1.5$	$q = 2.0$
0.5	0.5	3.56904	3.207953	2.899764	2.672588	9.913439	3.207953	1.530804	0.930927
0.5	1.14	3.710965	3.506662	3.323383	3.174421	10.78957	3.506662	1.620365	0.958182
0.5	1.79	3.772281	3.632691	3.504744	3.395856	11.18774	3.632691	1.653276	0.966488
0.5	2.43	3.807664	3.704554	3.608828	3.525082	11.42313	3.704554	1.670858	0.970551
0.5	3.07	3.830736	3.7511	3.676525	3.610083	11.57888	3.7511	1.681813	0.97295
0.5	3.71	3.846839	3.783473	3.723766	3.669904	11.68864	3.783473	1.68924	0.974521
0.5	4.36	3.858567	3.807026	3.758247	3.713868	11.76915	3.807026	1.694552	0.975617
0.5	5	3.867355	3.824692	3.784198	3.747156	11.82979	3.824692	1.69849	0.976415
1.14	0.5	2.995733	2.239252	1.814919	1.582957	6.944277	2.239252	1.192904	0.794633
1.14	1.14	3.346813	2.856338	2.531553	2.331521	8.66059	2.856338	1.43596	0.902852
1.14	1.79	3.502821	3.139817	2.878257	2.704275	9.525451	3.139817	1.525731	0.933081
1.14	2.43	3.595366	3.310734	3.094236	2.941923	10.0713	3.310734	1.574279	0.947236
1.14	3.07	3.657498	3.426644	3.244168	3.110114	10.45219	3.426644	1.605027	0.955404
1.14	3.71	3.702248	3.510719	3.354871	3.236353	10.73395	3.510719	1.626295	0.960693
1.14	4.36	3.735983	3.574439	3.439962	3.334765	10.95056	3.574439	1.641861	0.964377
1.14	5	3.76224	3.624251	3.50725	3.413548	11.1217	3.624251	1.65371	0.967076
1.79	0.5	2.303499	1.398214	1.033172	0.848814	4.327445	1.398214	0.806893	0.572078
1.79	1.14	2.862676	2.174861	1.841298	1.662909	6.368587	2.174861	1.203479	0.810413
1.79	1.79	3.124237	2.566303	2.263111	2.09227	7.537825	2.566303	1.354938	0.876593
1.79	2.43	3.28475	2.817793	2.542488	2.379942	8.334858	2.817793	1.439035	0.907444
1.79	3.07	3.395432	2.996919	2.746806	2.592858	8.92292	2.996919	1.493513	0.925194
1.79	3.71	3.476997	3.132194	2.904703	2.75938	9.377589	3.132194	1.531961	0.936669
1.79	4.36	3.539765	3.238324	3.031108	2.894263	9.740328	3.238324	1.560627	0.94466
1.79	5	3.589571	3.323866	3.134822	3.006196	10.03636	3.323866	1.582831	0.95052
2.43	0.5	1.669843	0.817967	0.519576	0.366747	2.609266	0.817967	0.45757	0.307015
2.43	1.14	2.368745	1.636076	1.344175	1.192074	4.537271	1.636076	0.978717	0.696409
2.43	1.79	2.712801	2.068746	1.78585	1.635251	5.764387	2.068746	1.181088	0.805097
2.43	2.43	2.931539	2.358862	2.086616	1.938158	6.661749	2.358862	1.295425	0.856031
2.43	3.07	3.086639	2.573485	2.312827	2.16706	7.360199	2.573485	1.370775	0.885486
2.43	3.71	3.203638	2.741141	2.492526	2.349907	7.924102	2.741141	1.424845	0.904622
2.43	4.36	3.295521	2.876731	2.640279	2.501179	8.390666	2.876731	1.465804	0.918012
2.43	5	3.369759	2.989064	2.764665	2.629378	8.783601	2.989064	1.498015	0.927877
3.07	0.5	1.158945	0.41269	0.156971	0.022816	1.570192	0.41269	0.150969	0.022558
3.07	1.14	1.931384	1.238313	0.984016	0.849811	3.253208	1.238313	0.777205	0.572504
3.07	1.79	2.32676	1.681838	1.429632	1.295656	4.401468	1.681838	1.021436	0.726282
3.07	2.43	2.585569	1.98488	1.735758	1.602229	5.285831	1.98488	1.160318	0.798553
3.07	3.07	2.773566	2.213625	1.96851	1.835659	6.00391	2.213625	1.252565	0.840492
3.07	3.71	2.918371	2.396008	2.155693	2.023759	6.604909	2.396008	1.319345	0.867842
3.07	4.36	3.034212	2.546524	2.311666	2.180889	7.118023	2.546524	1.37041	0.887059
3.07	5	3.129375	2.673705	2.444827	2.315438	7.562363	2.673705	1.410963	0.901277
3.71	0.5	0.762744	0.109936	-0.11886	-0.24108	0.928585	0.109936	-0.12246	-0.27263
3.71	1.14	1.568033	0.936782	0.708736	0.586316	2.380503	0.936782	0.596767	0.443627
3.71	1.79	1.990879	1.382608	1.15505	1.032632	3.411828	1.382608	0.877428	0.643932
3.71	2.43	2.273549	1.689102	1.46232	1.339977	4.233399	1.689102	1.0373	0.738148

The derivative with respect to λ is

$$\frac{\partial \ell(\alpha, \lambda)}{\partial \lambda} = -\frac{n}{\lambda} + (\alpha + 1) \sum_{i=1}^n \frac{t_i}{\lambda^2 (1 + \frac{t_i}{\lambda})} - \sum_{i=1}^n \frac{t_i}{\lambda^2 (1 + \frac{t_i}{\lambda}) \ln(1 + \frac{t_i}{\lambda})}. \quad (68)$$

Setting it equal to zero gives the equation for $\hat{\lambda}$:

$$n = (\alpha + 1) \sum_{i=1}^n \frac{t_i}{\lambda + t_i} - \sum_{i=1}^n \frac{t_i}{(\lambda + t_i) (1 + \frac{t_i}{\lambda})}. \quad (69)$$

The score equations (67) and (69) are nonlinear and have no closed-form solution, so we maximized the log-likelihood numerically with the L-BFGS-B algorithm (limited-memory Broyden–Fletcher–Goldfarb–Shanno with box constraints). The constraints $\alpha > 0$ and $\lambda > 0$

Rényi Entropy Surface for ETLx Distribution

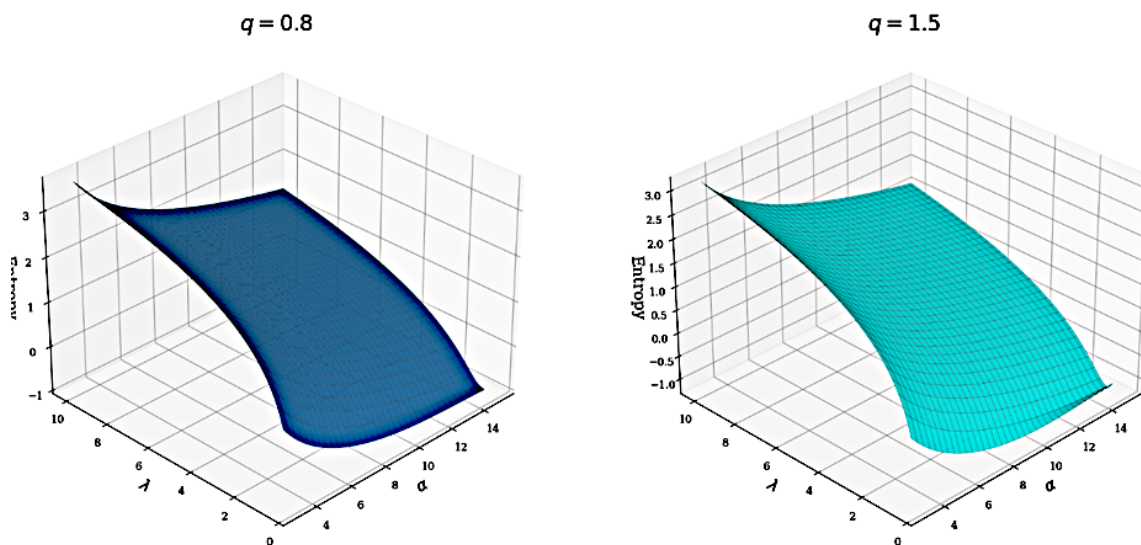


Fig. 10: Rényi entropy of the ETLx distribution as a function of α and λ for selected values of q .

Tsallis Entropy Surface for ETLx Distribution

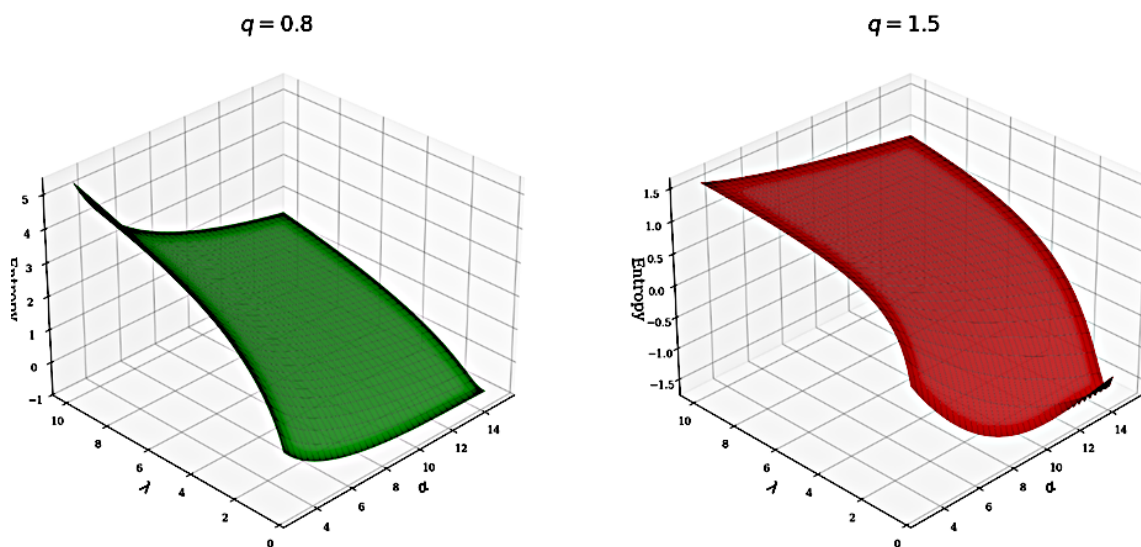


Fig. 11: Tsallis entropy of the ETLx distribution as a function of α and λ for selected values of q .

keep the estimates inside the parameter space of the ETLx distribution.

To avoid convergence problems, the algorithm was started from several positive initial values, chosen as rough guesses based on sample statistics. The iterations stopped when the largest absolute change in the estimates between two successive steps fell below 10^{-8} , that is,

when the log-likelihood no longer improved noticeably. All computations were done in Python 3.11.

4.1 Asymptotic Properties of the Maximum Likelihood Estimators

Under the usual regularity conditions, the maximum likelihood estimator

$$\hat{\theta} = (\hat{\alpha}, \hat{\lambda})^T$$

is asymptotically normal:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, I^{-1}(\theta)), \quad (70)$$

where $I(\theta) = -\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T}$ is the observed Fisher information matrix evaluated at the maximum likelihood estimates.

The variance-covariance matrix of the estimators is therefore estimated by

$$\hat{\Sigma} = I^{-1}(\hat{\theta}), \quad (71)$$

and the standard errors are the square roots of the diagonal elements of $\hat{\Sigma}$. Approximate 95% confidence intervals for the parameters are

$$\hat{\theta}_i \pm 1.96 SE(\hat{\theta}_i), \quad i = 1, 2. \quad (72)$$

These results can be used for interval estimation and hypothesis testing on the parameters of the ETLx distribution.

5 Monte Carlo Simulation Study

We carried out a Monte Carlo study to see how well the maximum likelihood estimators perform in finite samples, looking at their accuracy, bias and stability for different parameter values and sample sizes. Three parameter settings with different tail behavior were used:

Case I : $\alpha = 1.5, \lambda = 1.0$,

Case II : $\alpha = 0.9, \lambda = 1.0$,

Case III : $\alpha = 3.0, \lambda = 0.8$.

For each setting, samples of size $n = 50, 100, 200, 500$ and 1000 were drawn from the ETLx distribution by inverse-transform sampling, using the quantile function (29). Each combination of parameters and sample size was replicated $R = 1000$ times. In every replication the parameters were estimated by maximizing the log-likelihood with the L-BFGS-B procedure of Section 4, and the empirical mean, bias and mean squared error (MSE) of the estimates were then computed from (73). The steps are summarized in Algorithm 1, and the results are reported in Tables 5–7.

Algorithm 1. Monte Carlo simulation for the ETLx distribution.

1. Choose three parameter settings with different tail behavior:

Case I : $\alpha = 1.5, \lambda = 1.0$,

Case II : $\alpha = 0.9, \lambda = 1.0$,

Case III : $\alpha = 3.0, \lambda = 0.8$

2. Choose increasing sample sizes:

50, 100, 200, 500, and 1000.

3. For each combination of true parameter values and sample size, perform $R = 1000$ independent replications.

In each replication, for the given n :

- i. Generate a sample of size n from the ETLx distribution by the inverse transform method, using the quantile function (29).
- ii. Estimate α and λ by minimizing the negative log-likelihood numerically.
- iii. For each $\theta \in \{\alpha, \lambda\}$, compute the empirical mean, bias and MSE over the R replications:

$$\begin{aligned} \text{Mean}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R \hat{\theta}_r, \\ \text{Bias}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \theta), \\ \text{MSE}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \theta)^2, \end{aligned} \quad (73)$$

where $\hat{\theta}_r$ is the estimate of θ in the r th replication. These summaries describe the finite-sample behavior of the estimators and show whether they are consistent and efficient in practice.

Table 5 gives the results for Case I ($\alpha = 1.5, \lambda = 1.0$). As n grows, the mean estimates move towards the true values and both the bias and the MSE decrease steadily. The bias of $\hat{\alpha}$, for example, falls from 0.1936 at $n = 50$ to 0.0058 at $n = 1000$, and its MSE falls from 0.4428 to 0.0059. Table 6 gives the results for Case II ($\alpha = 0.9, \lambda = 1.0$), where the tail is heavier. The estimators again perform well: the bias stays small and the MSE decreases as the sample size increases. Table 7 gives the results for Case III ($\alpha = 3.0, \lambda = 0.8$), where the tail is lighter and the distribution is more concentrated. Here the bias and MSE are larger in small samples, especially for α (the MSE of $\hat{\alpha}$ is 22.94 at $n = 50$), but they drop quickly and the MSE is only 0.050 at $n = 1000$.

In all three cases the results agree with the asymptotic theory of maximum likelihood estimators under the usual regularity conditions. A simulation study cannot prove consistency or efficiency; it only shows that, for these settings, the estimators behave as the theory predicts.

6 Illustrative Applications of the ETLx Distribution

We illustrate the use of the ETLx distribution with three lifetime datasets of different sizes and tail weights. The data were generated from the ETLx distribution by inverse-transform sampling based on the quantile function (29). Generated data give a controlled setting in which the

Table 5: Simulation results for the ETLx distribution with $\alpha = 1.5$ and $\lambda = 1.0$ (Case I).

n	$\alpha = 1.5$			$\lambda = 1.0$		
	Mean	Bias	MSE	Mean	Bias	MSE
50	1.693563	0.193563	0.442769	1.24302	0.24302	0.669799
100	1.588085	0.088085	0.089289	1.122572	0.122572	0.144134
200	1.519261	0.019261	0.032203	1.036482	0.036482	0.054497
500	1.525719	0.025719	0.012738	1.030109	0.030109	0.019368
1000	1.505779	0.005779	0.005895	1.00777	0.00777	0.009613

Table 6: Simulation results for the ETLx distribution with $\alpha = 0.9$ and $\lambda = 1.0$ (Case II).

n	$\alpha = 0.9$			$\lambda = 1.0$		
	Mean	Bias	MSE	Mean	Bias	MSE
50	0.94788	0.04788	0.038314	1.191431	0.191431	0.401346
100	0.915778	0.015778	0.012925	1.05087	0.05087	0.113116
200	0.902978	0.002978	0.006875	1.023689	0.023689	0.076627
500	0.910335	0.010335	0.004002	1.045371	0.045371	0.071159
1000	0.905214	0.005214	0.002078	1.025047	0.025047	0.046158

Table 7: Simulation results for the ETLx distribution with $\alpha = 3.0$ and $\lambda = 0.8$ (Case III).

n	$\alpha = 3.0$			$\lambda = 0.8$		
	Mean	Bias	MSE	Mean	Bias	MSE
50	4.173518	1.173518	22.943631	1.245843	0.445843	3.312836
100	3.368483	0.368483	2.233826	0.955599	0.155599	0.394202
200	3.171685	0.171685	0.579977	0.869858	0.069858	0.093674
500	3.056503	0.056503	0.117681	0.818188	0.018188	0.01709
1000	3.021827	0.021827	0.050102	0.809599	0.009599	0.00773

flexibility and fit of the model can be checked, and they make all numerical results fully reproducible.

The third dataset (Data III) has 500 positive observations. Its tail is lighter than those of the other two datasets, but it is still positively skewed. The observations are:

1.845301843, 0.332517549, 2.066420803, 0.374760519, 0.114677995, 0.357387138, 0.464855863, 0.166617608, 0.521722623, 1.384375437, 0.501114819, 0.611191471, 14.11617816, 0.279924145, 0.690847235, 5.760304131, 0.614174159, 0.776442118, 0.695576876, 0.178748465, 1.076136706, 2.291577633, 2.522528742, 0.767354895, 0.426809292, 0.962803771, 2.91263134, 4.492852996, 0.349874363, 2.70126056, 1.120589475, 1.850687775, 0.231806691, 4.33107881, 1.200456822, 0.230838991, 0.68447451, 0.174860107, 1.102331407, 0.783728767, 0.751626824, 1.66281349, 1.26219446, 0.1538425, 1.232611502, 0.686726407, 0.253184873, 0.420668843, 0.634720522, 1.746106953, 1.941860761, 0.164948983, 0.538168856, 1.63801834, 0.797876529, 1.718742039, 0.78779276, 0.287748152, 0.624967817, 0.644020991, 2.066942551, 0.430564306, 0.299931025, 1.183676573, 0.508493846, 0.597306188, 4.358880979, 0.960589317, 0.953479997, 2.66340552, 1.076621101, 3.216365891, 0.862616194, 0.768773769, 0.351226669, 0.295647665,

0.670225069, 0.116850501, 0.232616982, 1.052932494, 0.971844095, 0.075977001, 0.227471405, 0.829419418, 0.493300098, 0.771733091, 0.19258922, 1.774086228, 0.324621869, 0.452524065, 0.547925964, 0.779163934, 4.725750679, 0.273969516, 0.226687497, 0.209505218, 1.282397824, 0.462197476, 2.5295696, 0.125722532, 0.670313491, 1.214144889, 0.631461599, 0.491971045, 1.45137965, 0.634580811, 2.21638768, 0.303630435, 0.790141001, 0.865209587, 2.224171058, 0.155504135, 1.422022205, 0.98359207, 1.759543345, 0.810940718, 0.553947715, 1.376787799, 0.351182017, 0.1124376, 1.187626781, 1.455826226, 1.54575329, 0.545640485, 1.128374263, 1.068600925, 0.355837779, 1.245709543, 3.593014411, 0.426338934, 0.339811163, 0.76203854, 1.903593664, 0.809789656, 3.901693684, 2.453759355, 1.059871463, 1.048613511, 0.242418065, 0.413665277, 1.444333475, 0.484281991, 0.292855258, 0.121694921, 0.287758917, 0.749453828, 2.479624426, 0.473238429, 1.345748605, 0.19268512, 1.511724704, 2.539874485, 0.624522247, 0.555326131, 2.237133918, 0.292613443, 2.421021722, 2.37163485, 3.315546091, 2.034937587, 0.660859639, 0.979049389, 0.672673217, 0.370044625, 6.436586857, 0.070719521, 0.316937948, 2.560431357, 3.1348318, 0.34548238, 26.36262164, 1.457231444, 1.291156791, 0.739793681, 0.476621204, 0.040990314,

0.042059554,	0.452977165,	0.142655621,	0.145984744,	0.429243563,	6.659120633,	0.195298135,	0.306962357,
0.329242791,	0.612117523,	1.279889442,	0.771567081,	1.350316441,	0.470360042,	0.536562073,	0.265470501,
0.685882997,	1.378053159,	1.668086806,	0.535073223,	0.220552663,	1.744673391,	0.777168907,	1.33106182,
0.807512035,	2.431548016,	0.140792331,	0.81103817,	0.443241664,	0.60073783,	2.346093204,	2.199679418,
1.289723699,	0.518545327,	1.18754253,	1.615312525,	0.784502342,	0.978121409,	0.135193853,	0.417076754,
1.241077757,	0.678958549,	1.101643912,	1.055777437,	0.731593295,	0.537914526,	0.376243611,	1.036821457,
0.463298946,	0.064629618,	3.054911319,	0.573346255,	0.300634692,	2.264057538,	0.271607082,	1.081541775,
0.366538276,	0.315908007,	0.483083919,	0.750873031,	0.612079993,	1.129471005,	0.490669629,	0.428224354,
0.648121921,	0.46188188,	0.138513002,	4.7511925,	0.887450234,	3.547334686,	0.997017767,	0.055406914,
0.054662115,	2.407168531,	0.081117376,	0.628909616,	0.07983417,	0.277888522,	0.427672968,	0.22376052,
1.327775957,	0.932885861,	1.045261886,	0.160047283,	3.173402159,	0.331762109,	0.560687165,	0.204799853,
1.786429605,	0.900029751,	2.840196844,	8.443126349,	0.373676054,	0.694060055,	0.507178383,	0.150751049,
1.204565234,	1.419214262,	3.453174359,	1.085895209,	1.180012886,	3.87489932,	0.300493869,	0.286266233,
0.816739502,	0.662831016,	0.656197295,	0.87741625,	1.20667473,	0.554541983,	0.093159037,	0.10334561,
0.843910635,	0.015360178,	0.213465864,	0.504873866,	4.819212219,	1.098413369,	0.644272616,	0.222444368,
0.35481523,	0.960262347,	6.443850347,	0.331825369,	2.267419812,	0.453440066,	0.399091093,	1.348093999,
1.447150306,	0.448902306,	1.099828694,	0.150617475,	1.486586314,	0.061154325,	0.701554046,	0.425093986,
0.842318467,	0.652590055,	1.548101834,	0.18216266,	3.915666622,	0.384023736,	0.368867259,	0.920291403,
0.654448274,	1.780992581,	1.014084915,	0.225045613,	2.180628054,	0.882150849,	0.71827388,	0.184456719,
1.705702595,	0.295845094,	0.133926132,	0.146399967,	0.322696204,	1.388578918,	0.200142055,	2.803591522,
1.322488666,	1.617400412,	1.929844924,	1.203645409,	2.986439021,	0.875394205,	0.370556608,	2.479066868,
2.066396845,	2.541134387,	0.679548919,	0.860323004,	1.382747017,	1.142691428,	0.20577325,	0.814212231,
3.171482906,	0.642519569,	0.305828582,	2.766099089,	1.482154616,	0.296684942,	0.32418417,	0.545411477,
0.106558106,	0.761074748,	2.575687661,	1.167864112,				
0.804115177,	0.600368797,	0.338889764,	0.724485674,				
1.148059594,	0.72072449,	1.249034142,	3.737953992,				
0.8762135,	1.051491671,	1.676931282,	0.706657945,				
0.355624564,	1.363681772,	1.523547711,	4.659013332,				
0.064503355,	0.631377669,	0.119463169,	0.615017447,				
0.587278929,	0.284432827,	16.37172563,	1.16257451,				
0.466594593,	1.370002927,	0.312071606,	0.351570381,				
1.289209732,	1.015284862,	1.324934171,	0.47056321,				
0.249189218,	0.527993268,	1.155184813,	1.656709909,				
2.161948285,	7.108420256,	0.382585787,	1.602684104,				
13.11004023,	0.346326141,	0.029772199,	0.206615238,				
1.294580722,	1.412905673,	0.32547175,	2.803200501,				

6.1 Exploratory Data Analysis and Graphical Summary

The three datasets were chosen to have different characteristics, from moderate skewness to very heavy tails, so that the flexibility of the model could be checked in several situations. Their summary statistics are given in Table 8.

The first dataset (Data I) has 200 positive observations and a moderately heavy tail:

0.636222419,	2.823079547,	1.680185594,	1.260465825,
1.201450357,	4.919064629,	9.605920329,	0.631693467,
2.943106796,	1.002043969,	20.86845498,	10.94279217,
2.786791594,	4.201060148,	1.93263982,	5.813428125,
1.589851239,	1.142460664,	0.935087508,	0.774909744,
1.994121392,	1.510309636,	3.045381159,	0.118491127,
1.586691658,	1.24009565,	0.683689038,	2.451811446,
1.530006715,	1.007690881,	0.724658007,	7.682583044,
5.07438489,	2.542842538,	1.165261218,	14.8589376,
2.229013889,	1.518564933,	9.24813755,	1.073634346,
3.404535385,	1.054578956,	0.883017716,	3.462911301,
0.77959022,	1.811908175,	2.343547305,	0.664897537,
3.867362634,	2.132224943,	2.662855474,	1.266808454,
1.463259001,	1.821033969,	1.693504363,	3.17483862,
2.323434299,	1.446356641,	0.041553202,	4.996570744,
1.956779855,	1.098792927,	1.848521753,	0.393618432,
9.571184376,	45.96458531,	0.28982763,	1.213835766,
3.850352597,	1.056014015,	2.253677722,	1.447644857,
4.585263138,	17.75495667,	8.443796388,	2.658111977,
0.583065702,	14.90847459,	0.16730208,	1.000063974,
0.947107641,	3.134376552,	1.781743129,	0.39200748,
0.118580215,	2.52409471,	1.803316029,	2.501364364,

2.233332061,	8.571200229,	10.79331785,	0.663023824,	10.18319985,	97.79658015,	72.51770954,	2.669230609,
13.96574184,	4.858964562,	2.815062166,	3.009661221,	6.506161443,	10.78077941,	1.471365495,	1.328418225,
2.116951658,	10.67971142,	0.795785535,	2.485514206,	62.5577112,	55.10166049,	6.098793905,	24.94025315,
5.488279916,	0.51101761,	2.613336072,	1.389681375,	10.43850152,	31.58248436,	4.632528354,	8.740204873,
4.432274409,	0.317559536,	2.543666655,	6.89014603,	107.6522777,	50.79847584,	4.533391112,	94.03670139,
2.720943868,	1.080571192,	2.972221146,	1.140085667,	6.483587191,	21.94165064,	25.62227902,	6.560160113,
3.100657296,	0.492923265,	0.941850434,	0.202415381,	55.27538566,	8.442337616,	398.0771112,	215.9807816,
0.365872651,	2.172458882,	0.791055613,	1.937912306,	118.4344906,	132.6517573,	2.068672598,	18.52243254,
3.009287236,	8.02732724,	1.201985554,	1.071336388,	30.36119819,	10.91431468,	10.93517916,	2.783101698,
1.054051704,	0.46327796,	3.045408824,	5.635022531,	2.2419761,	39.71413067,	2.10063585,	28.14665694,
1.708579096,	2.195428273,	17.75974373,	3.439532157,	1.23098849,	6.840181078,	0.566606454,	19.64067254,
4.076305691,	8.511403918,	0.743481636,	2.918114659,	4.220855322,	8.521772617,	4.106537886,	10.34658166,
3.674418626,	6.334044232,	1.160468844,	3.320521717,	19.37945904,	2.320670335,	38.10694396,	39.5459308,
2.524756195,	1.41698076,	0.417307431,	1.194503569,	242.8270439,	6.036244238,	85.36125308,	1.426294515,
19.45216309,	0.745654335,	0.451346675,	1.249633822,	3932.942373,	2.935003104,	5.873998811,	1.424153685,
0.41528903,	0.900853988,	10.4089974,	1.071693684,	59.16853268,	9.167261014,	421.1872342,	2.038436016,
2.385580772,	0.812427398,	3.094325269,	0.490982342,	106.1255817,	5.547699652,	3.289494198,	19.32983596,
4.903680618,	1.475220401,	0.64978455,	4.294436789,	0.653082535,	10.76706752,	8.725151988,	0.888853043,
2.169309874,	1.542800695,	0.81816625,	4.726713915,	7.459851211,	4.439163848,	3.480236814,	190.4569909,
1.240492596,	0.238951531,	20.19100376,	0.194038584,	14.69668923,	2.157838192,	33.48802738,	5.77454182,
0.470686884,	1.673052728,	0.720067245,	1.129259576,	14.73299792,	6.018024568,	17.67671387,	21.89168731,
2.257166036,	17.56766002,	13.01248398,	0.336218208,	4.055747869,	57.13998108,	16.07386425,	15.1624297,
6.740488707,	0.60450334,	5.414795741,	0.995525654,	3.018391682,	3.564261621,	6.39834029,	0.702513549,
3.680311375,	3.693208862,	6.63461511,	29.10533512,	0.965003193,	19.8658138,	4.657345143,	26.90015059,
0.927299796,	2.720604582,	3.986663692,	0.353272967,	1.204578522,	30.18367123,	2.963220031,	82.14642552,
3.043171658,	1.081190489,	0.586453153,	2.13849311,	129.2079462,	22.74536875,	6.370758805,	0.590038802,
1.653948377,	2.32274903,	2.841315758,	1.841583537,	34.34065946,	12.59188541,	1.292409767,	13.02667671,
9.606465119,	0.633299336,	0.970546404,	1.206314748,	0.258038382,	7.985862484,	15.72017287,	45.64738753,
The second dataset (Data II) has 300 positive observations and a much heavier tail than Data I:							
37.43388883,	18.86241131,	2.852977692,	30.97363948,	4.179764096,	3.225406088,	4.945151911,	1.018871558,
36.73579817,	36.42447684,	1.246339842,	2.934027557,	194.8540581,	92.81458034,	14.95082015,	1.201760974,
111.0464798,	91.54875961,	2.841020496,	11.99517601,	1.308112338,	197.0427942,	1.798474481,	2.051519672,
3.704875538,	6.950115738,	0.521444877,	4.025048935,	40.3589456,	4.376720054,	6.821330408,	6.780164842,
3.836936944,	156.6470478,	14.04042838,	2.459120317,	24.077433,	8.680864269,	49.06823328,	9.544633808,
7.267094789,	9.429561415,	17.57786196,	11.08230558,	4.177171521,	2.68847225,	33.047375,	3.040189197,
7.080688523,	4.549025722,	13.26850982,	733.1210071,	15.41323801,	14.90192861,	433.4901023,	1.378515647,
12.07705727,	100.0370056,	0.432158846,	2.80633827,	2.868455465,	19.75530164,	4.941236119,	1.841817585,
78.15062995,	12.95601582,	9.227386623,	5.75699604,	111.0229279,	28.68559835,	3.084617954,	1.444231929,
4.262955536,	21.5456551,	5.686844527,	7.490987521,	63.07827634,	12.98329152,	187.9286675,	10.40638984,
38.05564228,	266.4784236,	3748.103849,	9.129830774,	3.207910428,	1.902389161,	6.994997018,	22.12778934,
1.958813906,	0.579720436,	9.495297141,	1.222587466,	0.212048184,	6.397700384,	6.616041029,	20.70840339,
1.71091793,	3.267499179,	2832.021786,	2.260415657,	2.693748913,	88.15983816,	4.825214222,	55.14469703,
1.620542508,	148.7825235,	16.36747157,	30.14227137,	55.5184826,	97.5214604,	3.642303815,	61.66279641,
5.44157896,	203.8740281,	743.3802949,	12.36015879,	3.202360343,	6.502200238,	2.461320021,	9.101101976,
1549.918963,	9.933727333,	7.625807304,	40.34882953,	1.20891549,	353.5442057,	0.393396406,	9.570022435,
18.16277989,	2.770582605,	10.31127507,	0.451608397,	60.70853893,	1.889615753,	14.61873905,	11.01976714,
87.75902378,	11.55354156,	0.86012316,	220.7749472,	33.57327111,	23.88865418,	54.3353213,	42.63794221,
64825.31607,	29.68201209,	7.37594775,	14.25532321,	13.72962667,	5.338160669,	1.588461119,	2.280795442,
0.342964947,	1.600566951,	4.943251274,	6.813634143,	Data III, listed above, has 500 observations; its tail is lighter than that of Data II, but the data are still clearly skewed.			
33.24090269,	2.370163519,	1.691687405,	23.74500874,	Table 8 shows that all three datasets are right-skewed and leptokurtic. For Data I the mean (3.591945) exceeds the median (1.890581), the skewness is 4.093109 and the kurtosis is 23.52151. Data II is far more extreme: its mean (296.4995) is about thirty times its median (9.557328), with skewness 17.06976 and kurtosis 294.0157. For Data III the mean is 1.204326 and the median 0.746928, with			
101.2788369,	91.79397532,	30.50708874,	1180.852278,				
10.61832553,	37.45549434,	6.974687102,	5.409350335,				
137.5735692,	7.357982602,	31.35624559,	9.586543941,				
1.645283233,	1.821147682,	10.11037459,	25.08932214,				
30.61109293,	5.038723355,	0.32312265,	11.12415909,				
2.620230682,	4.727771009,	12.78896368,	6.309104987,				
8.231537286,	0.9128974,	2.532121746,	62.78686185,				

Table 8: Summary statistics of Data I, II and III.

Statistics	Data I	Data II	Data III
Minimum	0.041553	0.212048	0.01536
First Quartile	0.998929	3.221032	0.368285
Median	1.890581	9.557328	0.746928
Mean	3.591945	296.4995	1.204326
Third Quartile	3.515788	33.09576	1.328597
Maximum	45.96459	64825.32	26.36262
Std. Deviation	5.307313	3756.739	1.907795
Skewness	4.093109	17.06976	7.224677
Kurtosis	23.52151	294.0157	75.18231

skewness 7.224677 and kurtosis 75.18231. The datasets therefore differ considerably in skewness and tail weight.

Figures 12–14 show a histogram, a kernel density estimate, a box plot and a violin plot for each dataset. The histograms are unimodal, with right tails of different lengths. Data II is very widely spread, whereas most of Data III lies in a narrow central region with a few large values. The kernel density estimates are also unimodal, and their right tails decrease gradually rather than abruptly, which points to tails heavier than those of exponential-type models.

In the box plots the whiskers are asymmetric and longer on the upper side, and there are many large outlying values, which again indicates long right tails. The violin plots, which combine a density estimate with summary statistics, have elongated upper parts, supporting the presence of heavy tails.

Taken together, the numerical and graphical summaries show skewed, dispersed data whose tails are unlikely to be described well by light-tailed models such as the exponential distribution. We therefore compare the ETLx distribution with the Weibull, Lomax, Gamma, Lognormal and Exponential distributions to see how well it fits these data.

6.2 Model Comparison and Goodness-of-Fit Analysis

The ETLx distribution was compared with five common lifetime distributions: the Lomax, Weibull, Gamma, Lognormal and Exponential. These are standard benchmarks in reliability, survival analysis, actuarial science and risk modeling. The Lomax distribution is the baseline of the ETLx model, so comparing the two shows directly what the entropy transformation adds. The Weibull and Exponential distributions are classical lifetime models, while the Gamma and Lognormal distributions are flexible, positively skewed models that allow for different degrees of heterogeneity and tail weight.

For each fitted model we computed the log-likelihood (LogLik) and the Akaike (AIC), Bayesian (BIC),

Hannan–Quinn (HQIC) and consistent Akaike (CAIC) information criteria. A better model has a larger log-likelihood and smaller values of the four criteria. The results for the three datasets are given in Tables 9, 11 and 13.

We also checked the fit with the Kolmogorov–Smirnov (KS) statistic and its p-value, the Cramér–von Mises (CVM) statistic and the Anderson–Darling (AD) statistic. Smaller KS, CVM and AD values, along with larger KS p-values, indicate closer agreement between the fitted model and the data. These results are reported in Tables 10, 12 and 14.

For Data I, the ETLx model has log-likelihood –434.402, AIC 872.804, BIC 879.401, HQIC 875.474 and CAIC 881.401 (Table 9). Its KS statistic is 0.0474 (p-value 0.742), and its CVM and AD statistics are 0.0546 and 0.3610 (Table 10). All of these are better than the values for the competing models, so the ETLx distribution gives the best fit to Data I.

For Data II the ETLx model gives log-likelihood –1323.918, AIC 2651.836, BIC 2659.244, HQIC 2654.801 and CAIC 2661.244 (Table 11), with KS = 0.0229 (p-value 0.997), CVM = 0.0187 and AD = 0.1402 (Table 12). Again, no competing distribution does better on any of these measures.

For Data III the corresponding values are –545.263 for the log-likelihood, 1094.526 for AIC, 1102.956 for BIC, 1097.834 for HQIC and 1104.956 for CAIC (Table 13), together with KS = 0.0242 (p-value 0.925), CVM = 0.0347 and AD = 0.2119 (Table 14). The ETLx model is once more the best of the six.

Overall, the ETLx distribution yields the most favorable model selection criteria and goodness-of-fit statistics for all three datasets. The entropy transformation thus gives a good trade-off between flexibility and parsimony, and the ETLx distribution is a useful model for positively skewed, heavy-tailed lifetime data in reliability analysis, survival studies and risk modeling.

Note: Bold entries mark the best model for each criterion.

Note: Bold entries mark the best model for each criterion.

We also assessed the fits graphically. For each dataset the fitted densities were drawn over the histogram (Figures 15–17), and the fitted CDFs were compared with the empirical CDF (Figures 18–20). The ETLx density follows the shape of the histogram closely for all three datasets, both at the peak and along the slowly decaying right tail. The Weibull, Lomax, Gamma, Lognormal and Exponential densities deviate more, mainly in the tails.

The CDF plots show the same pattern. The fitted ETLx CDF is very close to the empirical CDF. The Exponential distribution departs from it markedly and the Weibull and Gamma distributions moderately, while the Lognormal and Lomax distributions fit better than these but still less well than the ETLx model. The graphical results therefore agree with the numerical comparison.

Graphical Summary of Data I

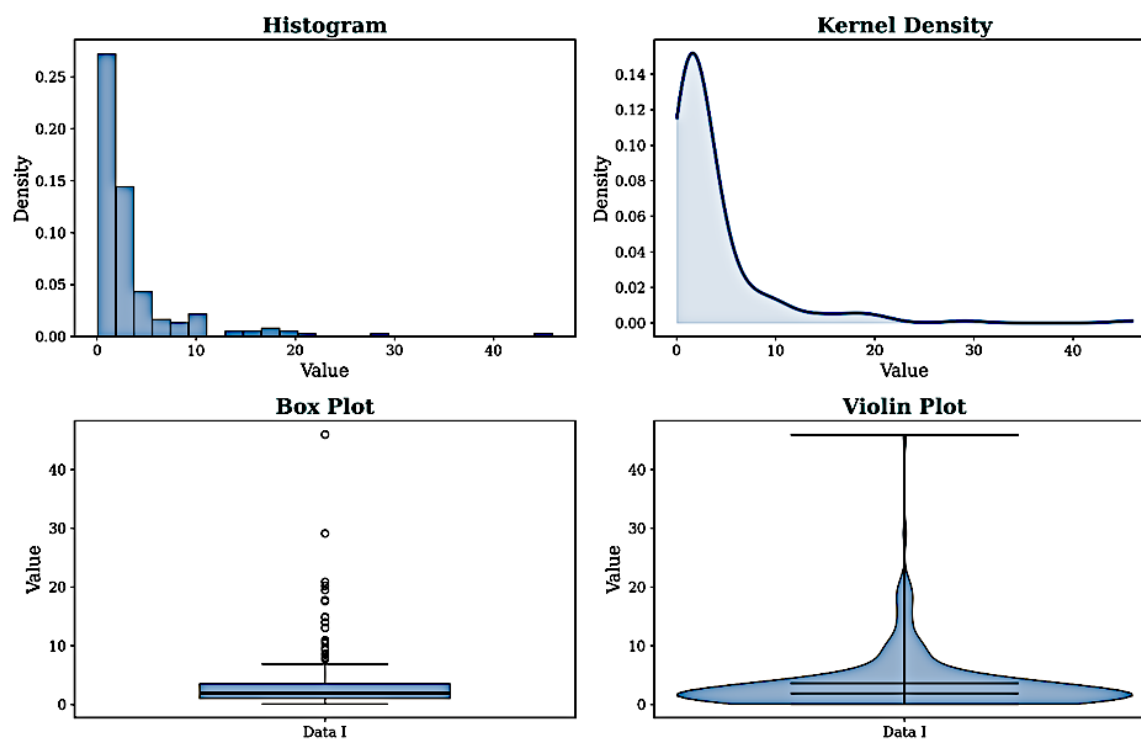


Fig. 12: Histogram, kernel density, box plot and violin plot of Data I.

Table 9: Model selection criteria for Dataset I.

Model	Parameter Estimates	LogLik	AIC	BIC	HQIC	CAIC
ETLx	$\hat{\alpha}=2.3724$					
$\hat{\lambda}=1.8638$	-434.4022	872.8045	879.4011	875.4740	881.4011	
Lognormal	$\hat{\sigma}=1.1099$					
$\hat{\mu}=1.9217$	-435.2885	874.5770	881.1737	877.2466	883.1737	
Lomax	$\hat{\alpha}=3.5108$					
$\hat{\lambda}=8.9646$	-444.4532	892.9065	899.50311	895.5760	901.5031	
Weibull	$\hat{k}=0.8924$					
$\hat{\lambda}=3.3568$	-452.9665	909.9330	916.52962	912.6026	918.5296	
Exponential	$\hat{\lambda}=3.5911$	-455.7388	913.4776	916.7759	914.8123	917.7759
Gamma	$\hat{\alpha}=0.9308$					
$\hat{\beta}=3.8591$	-455.3958	914.7917	921.3883	917.4612	923.3883	

Table 10: Goodness-of-fit statistics for Dataset I.

Model	KS	KS (p-value)	CVM	CVM (p-value)	AD
ETLx	0.0474	0.7419	0.0546	0.8483	0.3610
Lognormal	0.0555	0.5497	0.0792	0.6973	0.4986
Lomax	0.0885	0.0822	0.3842	0.0795	2.6944
Weibull	0.1162	0.0083	0.6595	0.0159	4.0106
Exponential	0.1432	0.0005	1.0461	0.0019	5.3306
Gamma	0.1367	0.0010	0.9025	0.0041	4.8217

Graphical Summary of Data II

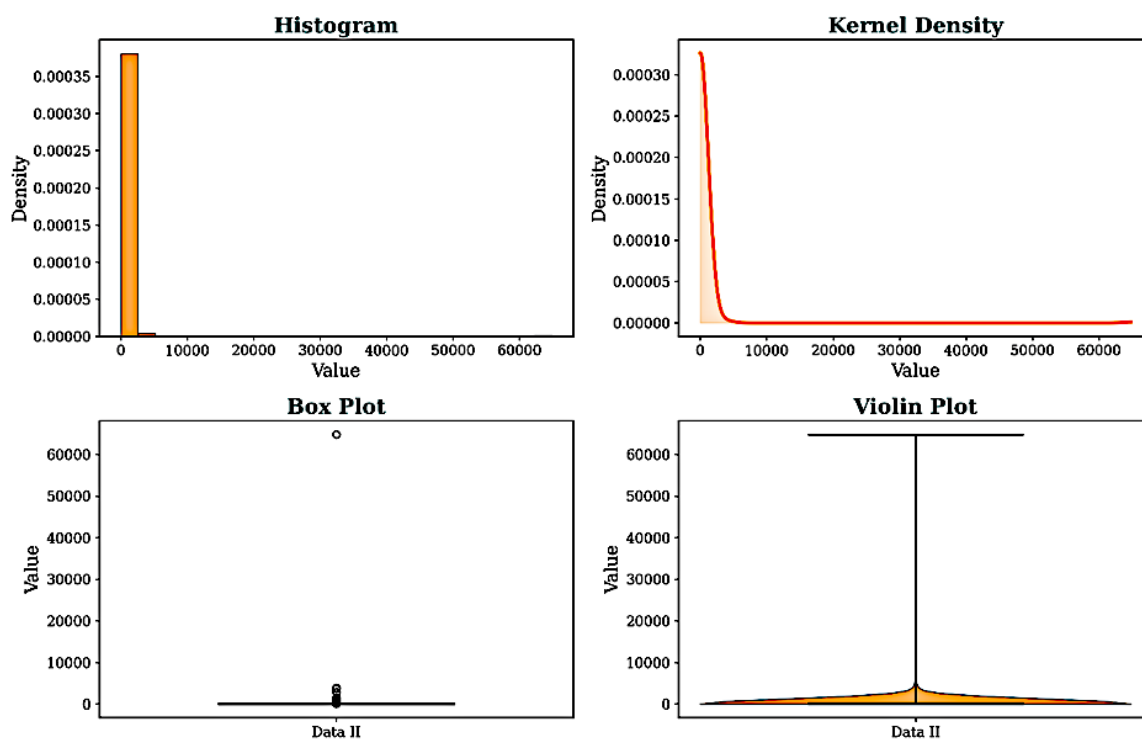


Fig. 13: Histogram, kernel density, box plot and violin plot of Data II.

Table 11: Model selection criteria for Dataset II.

Model	Parameter Estimates	LogLik	AIC	BIC	HQIC	CAIC
ETLx	$\hat{\alpha}=0.9456$					
$\hat{\lambda}=1.9462$	-1323.9182	2651.8365	2659.2441	2654.8010	2661.2441	
Lomax	$\hat{\alpha}=0.8663$					
$\hat{\lambda}=8.4541$	-1329.7380	2663.4759	2670.8835	2666.4405	2672.8835	
Lognormal	$\hat{\sigma}=1.7956$					
$\hat{\mu}=11.5742$	-1335.9108	2675.8216	2683.2291	2678.7861	2685.2292	
Weibull	$\hat{k}=0.4387$					
$\hat{\lambda}=30.0321$	-1407.2942	2818.5884	2825.9960	2821.5529	2827.9960	
Gamma	$\hat{\alpha}=0.2230$					
$\hat{\beta}=1329.34$	-1541.5652	3087.13042	3094.5380	3090.0949	3096.5380	
Exponential	$\hat{\lambda}=296.499$	-2007.6136	4017.2273	4020.9310	4018.7096	4021.9311

Table 12: Goodness-of-fit statistics for Dataset II.

Model	KS	KS (p-value)	CVM	CVM (p-value)	AD
ETLx	0.0229	0.9965	0.0187	0.9981	0.1402
Lomax	0.0488	0.4592	0.0857	0.6601	0.8087
Lognormal	0.0655	0.1460	0.2517	0.1863	1.5248
Weibull	0.1562	0.0000	2.1187	0.0000	13.5366
Gamma	0.2994	2.4E-24	9.9351	1.6E-09	49.3398
Exponential	0.6694	4.3E-13	58.6327	2.4E-08	449.9957

Graphical Summary of Data III

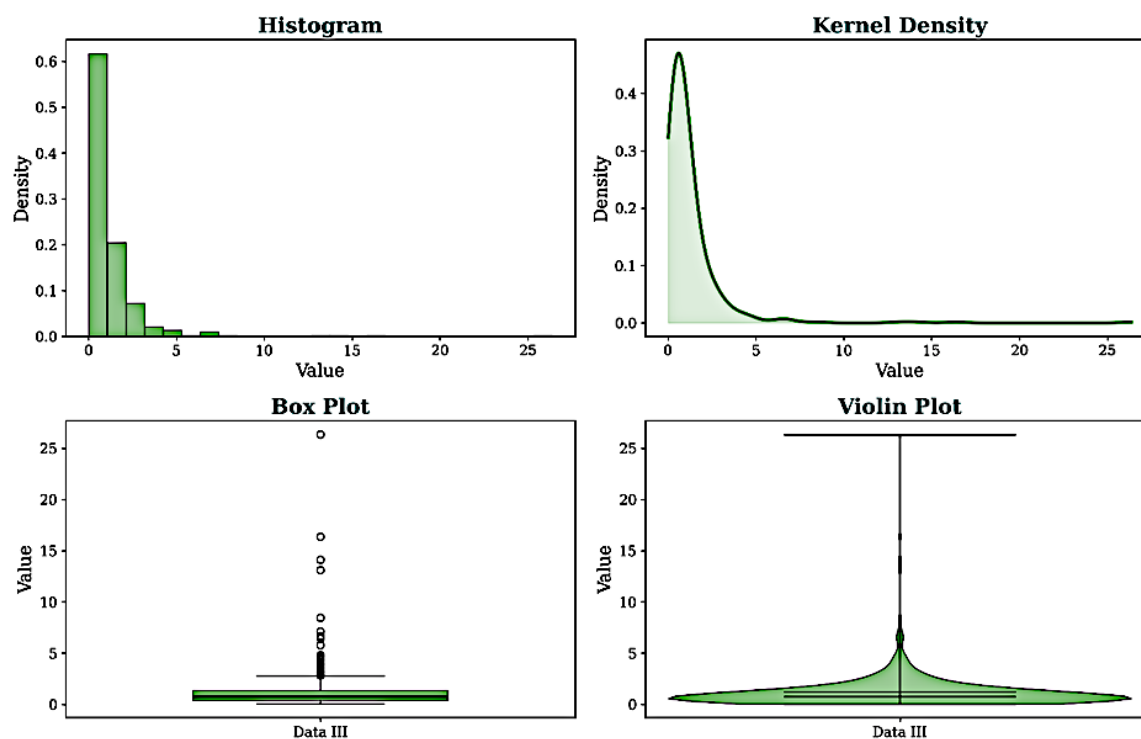


Fig. 14: Histogram, kernel density, box plot and violin plot of Data III.

Table 13: Model selection criteria for Dataset III.

Model	Parameter Estimates	LogLik	AIC	BIC	HQIC	CAIC
ETLx	$\hat{\alpha}=3.1146$					
$\hat{\lambda}=1.0099$	-545.2632	1094.5263	1102.9556	1097.8340	1104.9556	
Lognormal	$\hat{\sigma}=1.0335$					
$\hat{\mu}=0.7017$	-548.7803	1101.5607	1109.9899	1104.8683	1111.9899	
Lomax	$\hat{\alpha}=5.4397$					
$\hat{\lambda}=5.2305$	-572.3179	1148.6358	1157.0650	1151.9434	1159.0650	
Weibull	$\hat{k}=0.9496$					
$\hat{\lambda}=1.1699$	-591.4761	1186.9521	1195.3813	1190.2597	1197.3813	
Exponential	$\hat{\lambda}=1.2043$	-592.9600	1187.9200	1192.1346	1189.5738	1193.1346
Gamma	$\hat{\alpha}=1.0612$					
$\hat{\beta}=1.1349$	-592.4070	1188.8139	1197.2431	1192.1215	1199.2431	

Table 14: Goodness-of-fit statistics for Dataset III.

Model	KS	KS (p-value)	CVM	CVM (p-value)	AD
ETLx	0.0242	0.9253	0.0347	0.9585	0.2119
Lognormal	0.0329	0.6398	0.1233	0.4822	0.7402
Lomax	0.0848	0.0014	0.9490	0.0032	6.7637
Weibull	0.0823	0.0022	1.0317	0.0021	7.0533
Exponential	0.0897	0.0006	1.0879	0.0015	6.6584
Gamma	0.0916	0.0004	1.1578	0.0011	6.5435

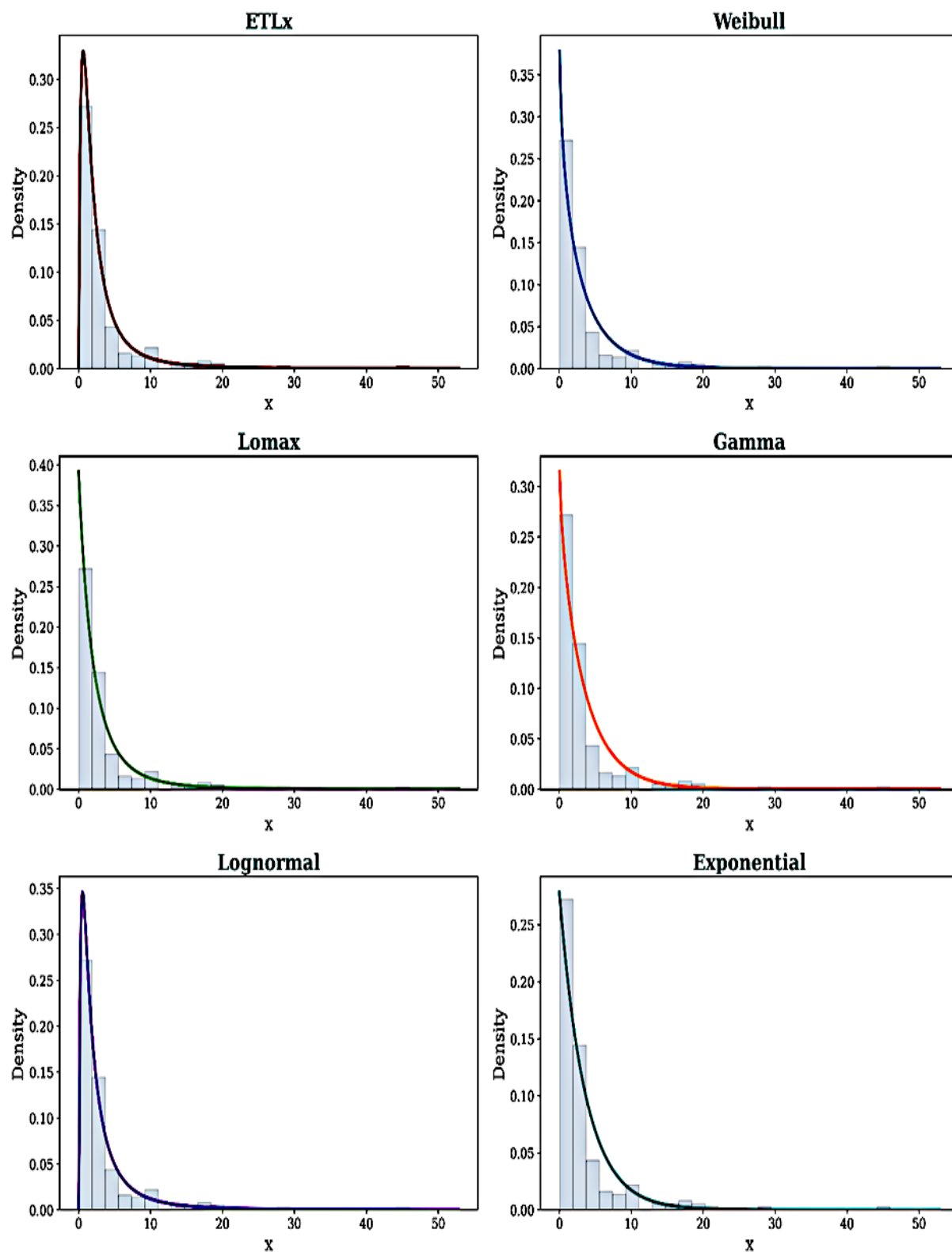


Fig. 15: Histogram of Data I with the fitted densities.

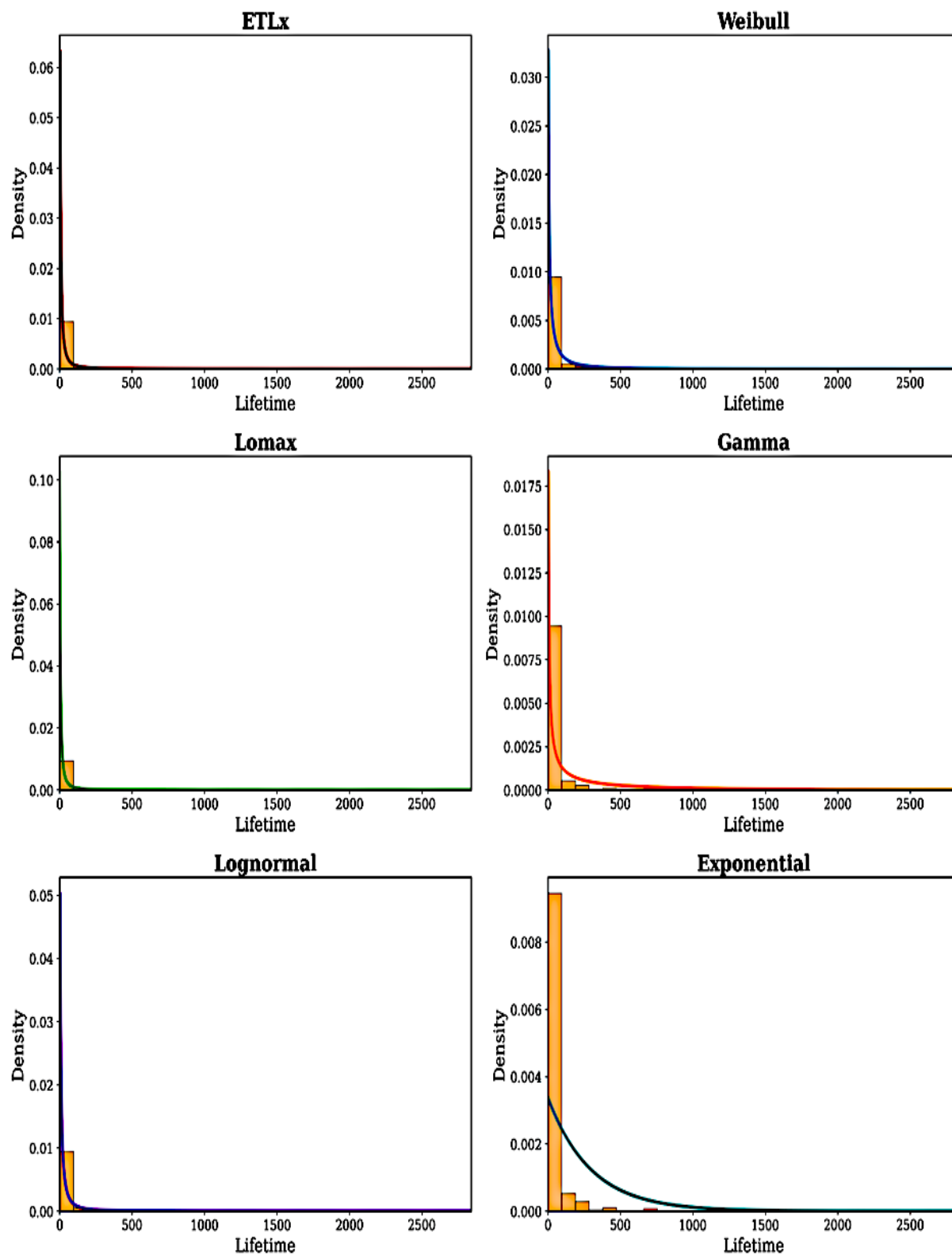


Fig. 16: Histogram of Data II with the fitted densities.

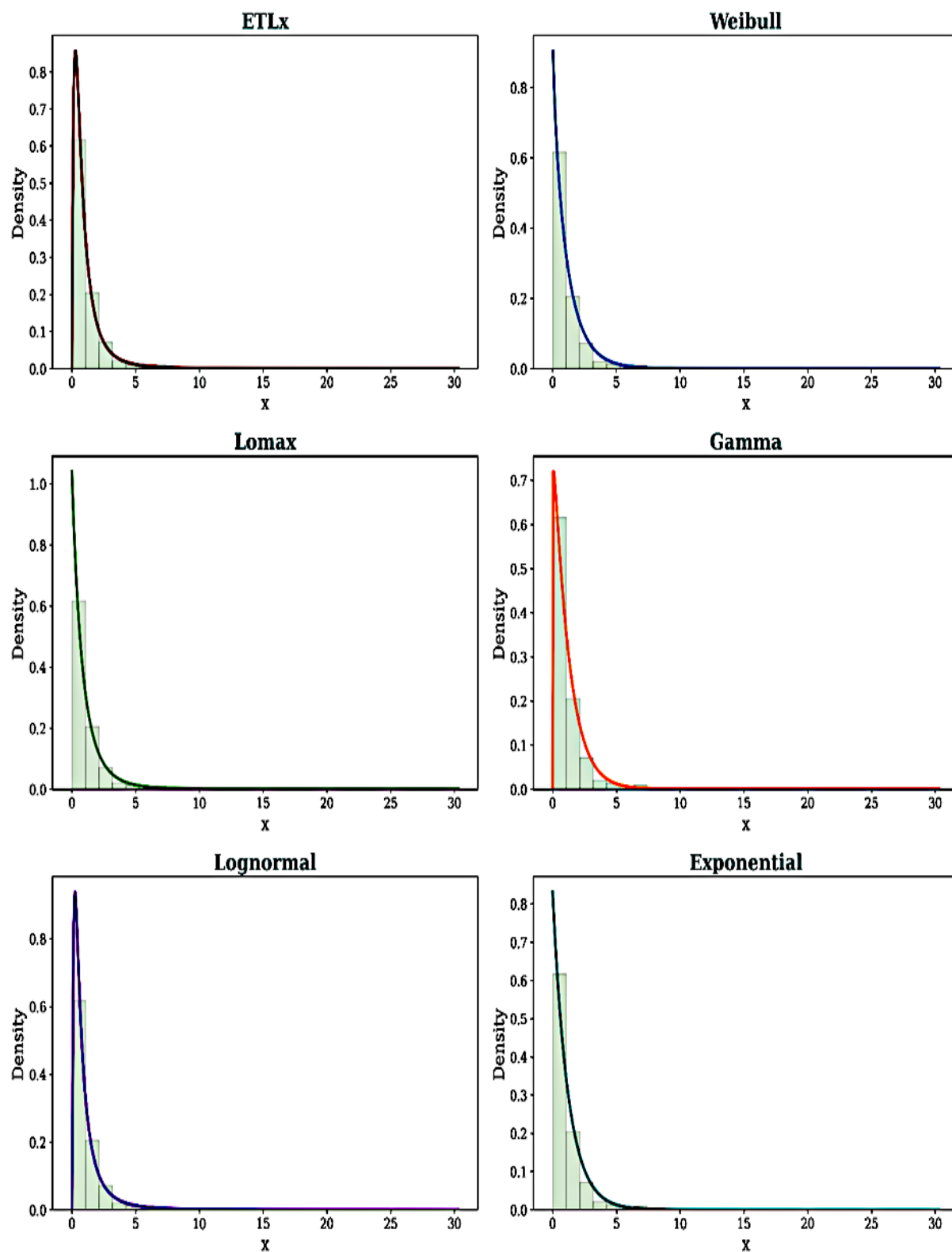


Fig. 17: Histogram of Data III with the fitted densities.

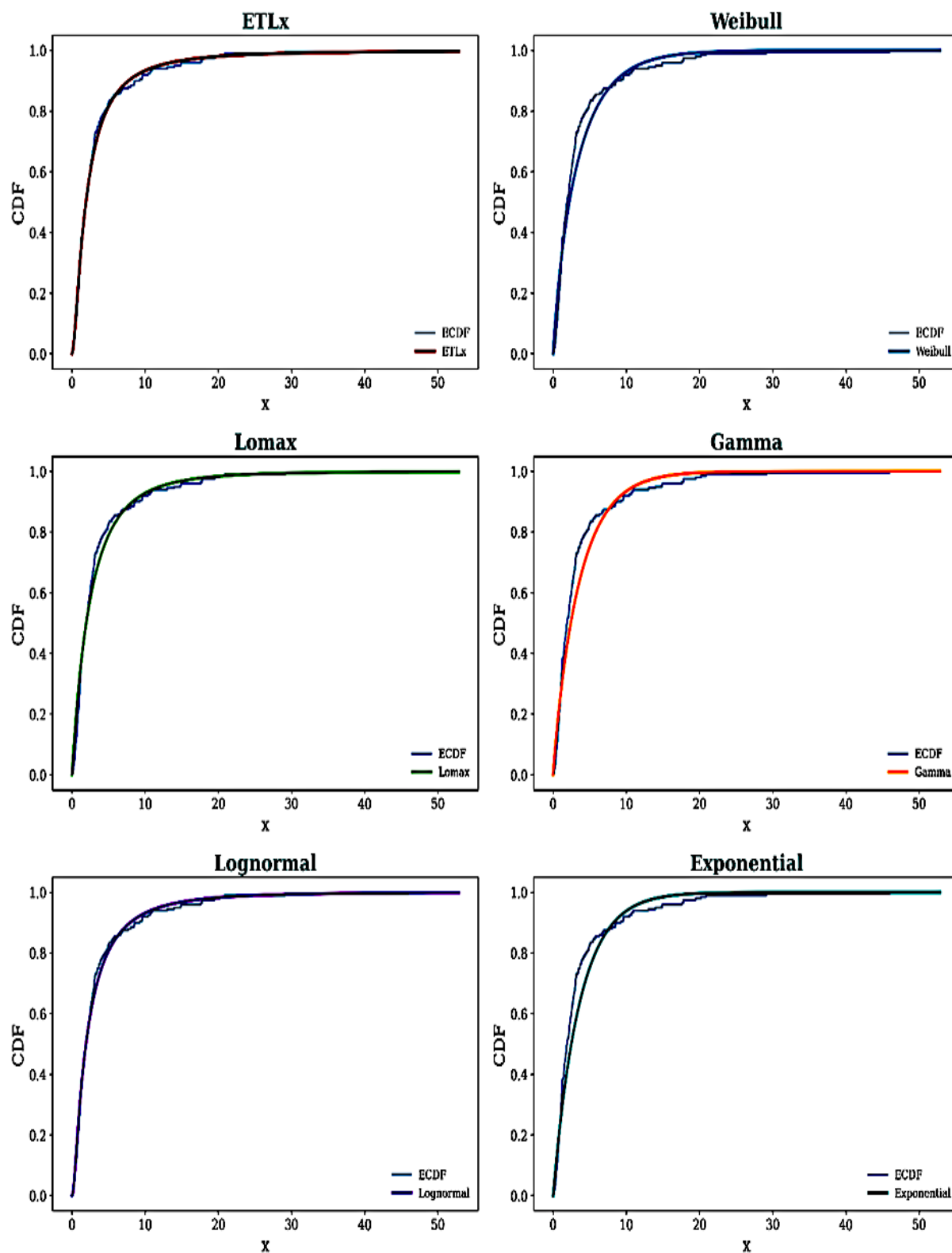


Fig. 18: Empirical and fitted CDFs for Data I.

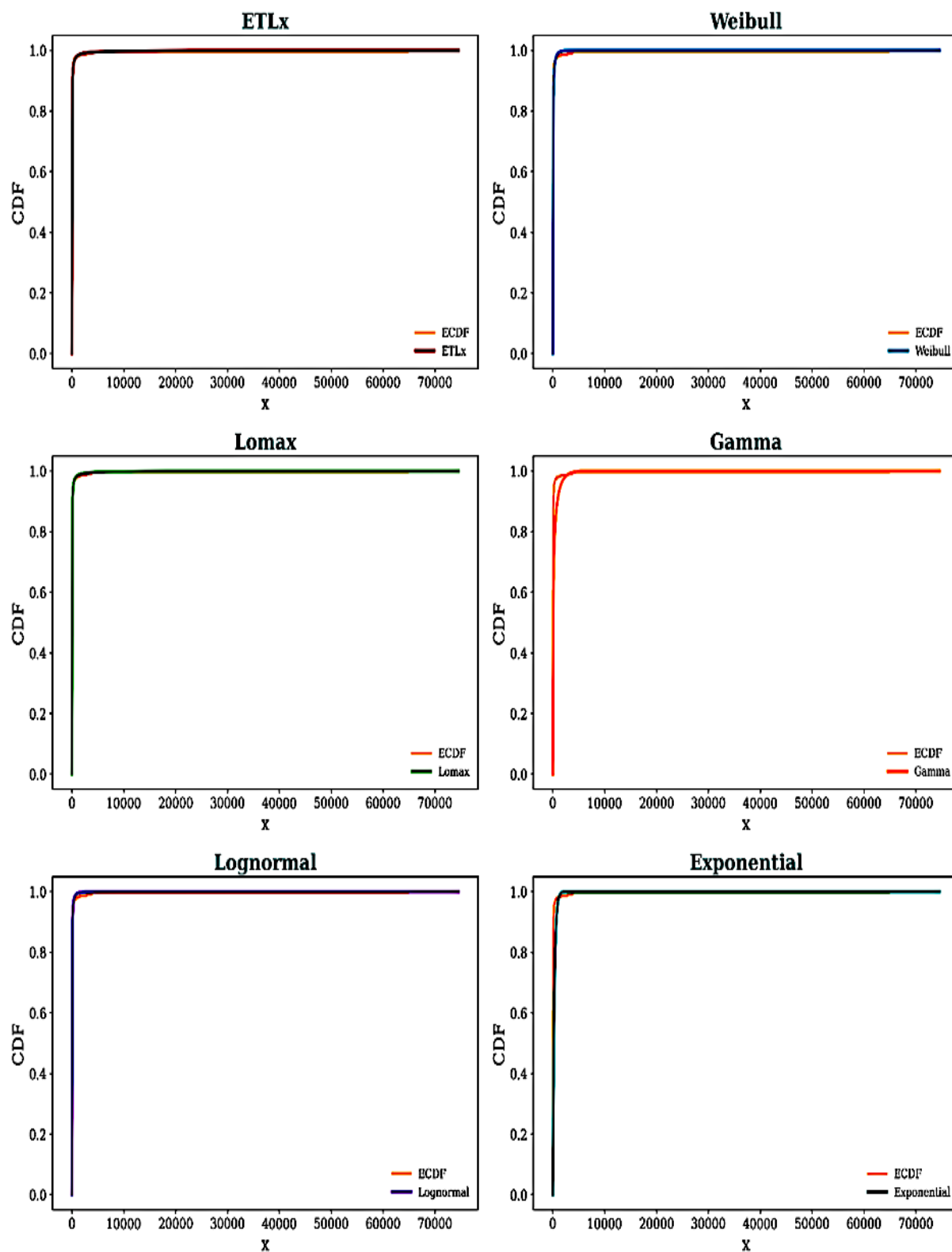


Fig. 19: Empirical and fitted CDFs for Data II.

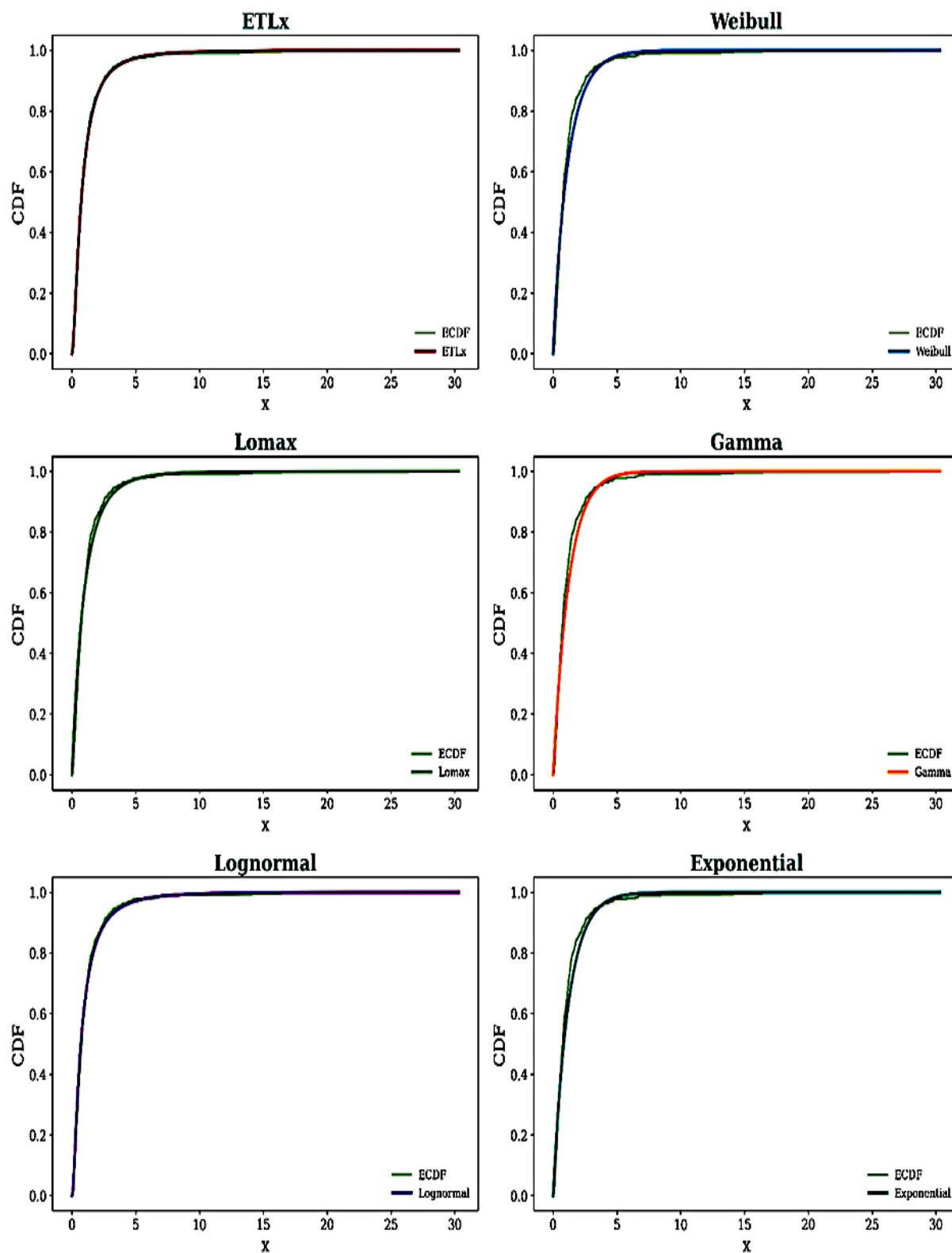


Fig. 20: Empirical and fitted CDFs for Data III.

The five competing distributions were chosen because they are widely used in reliability and lifetime analysis. As the parent model, the Lomax distribution is the most direct benchmark for judging what the entropy transformation contributes. The Weibull and Exponential distributions are established reliability models with simple hazard functions, and the Gamma and Lognormal distributions are common choices for positively skewed, heterogeneous lifetimes. Together they form a reasonable reference set for assessing the ETLx distribution.

6.3 Discussion

The theoretical results, the simulation study and the data applications together give a clear picture of how the ETLx distribution behaves. Our aim was a two-parameter model for skewed and moderately heavy-tailed lifetime data, and the results suggest that this aim has been met. On the theoretical side, the ETLx distribution extends the Lomax distribution through a transformation that adds flexibility to the shapes of the density and the hazard rate without adding parameters. The derived density, distribution, survival and hazard functions show that the model can produce slowly decaying tails and different degrees of skewness, as is often seen in reliability data.

The simulation study shows that the maximum likelihood estimators work well: their bias and mean squared error decrease steadily as the sample size grows, as expected from consistent and efficient estimators. In the applications, the ETLx model had the largest log-likelihood and the smallest information criteria for all three datasets, so it offered the best compromise between fit and complexity among the models considered. Its Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling statistics were also clearly smaller than those of the other models.

The behavior in the tails deserves attention. The Anderson–Darling statistic gives more weight to tail deviations, and it favored the ETLx model in every case, which suggests that the model captures the tail behavior that matters in reliability and risk analysis. The histogram and CDF comparisons show that the fitted ETLx distribution follows the data over the whole range, including the centre and the upper tail. The logarithmic factor in the density is what allows a wider range of tail shapes than the Exponential or Gamma distributions, while the two-parameter form keeps the model easy to interpret and to compute.

The ETLx distribution therefore adds a practical option to the existing lifetime models for heavy-tailed reliability data, and it can be used alongside the traditional distributions when skewness and tail behavior are important.

7 Conclusion

We have introduced the ETLx distribution, a two-parameter lifetime model obtained by applying an entropy-based transformation to the Lomax distribution. We derived its density, distribution, survival, hazard rate and quantile functions, its ordinary and incomplete moments, skewness and kurtosis, and its Rényi and Tsallis entropies. The parameters were estimated by maximum likelihood, and the finite-sample behavior of the estimators was examined in a Monte Carlo study. In applications to three generated lifetime datasets, the ETLx distribution was competitive with, and in most cases better than, the Lomax, Weibull, Gamma, Lognormal and Exponential distributions.

The study has some limitations. Inference was based only on maximum likelihood for complete data, and the consistency and asymptotic efficiency of the estimators were not proved. The applications used generated data and a set of standard benchmark models, so the conclusions apply to the models and data considered here. Several extensions are possible. Bayesian, maximum product spacing, robust and percentile-based estimators could be developed. The model could be adapted to right-, left- and interval-censored data, which are common in reliability and survival studies. Regression models based on the ETLx distribution would make it possible to analyze lifetimes with explanatory variables, and bivariate, multivariate and copula-based versions could be used for dependent lifetimes. Future work may also study the estimators in very small samples (e.g. $n < 50$) and compare them with other estimation methods.

It would also be useful to compare the ETLx distribution with recent Lomax extensions and with other entropy-based lifetime models, and to apply it to publicly available data from reliability engineering, the biomedical and environmental sciences, actuarial science, finance and risk analysis.

Acknowledgments

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4290034GSSR06.

Funding

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4290034GSSR06.

Data Availability Statement

The datasets analyzed in this study were generated from the ETLx distribution by the inverse transform method

described in Section 5. They are listed in full in Section 6, so that all results can be reproduced.

Conflicts of Interest

The authors declare no conflict of interest.

References

- [1] M. Aboraya, M. M. Ali, H. M. Yousof and M. Ibrahim, A novel Lomax extension with statistical properties, copulas, different estimation methods and applications, *Bull. Malays. Math. Sci. Soc.*, **45**, 85–120 (2022).
- [2] A. Z. Afify, Z. M. Nofal, H. M. Yousof, Y. M. El Gebaly and N. S. Butt, The transmuted Weibull Lomax distribution: properties and application, *Pak. J. Stat. Oper. Res.*, 135–152 (2015).
- [3] A. A. H. Ahmadini, B. Elkalzah, K. Elsharkawy and G. A. Marei, Reliability estimation of s -out-of- k systems under inverted Z-Lindley stress-strength models with progressive censoring: an engineering application, *Qual. Reliab. Eng. Int.*, (2026), <https://doi.org/10.1002/qre.70391>.
- [4] A. A. H. Ahmadini, M. Elwahab, M. Elgarhy, H. Aljohani and G. A. Marei, Modulating early-life risk without breaking Weibull structure: an odds-based Weibull model for engineering failure-time data, *Rev. Int. Métodos Numér. Cál. Diseño Ing.*, **42**(6), (2026), <https://doi.org/10.23967/j.rimni.2026.10.82546>.
- [5] A. A. Al-Babtain, A. S. Hassan, A. N. Zaky, I. Elbatal and M. Elgarhy, Dynamic cumulative residual Rényi entropy for Lomax distribution: Bayesian and non-Bayesian methods, *AIMS Math.*, **6**, 3889–3914 (2021).
- [6] A. H. Alenezy, B. Elkalzah, A. F. I. Alahshan and G. A. Marei, Robust inference for stress-strength reliability of the Garhy distribution under diverse record schemes with engineering and medical applications, *Mathematics*, **14**, 1940 (2026).
- [7] A. H. Alenezy, K. A. Rashedi, A. B. Ghorbal and G. A. Marei, Bridging Markov chain Monte Carlo techniques and Tierney–Kadane approximations for progressively censored Garhy reliability models: simulation insights and a medical application, *Mathematics*, **14**, 1777 (2026).
- [8] A. H. Alenezy, L. S. Diab, K. A. Rashedi and G. A. Marei, Inference for a step-stress model under the inverted Z-Lindley distribution with progressive Type-II censoring, *AIMS Math.*, **11**, 31919–31962 (2026).
- [9] L. A. Al-Essa, M. S. Eliwa, M. El-Morshedy, H. Alqifari and H. M. Yousof, Flexible extension of the Lomax distribution for asymmetric data under different failure rate profiles: characteristics with applications for failure modeling and service times for aircraft windshields, *Processes*, **11**, 2197 (2023).
- [10] H. M. Almongy, E. M. Almetwally and A. E. Mubarak, Marshall-Olkin alpha power Lomax distribution: estimation methods, applications on physics and economics, *Pak. J. Stat. Oper. Res.*, 137–153 (2021).
- [11] N. Alotaibi, I. Elbatal, M. Shrahili, A. S. Al-Moisheer, M. Elgarhy and E. M. Almetwally, Statistical inference for the Kavya–Manoharan Kumaraswamy model under ranked set sampling with applications, *Symmetry*, **15**, 587 (2023).
- [12] R. Alotaibi, H. Okasha, H. Rezk, A. M. Almarashi and M. Nassar, On a new flexible Lomax distribution: statistical properties and estimation procedures with applications to engineering and medical data, *AIMS Math.*, **6**, 13976–13999 (2021).
- [13] N. Alsadat, M. Elgarhy, A. H. Tolba, A. S. Elwehidy, H. Ahmad and E. M. Almetwally, Classical and Bayesian estimation for the extended odd Weibull power Lomax model with applications, *AIP Adv.*, **13** (2023).
- [14] M. Anwar and J. Zahoor, The half-logistic Lomax distribution for lifetime modeling, *J. Probab. Stat.*, **2018**, 3152807 (2018).
- [15] S. Ashour and M. Eltehiwy, Transmuted Lomax distribution, *Amer. J. Appl. Math. Stat.*, **1**, 121–127 (2013).
- [16] R. A. Bantan, M. Elgarhy, C. Chesneau and F. Jamal, Estimation of entropy for inverse Lomax distribution under multiple censored data, *Entropy*, **22**, 601 (2020).
- [17] R. Clausius, Ueber verschiedene für die Anwendung bequeme Formen der Hauptgleichungen der mechanischen Wärmetheorie, Verlag nicht ermittelbar, (1865).
- [18] G. M. Cordeiro, E. M. Ortega and B. V. Popović, The gamma-Lomax distribution, *J. Stat. Comput. Simul.*, **85**, 305–319 (2015).
- [19] M. Elgarhy, G. Abdalla, A. Hassan and E. Almetwally, Bayesian and non-Bayesian analysis of the novel unit inverse exponentiated Lomax distribution using progressive censoring schemes with optimal scheme and data application, *Comput. J. Math. Stat. Sci.*, **5**, 78–108 (2026).
- [20] M. Elgarhy, Garhy distribution with different estimation methods and applications to engineering and medical data, *Rev. Int. Métodos Numér. Cál. Diseño Ing.*, **42**, 90 (2026).
- [21] B. Elkalzah, M. O. Mohamed, K. Elsharkawy, E. S. Osman, A. Aldukeel and G. A. Marei, Classical and Bayesian estimation of stress-strength reliability under the discrete alpha-power Weibull distribution with incomplete and record data: application to high-voltage capacitors, *AIMS Math.*, **11**, 6374–6399 (2026).
- [22] M. A. Fawzy, H. Athar and Y. F. Alharbi, Inference based on Marshall-Olkin extended Rayleigh Lomax distribution, *Appl. Math. E-Notes*, **21**, 1–11 (2021).
- [23] H. H. Hadi and N. H. Al-Noor, Truncated exponential Marshall Olkin Lomax distribution: properties, entropies, and applications, *AIP Conf. Proc.*, **2414**, 040054 (2023).
- [24] H. Haj Ahmad, E. M. Almetwally, M. Elgarhy and D. A. Ramadan, On unit exponential Pareto distribution for modeling the recovery rate of COVID-19, *Processes*, **11**, 232 (2023).
- [25] A. S. Hassan and M. A. Abdelghafar, Exponentiated Lomax geometric distribution: properties and applications, *Pak. J. Stat. Oper. Res.*, 545–566 (2017).
- [26] A. S. Hassan, E. M. Almetwally, S. C. Gamoura and A. S. Metwally, Inverse exponentiated Lomax power series distribution: model, estimation, and application, *J. Math.*, **2022**, 1998653 (2022).
- [27] M. Ijaz, S. M. Asim and Alamgir, Lomax exponential distribution with an application to real-life data, *PLoS ONE*, **14**, e0225827 (2019).

- [28] M. Imran, M. A. F. Elbarkawy, F. Jamal and G. A. Marei, An extended Lindley model with applications to entropy-based loss and actuarial risk measures, *Comput. J. Math. Stat. Sci.*, **5**, 1–33 (2026).
- [29] A. Isaic-Maniu and V. G. Voda, Some comments on an entropy-like transformation of Soleha and Sewilam, *Econ. Comput. Econ. Cybern. Stud. Res.*, **42**, 5–11 (2008).
- [30] A. I. Ishaq, A. A. Suleiman, H. N. Alqifari, A. Almohaimeed, H. Daud and S. I. Abba, Developing new bivariate distributions with advanced estimation methods for interdisciplinary data analysis, *J. Math.*, **2026**, 5518563 (2026).
- [31] A. I. Ishaq et al., A new log-Lomax distribution, properties, stock price, and heart attack predictions using machine learning techniques, *AIMS Math.*, **10**, 12761–12807 (2025).
- [32] A. A. Khalaf, A. R. El-Saeed, M. A. Khaleel and A. H. Tolba, The four-parameter odd generalized Rayleigh Lomax distribution: theory, simulation, and applications, *Symmetry*, (2026).
- [33] N. Kilany, *Weighted Lomax distribution*, SpringerPlus, **5**, 1862 (2016).
- [34] N. Kumar and V. Vijay, On estimation of Tsallis entropy for exponentially distributed doubly type-II censored sample, *Comput. J. Math. Stat. Sci.*, **5**, 34–77 (2026).
- [35] A. Kumari, S. Kumar and K. Kumar, Inference for reliability in a multicomponent stress-strength model from generalized inverted exponential lifetime distribution under progressive first failure censoring, *J. Stat. Comput. Simul.*, **93**, 863–887 (2023).
- [36] A. J. Lemonte and G. M. Cordeiro, An extended Lomax distribution, *Statistics*, **47**, 800–816 (2013).
- [37] J. Miśkiewicz, Improving quality of sample entropy estimation for continuous distribution probability functions, *Physica A*, **450**, 473–485 (2016).
- [38] N. A. Noori, M. A. Khaleel, S. A. Khalaf and S. Dutta, Analytical modeling of expansion for odd Lomax generalized exponential distribution in framework of neutrosophic logic: a theoretical and applied on neutrosophic data, *Innov. Stat. Probab.*, **1**, 47–59 (2025).
- [39] G. O. Orji, H. O. Etaga, E. M. Almetwally, C. P. Igboke, O. C. Aguwa and O. J. Obulezi, A new odd reparameterized exponential transformed-X family of distributions with applications to public health data, *Innov. Stat. Probab.*, **1**, 88–118 (2025).
- [40] A. A. Osi, G. S. S. Abdalla, N. S. Sawaran Singh and A. A. Suleiman, Advancing lifetime data modeling via the Marshall-Olkin cosine Topp-Leone distribution family, *Comput. J. Math. Stat. Sci.*, **4**, 476–499 (2025).
- [41] U. Panitanarak, A. I. Ishaq, A. A. Abiodun, H. Daud and A. A. Suleiman, A new Maxwell-log logistic distribution and its applications for mortality rate data, *J. Niger. Soc. Phys. Sci.*, 1976 (2025).
- [42] U. Panitanarak et al., A new beta distribution with interdisciplinary data analysis, *AIMS Math.*, **10**, 8495–8527 (2025).
- [43] S. M. Papalexiou and D. Koutsoyiannis, Entropy based derivation of probability distributions: a case study to daily rainfall, *Adv. Water Resour.*, **45**, 51–57 (2012).
- [44] K. A. Rashedi, L. S. Diab, A. H. Alenezy and G. A. Marei, A one-parameter decreasing-hazard lifetime model with exact inference on a bounded-below support, *AIMS Math.*, **11**, 24445–24489 (2026).
- [45] K. A. Rashedi, L. S. Diab, A. H. Alenezy and G. A. Marei, Inference for stress-strength reliability under unified hybrid censoring: a one-parameter model with applications, *Mathematics*, **14**, 2041 (2026).
- [46] S. Saini and R. Garg, Reliability inference for multicomponent stress-strength model from Kumaraswamy-G family of distributions based on progressively first failure censored samples, *Comput. Stat.*, **37**, 1795–1837 (2022).
- [47] L. P. Sapkota and V. Kumar, Odd Lomax generalized exponential distribution: application to engineering and COVID-19 data, *Pak. J. Stat. Oper. Res.*, 883–900 (2022).
- [48] H. Semaary et al., A new modified sine-Weibull distribution for modeling medical data with dynamic structures, *J. Radiat. Res. Appl. Sci.*, **18**, 101427 (2025).
- [49] Z. Shah, Z. Ahmad, Z. Almaspoor, F. Khan, C. K. Onyekwere, G. S. Rao and S. K. Khosa, A novel flexible logarithmic cosine-G family of distributions with applications to event time data, *Mod. J. Stat.*, **2**, 81–102 (2026).
- [50] C. E. Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.*, **27**, 379–423 (1948).
- [51] M. Shrahili, A. R. El-Saeed, A. S. Hassan, I. Elbatal and M. Elgarhy, Estimation of entropy for log-logistic distribution under progressive type II censoring, *J. Nanomater.*, **2022**, 2739606 (2022).
- [52] B. P. Singh, M. Ailani and U. D. Das, Some statistical properties and applications of Lomax generated distribution, *Reliab. Theory Appl.*, **20**, 599–617 (2025).
- [53] A. A. Suleiman, H. Daud, A. G. Usman, S. I. Abba, M. Othman and M. Elgarhy, A new two-parameter half-logistic distribution with numerical analysis and applications, *J. Stat. Sci. Comput. Intell.*, **1**, 1–28 (2025).
- [54] A. A. Suleiman, Z. F. Althobaiti, H. Daud, N. Waeleh, M. E. Abd Elwahab and R. Idress, Development of the new beta prime-G family of distributions for advanced modeling of radiation data, *J. Radiat. Res. Appl. Sci.*, **19**, 102432 (2026).
- [55] A. A. Suleiman et al., The modified gamma distribution and machine learning for modeling and classification of groundwater potability, *Math. Model. Eng. Probl.*, **12** (2025).
- [56] P. S. Swetha, H. Hamdani, M. N. Ouertani, V. B. Nagarjuna, S. Nasiru and M. Elgarhy, Kumaraswamy modified Kies power Lomax distribution: properties, actuarial measures, and applications, *Eng. Rep.*, **7**, e70390 (2025).
- [57] M. H. Tahir, G. M. Cordeiro, M. Mansoor and M. Zubair, The Weibull-Lomax distribution: properties and applications, *Hacet. J. Math. Stat.*, **44**, 455–474 (2015).
- [58] M. H. Tahir, M. A. Hussain, G. M. Cordeiro, G. Hamedani, M. Mansoor and M. Zubair, The Gumbel-Lomax distribution: properties and applications, *J. Stat. Theory Appl.*, **15**, 61–79 (2016).
- [59] M. Yusuf, A. E. Aly and N. Al-Mutairi, Bayesian estimation of the Lomax distribution parameters under progressive Type-II censoring, *Adv. Basic Appl. Sci.*, **5**, 29–34 (2025).
- [60] M. Zubair, G. M. Cordeiro, M. Tahir, M. Mahmood and M. Mansoor, A study of logistic-Lomax distribution and its applications, *J. Probab. Stat. Sci.*, **15**, 29–46 (2017).

Hazar A. Khogeer B.Sc. Mathematics, Arizona State University, Tempe, AZ, USA. M. Sc. Statistics, Arizona State University, Tempe, AZ, USA, May 2014. Ph.D. Statistics, Arizona State University, Tempe, AZ, USA, May 2020.



Mustafa Kamal is a Professor in the Department of Basic Sciences, College of Science and Theoretical Studies, Prince Musaid bin Abdulrahman University, Saudi Arabia. He received his Ph.D. in Statistics from Aligarh Muslim University, India, in 2013. He has

authored and co-authored more than 70 research papers in reputed journals and conference proceedings, along with a book published by Springer and several book chapters. His research interests include statistical estimation and its applications, reliability theory, accelerated life testing, and statistical modelling, with a particular focus on developing statistical methods for complex real-world problems.

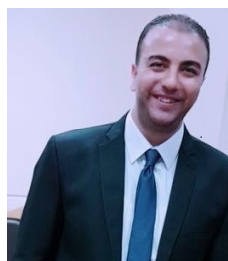
Yousef Alharbi earned his Ph.D. in 2022 and M.Sc. in Statistics in 2021 from the University of Missouri–Kansas City, an M.Sc. in Applied Mathematics from Kent State University in 2017, and a B.Sc. in Mathematics from Qassim University in 2010.

Eslam Hussam got his Phd from Helwan university in 2018 and Msc in 2013 and his bachelors in 2007 he has many papers in distribution theory and statistical inference.

Mohamed Said Mohamed is an Associate Professor in the Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim 16511, Saudi Arabia. His research interests include statistical modeling and data analysis, estimation theory, distribution theory, reliability theory, statistical computing, ordered random variables, concomitants of ordered random variables, and information theory.

Ahmad Abubakar Suleiman Department of Statistics, Faculty of Computing and Mathematical Sciences, Aliko Dangote University of Science and Technology, Wudil 713281, Nigeria.

Maryam Ibrahim Habadi is an Assistant Professor in the Department of Statistics at King Abdulaziz University, Jeddah, Saudi Arabia. She earned her Ph.D. in Statistics from the University of South Florida in 2019. Her research interests include statistical modeling, time series analysis, and applications of machine learning in environmental and health sciences. She has published several papers in international journals and conferences, focusing on climate change, Alzheimer's disease, and predictive modeling.



Mohammed Elgarhy is an Associate Professor of Statistics in Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt. He received his M.Sc. and Ph.D. in Statistics in 2014 and 2017 from Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt. His current research interests include generalized classes of distributions, distribution theory, statistical inference, entropy, ranked set sampling, and lifetime test. He has published more than 300 research articles in reputed international journals of mathematical and engineering sciences. He is a referee and editor of mathematical journals.