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Properties of Motion of the Infinitesimal Variable Mass Body in the Well Known Circular Restricted Three-Body Problem with Newtonian and Yukawa Potential

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Abstract: The effects of Newtonian and Yukawa gravitational potentials are studied on the circular restricted three-body system under the assumption that infinitesimal body varies its mass according to Jeans law. The equations of motion are determined under these perturbations. The numerical studies are conducted where locations of equilibrium points, regions of motion, trajectories with Poincaré surfaces of section and the basins of attraction have been investigated by well known software Mathematica. Moreover, the stability of the locations of equilibrium points are determined and it was found that all these points are unstable.

Keywords: Attracting domain, Newtonian potential, Variable mass, Trajectories, Yukawa potential.

1 Introduction

The classical circular restricted three-body problem is the most intriguing problem in celestial mechanics and dynamic astronomy. The restricted three-body problem describes the movement of a third body in the combined gravity field of two primary bodies with an infinitesimal mass (serving as the particle testing), according to [1]. Several practical applications exist in this subject ranging from theory of molecular physics, chaos, planetary physics and galactic dynamics.

In the last decade, the classical three-body problem has changed considerably to explain the motion existence of mass-free measuring particles in the solar system, considering more dynamic parameters. Particularly, several additional forces have increased the effective potential of the classical restricted three-body problem .

In the conventional version of the restricted three body problem, the two primaries are spherical and homogeneous. However, some celestial bodies (e.g. Saturn and Jupiter) in our solar system also have an oblate shape. The parameter of oblateness has been added to achieve a more accurate definition of the motion of the test particle. Numerous articles, such as: [2], [3], [4], [5], [6], [7], [8], [9], have explored the effect of oblateness on the character of motion.

The restricted three-body problem explains how the two finite masses (i.e. primaries) move in circular orbits around their center of mass due to their reciprocal attraction and infinitesimal mass body that does not affect the motion of primaries. It was first conceived because of the nearly circular orbits of the planets around the sun, the tiny masses of the asteroids as well as planetary satellites relative to the masses of those planets were considered. The restricted problem is also addressed in other configurations as three-body problem (with special configurations as copenhagen problem, Robe's problem), four-body problem, five-body problem and six-body problem by many researchers, including [10], [11], [12], [13], [14], [15] etc. The infinitesimal body has been seen to experience a position of rest at a certain point in the motion plane at some particular points in case of zero velocity and zero acceleration. These points are known as balance points (stationary points) and are five in classical case. Three of these points are called hill balance points because they lie on the x-axis and on the line that connects the primaries. These three points are referenced respectively by L_1, L_2 and L_3. The hillside balance points

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were usually unstable. The other two points are called triangular points of balance since they form a triangular equilateral arrangement with the primaries and are marked by L_4 and L_5. For values of the mass parameter μ below the critical value of Routh, triangular balancing points are stable. Over the years, periodic orbits around the balance points were studied both on the plane and perpendicular to the plane of motion. The same problem including oblateness of the primaries, radiation pressure effect, albedo effect, heterogeneous shapes, viscous force, charged body, finite straight segments, perturbations in Coriolis and centrifugal forces etc. are studied by [16], [17], [18], [19], [20], [21], [22], [23], [24], [25] etc.

Many researchers have investigated the variables mass of the test particle in the restricted three-body problem. [26] studied the two body problem with a variable mass when investigating the growth of double star. [27] worked on the mechanics of bodies with a variable mass. The effect of perturbations on the locations and stability of triangular equilibrium points in the restricted three-body problem with variable mass were investigated by [28], under the assumption that the third infinitesimal mass is variable and primaries are spherical with constant masses. They explored the problem by means of the Jeans law and space-time transformation, connecting it with the Meshcherskii transformations. The stability of libration points and the effect of perturbations on nonlinear stability of triangular points in the restricted three-body problem with variable mass were investigated by [29]. Furthermore, [30] examined the stability of the triangular equilibrium points using space-time transformation of Meshcherskii in the restricted three-body system where the third infinitesimal body with variable mass and two luminous primaries with constant masses exist. The problem of the motion of a star inside a layered inhomogeneous rotating elliptical galaxy with a variable mass was studied by [31] that found seven libration points of the autonomized equations located (except for one) outside the gravitating galaxy. He discussed the stability of these points using the Lyapunov Characteristic Number (LCN), and found that negative exponent solutions remained stable. [32] considered an analytical study of the dynamics of the third body in the restricted three-body problem with a variable mass, and derived the equation of motion when the loss of mass is non-isotropic. He studied the locations of the out-of-plane equilibrium for a non isotropic variation of the mass and proved the forbidden motions and the regions of possibility in restricted body problem.
This paper is organized, as follows: Section 2 presents the formation of the problem and equations of motion. Numerical explorations which contain the investigation of equilibrium points, regions of motion, Poincaré surfaces of section with trajectories allocation and basins of attraction are addressed in section 3. Stability of the equilibrium points are performed in Section 4. Conclusion is presented in Section 5.

2 Model formation and equations of motion

Let m_1, m_2 and m be three masses which form the restricted three-body problem under the assumption that the first two bodies known as primaries are moving in circular orbits around their common center of mass which is taken as origin and imposing the Newtonian as well as Yukawa potential. The third infinitesimal body which is varying its mass according to Jeans law and moving under the gravitational influences of the primaries but not affecting them. Let xy be the synodic coordinate system where the line joining to the primaries is taken as $x-$axis and line perpendicular to the $x-$axis and passing through the origin O, is known as $y-$axis. Let ρ_1 and ρ_2 be the distances of the primaries from the infinitesimal body. For the non-dimensional units let the distance between primaries, the sum of the masses of the primaries and the unit of time chosen as G be unity. Hence let $m_1 = \mu$, $m_2 = 1-\mu$ and mean motion be unity. The equations of motion of infinitesimal body with constant mass can be written as [33]:

$$\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \frac{\partial \Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \frac{\partial \Omega}{\partial y}, \end{aligned} \tag{1}$$

where

$$\Omega = \frac{1}{2}(x^2+y^2) + \frac{1}{k}\left(\frac{\mu}{\rho_1}q_1 + \frac{1-\mu}{\rho_2}q_2\right),$$

with

$$k = 1+\alpha(1+\lambda)e^{-\lambda},$$

$$q_i = 1+\alpha e^{-\lambda\rho_i}, \quad (i=1,2)$$

$$\rho_1 = \sqrt{(x+\mu-1)^2+y^2},$$

$$\rho_2 = \sqrt{(x+\mu)^2+y^2},$$

and α as well as λ are the constant parameters due to Yukawa potential.

If the mass of infinitesimal body varies with time, the equations of motion can be written as [32];

$$\begin{aligned} \frac{\dot{m}}{m}(\dot{x}-y) + (\ddot{x}-2\dot{y}) &= \Omega_x, \\ \frac{\dot{m}}{m}(\dot{y}+x) + (\ddot{y}+2\dot{x}) &= \Omega_y, \end{aligned} \tag{2}$$

where dot (.) denotes the differentiation w. r. to time t. We will simplify Eq. (2) using Jeans law [26] and

Meshcherskii space time transformations [34] i.e.

$$
\begin{aligned}
m &= m_0 e^{-\beta_1 t}, \\
dt &= d\tau \\
(x,y) &= \beta_2^{-1/2}(\xi,\eta) \\
(\dot{x},\dot{y}) &= \beta_2^{-1/2}\left((\dot{\xi},\dot{\eta}) + \frac{\beta_1}{2}(\xi,\eta)\right), \\
(\ddot{x},\ddot{y}) &= \beta_2^{-1/2}\left((\ddot{\xi},\ddot{\eta}) + \beta_1(\dot{\xi},\dot{\eta}) + \frac{\beta_1^2}{4}(\xi,\eta)\right).
\end{aligned} \tag{3}
$$

where β_1 is constant, $\beta_2 = \dfrac{m}{m_0}$ and m_0 is the initial mass.
Then equations of motion will be

$$
\begin{aligned}
\ddot{\xi} - 2\dot{\eta} &= U_\xi, \\
\ddot{\eta} + 2\dot{\xi} &= U_\eta,
\end{aligned} \tag{4}
$$

where

$$U = \left(\frac{1}{2} + \frac{\beta_1^2}{8}\right)(\xi^2+\eta^2) + \frac{\beta_2^{3/2}}{k}\left(\frac{\mu}{r_1}p_1 + \frac{1-\mu}{r_2}p_2\right),$$

$$p_i = 1 + \alpha e^{-\lambda \beta_2^{-1/2} r_i}, \quad i = 1,2,$$

$$r_1 = \sqrt{(\xi-(1-\mu)\beta_2^{1/2})^2 + \eta^2},$$

$$r_2 = \sqrt{(\xi+\mu\beta_2^{1/2})^2 + \eta^2}.$$

The equations of motion (4) admits as

$$\dot{\xi}^2 + \dot{\eta}^2 = 2U - C - 2\int_{t_0}^{t} \frac{\partial U}{\partial t} dt, \tag{5}$$

where left hand side of Eq. (5) represents the velocity of the infinitesimal body and C is the quasi-Jacobian constant for the system.

3 Numerical Expolarations

In this part of the paper, we numerically investigate the locations of equilibrium points, regions of possible motions , trajectories allocations, Poincaré surfaces of sections and basins of attractions in various subsections 3.1, 3.2, 3.3, 3.4 and 3.5 respectively using well known software Mathematica.

3.1 Locations of equilibrium points

We will show the locations of equilibrium points graphically by solving Eq. (4) under assumptions that all the derivatives with respect to time t must be zero. Hence

$$
\begin{aligned}
U_\xi &= 0, \\
U_\eta &= 0.
\end{aligned} \tag{6}
$$

We solved Eq. (6) for two cases and found five equilibrium points (L_1, L_2, L_3, L_4 and L_5) out of which three are collinear (L_1, L_2 and L_3)and two are non-collinear (L_4 and L_5), like classical case of circular restricted three-body problem (Figures 1a & 1b). We and Kokubun revealed that Yukawa parameters (α & λ) have very small effect which can not be shown in figures so we have given it in Table-1.

Figure (1a) represents the locations of equilibrium points in two colors. Magenta color locations show without effect of mass variation (Outer locations). Orange color locations show with the effect of mass variation (Inner locations). We observed from this figure (1a) that as we consider the effect of variation of mass, the location of equilibrium points move towards origin.

Figure (1b) shows that as the value of the mass variation parameter β_2 (= 0.4 (green), 0.8 (blue) & 1.2 (red)), the locations of equilibrium points move away from the origin.

3.2 Regions of possible motion

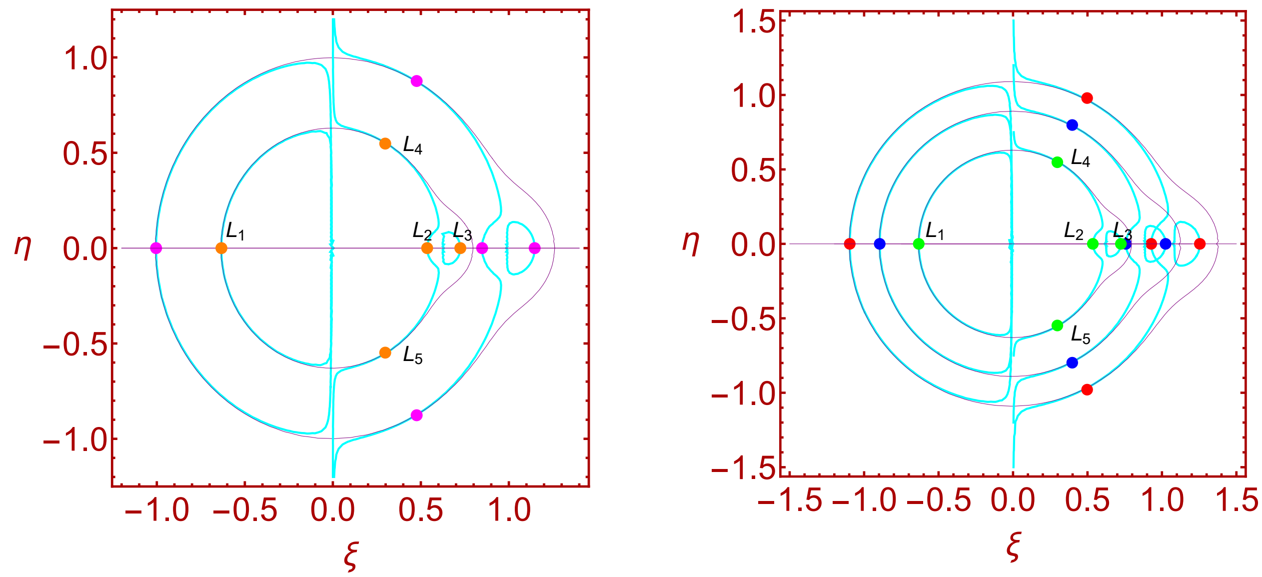
Following the procedure given by [35], we will investigate the regions of possible and forbidden motions under the effect of Yukawa parameters (α & λ) as well as mass variation parameters (β_1 & β_2) as presented in figure 2. First, we will evaluate the value of Jacobian constant (C) corresponding to each equilibrium points, then we will draw the regions of motion corresponding to each equilibrium points. Figure 2(a) corresponds to equilibrium point L_1 where we observed that the third body can move everywhere except near equilibrium points L_4 & L_5. Figure 2(b) corresponds to equilibrium point L_2, where we observed that the third body can move only near equilibrium point L_2. Figure 2(c) corresponds to equilibrium point L_3, where we observed that the third body can not move near equilibrium points L_1, L_4 & L_5 and other places can move freely. Figure 2(d) corresponds to equilibrium point L_4 or L_5, where we observed that the third body can move freely everywhere.

3.3 Trajectories allocations

Trajectories allocations depend on the potential of the system so if we change the potential, the motion of the third body will be changed. We have already mentioned in subsection 3.1 that inclusion of Yukawa term has very low effect on the properties of the motion but the inclusion of mass variation has much effect on the

Table 1: Coordinates of equilibrium points for various values of parameters used.

	Table-1	
Cases	Parameter α	Locations of equilibrium points
$\beta_1 = 0$, $\beta_2 = 1$ $\lambda = 0.1$, $\mu = 0.01$	0.34	(-1.0041683, 0), (0.8480637, 0), (1.1467899, 0) (0.4763108, $\pm$ 0.8765036)
	0.1	(-1.0041672, 0), (0.8480733, 0), (1.1467739,0) (0.4763014, $\pm$ 0.8764161)
	-0.1	(-1.0041658, 0), (0.8480852, 0), (1.1467541, 0) (0.4762537, $\pm$ 0.8762073)
	-0.25	(-1.0041643, 0), (0.8480984, 0), (1.1467324,0) (0.4761158, $\pm$ 0.8760816)
$\beta_1 = 0.2$, $\beta_2 = 0.4$ $\lambda = 0.1$, $\mu = 0.01$	0.34	(-0.6330027, 0), (0.5358793, 0), (0.7243348, 0) (0.2976472, $\pm$ 0.5481581)
	0.1	(-0.6330009, 0), (0.5358851, 0), (0.7243243, 0) (0.2976301, $\pm$ 0.5481408)
	-0.1	(-0.6329988, 0), (0.5358923, 0), (0.7243113, 0) (0.2976256, $\pm$ 0.5481374)
	-0.25	(-0.6329964, 0), (0.5359002, 0), (0.7242970,0) (0.2976109, $\pm$ 0.5481208)



(a) Magenta color (Outer) ($\beta_1 = 0$ & $\beta_2 = 1$) and Orange color(Inner) ($\beta_1 = 0.2$ & $\beta_2 = 0.4$) locations without and with effect of mass variation respectively

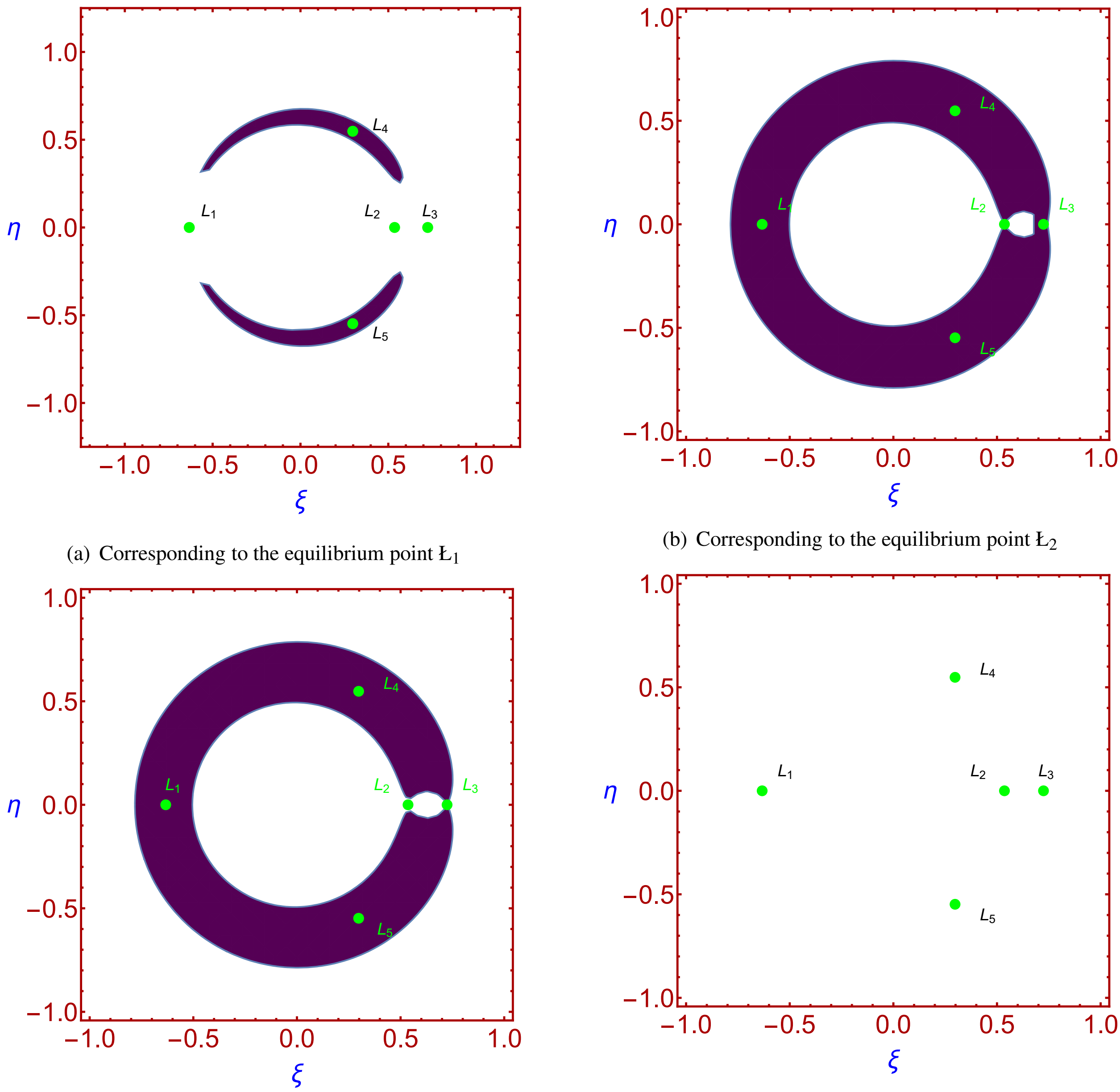
(b) For different values of variation parameter $\beta_2 = 0.4(green), 0.8(blue) \& 1.2(red)$

Fig. 1: Locations of equilibrium points at $\alpha = 0.1$, $\lambda = 0.1$, $\beta_1 = 0.2$ and $\mu = 0.01$.

properties of motion. Thus, the motion of the third body can be drawn graphically where we can see the exact location of the path for different values of mass variation parameters (β_1 & β_2) given in figure 3 ($\beta_2 = 0.4$ (3a), $\beta_2 = 0.8$ (3b) & $\beta_2 = 1.2$ (3c)). From these figures we observed that as the value of β_2 increases, the number of trajectories increases with the same interval of time. Hence, the mass variation parameter has great impact on the properties of the motion of the third body.

3.4 Poincaré surfaces of section

To reveal the dynamical properties of the path either chaos or regular, we have to draw the Poincaré surfaces of section. Thus, we must draw the graph between ξ and $\dot{\xi}$ when $\eta = 0$ whenever the orbit intersects the plane at $\dot{\eta} \geq 0$. We have drawn the Poncaré surfaces of section for three values of β_2 (= 0.4 (fig. 4a), 0.8 (fig. 4b) & 1.2 (fig. 4c)). These figures show no chaos and as the value of β_2



(a) Corresponding to the equilibrium point Ł$_1$

(b) Corresponding to the equilibrium point Ł$_2$

(c) Corresponding to the equilibrium point Ł$_3$

(d) Corresponding to the equilibrium point Ł$_4$ & Ł$_5$

Fig. 2: Regions of possible motion at $\alpha = 0.1$, $\lambda = 0.1$, $\beta_1 = 0.2$, $\beta_2 = 0.4$ and $\mu = 0.01$.

increases, the path of the motion becomes more regular or prominent.

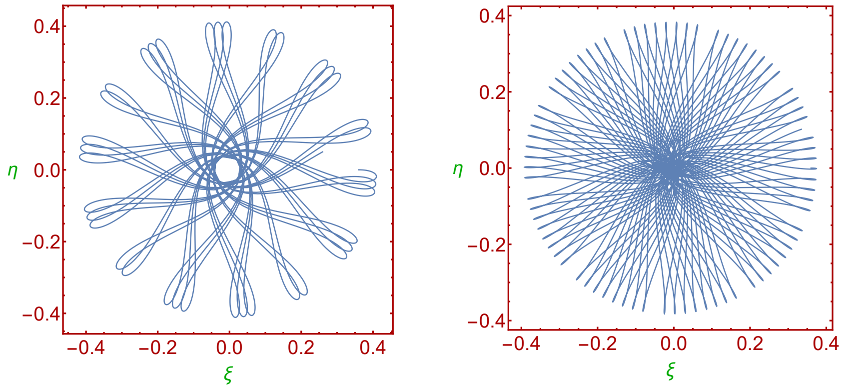
3.5 Basins of Attraction

The qualitative behaviour of the dynamical system can also be studied by the basins of attraction. To study this we will use well known N-R iterative method. By this iterative method, we have illustrated the attracting domain in $\xi - \eta$-plane. The algorithm of this problem is given as:

$$\xi_{n+1} = \xi_n - \left(\frac{U_\xi U_{\eta\eta} - U_\eta U_{\xi\eta}}{U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta} U_{\eta\xi}} \right)_{(\xi_n, \eta_n)}, \tag{7}$$

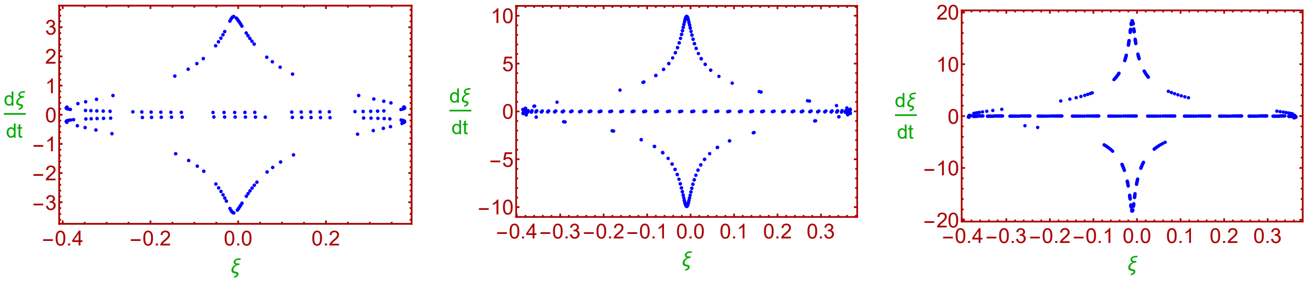
$$\eta_{n+1} = \eta_n - \left(\frac{U_\eta U_{\xi\xi} - U_\xi U_{\eta\xi}}{U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta} U_{\eta\xi}} \right)_{(\xi_n, \eta_n)}, \tag{8}$$

where ξ_n, η_n are the values of ξ and η coordinates of the n^{th} step of iterative process. The point (ξ, η) will be a



(a) Trajectories at $\beta_2 = 0.4$ (b) Trajectories at $\beta_2 = 0.8$ (c) Trajectories at $\beta_2 = 1.2$

Fig. 3: Trajectories allocations for the different values of mass variation parameter β_2.



(a) For $\beta_2 = 0.4$ (b) For $\beta_2 = 0.8$ (c) For $\beta_2 = 1.2$

Fig. 4: Poincaré surfaces of section for the three different values of mass variation parameter

member of the attracting domain, if the initial point converges rapidly to one of the equilibrium points. This process stops when the successive approximation converges to an equilibrium point. We also declare that the basins of attracting domain is unrelated to the classical basins of attracting domain in dissipative system. We used a color code for the classification of different equilibrium points on the $\xi-\eta$-plane.

We have performed the basins of attracting domain for the parameter $\alpha = 0.1$, $\lambda = 0.1$, $\beta_1 = 0.2$, $\beta_2 = 0.4$ & $\mu = 0.01$ in $\xi-\eta$-plane and given in figure 5. From this figure, we observed that there are five attracting points $L_{1,2,3,4,5}$ out of which $L_{1,2,3}$ corresponds to the cyan color regions, L_4 corresponds to the blue color region and L_5 corresponds to the green color region. All these regions extended to infinity. Orange dots denote the locations of the attracting points.

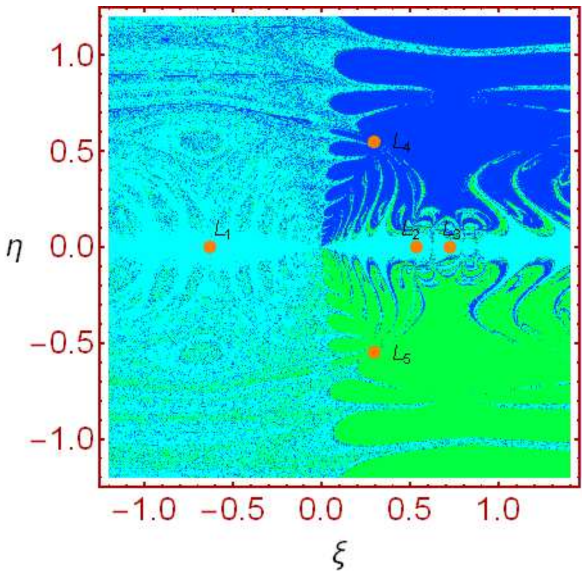


Fig. 5: Basins of attracting domain in $\xi-\eta$-plane

4 Linear Stability

In this section, we explore the linear stability of equilibrium points for the infinitesimal variable mass body in the neighbourhood of $(\xi_0+\xi_1, \eta_0+\eta_1)$ under the effect of the Newtonian and Yukawa forces of the primaries. Where (ξ_1, η_1) are small displacements from the equilibrium point (ξ_0, η_0).

The system (4) can be written in the form as:

$$\begin{aligned}\dot{\xi_1} &= \xi_2,\\ \dot{\eta_1} &= \eta_2,\\ \dot{\xi_2} &= 2\eta_2+(U_{\xi\xi})^0\xi_1+(U_{\xi\eta})^0\eta_1,\\ \dot{\eta_2} &= -2\xi_2+(U_{\eta\xi})^0\xi_1+(U_{\eta\eta})^0\eta_1,\end{aligned} \tag{9}$$

where the superscript 0 denotes the value of the second derivative of U at the corresponding equilibrium point (ξ_0, η_0).

When $\beta_1=0$, then the above system (9) will reduce to the classical case i.e. the mass of the infinitesimal body will be assumed to be constant. However, when $\beta_1\neq 0$, we can not examine the stability by ordinary method because the distances of the equilibrium point to the primaries vary with time. Therefore we will use Meshcherskii-space-time inverse transformations as:

$$\begin{aligned}\xi_3 &= \beta_2^{-1/2}\xi_1, \quad \eta_3=\beta_2^{-1/2}\eta_1,\\ \xi_4 &= \beta_2^{-1/2}\xi_2, \quad \eta_4=\beta_2^{-1/2}\eta_2.\end{aligned} \tag{10}$$

Using the transformations given in Eq. (10), the system (9) can be written as follows:

$$\dot{\sqcup}=M\sqcup, \tag{11}$$

where

$$\dot{\sqcup}=\begin{pmatrix}\dot{\xi_3}\\ \dot{\eta_3}\\ \dot{\xi_4}\\ \dot{\eta_4}\end{pmatrix}, \sqcup=\begin{pmatrix}\xi_3\\ \eta_3\\ \xi_4\\ \eta_4\end{pmatrix} \tag{12}$$

and

$$M=\begin{pmatrix}\frac{1}{2}\beta_1 & 0 & 1 & 0\\ 0 & \frac{1}{2}\beta_1 & 0 & 1\\ (U_{\xi\xi})^0 & (U_{\xi\eta})^0 & \frac{1}{2}\beta_1 & 2\\ (U_{\eta\xi})^0 & (U_{\eta\eta})^0 & -2 & \frac{1}{2}\beta_1\end{pmatrix} \tag{13}$$

The characteristic equation for the matrix M is

$$\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0=0, \tag{14}$$

where

$$\begin{aligned}\alpha_3 &= -2\beta_1,\\ \alpha_2 &= 4-(U_{\xi\xi})^0-(U_{\eta\eta})^0+\frac{3}{2}\beta_1^2,\\ \alpha_1 &= \beta_1\left((U_{\xi\xi})^0+(U_{\eta\eta})^0-4-\frac{1}{2}\beta_1^2\right),\\ \alpha_0 &= \beta_1^4+(U_{\xi\xi})^0(U_{\eta\eta})^0-\left((U_{\xi\eta})^0\right)^2\\ &\quad -\frac{1}{4}\beta_1^2\left((U_{\xi\xi})^0+(U_{\eta\eta})^0-4\right).\end{aligned} \tag{15}$$

Equation (14) numerically solved for the different values of parameters and evaluated the characteristic roots for all equilibrium points which are given in Table 2. It shows that all the equilibrium points are unstable because at-least one characteristic root is either positive real number or positive real part of the complex characteristic root. While in the classical case collinear equilibrium points are always unstable and triangular equilibrium points are stable [1]. Therefore, the parameters used change the stability of equilibrium points to instability.

5 Conclusion

Dynamical evolution of the infinitesimal variable mass has been investigated under the effect of Newtonian and Yukawa potentials. Our equations of motions are different from [33] because of the variable mass effects with variation parameters β_1 and β_2. We have determined the Jacobian quasi-integral by which we have studied the regions of possible motion and Poincaré surfaces of section to follow the procedure given by [35]. Numerical works have been done by the system of equations of motion and shown graphically. The locations of equilibrium points have been shown. Five equilibrium points like the classical restricted three-body problem have been found. Also we have indicated that Yukawa parameters α and λ have very little effect on the model but variation parameters have great effect on the location of equilibrium points (for more details, see subsection 3.1, Table-1 and Figure-1). The regions of possible motion have been studied in subsection 3.2 and given in figure 2 where shaded regions are prohibited regions and third body can move freely in white regions. The trajectory allocations showed the actual path of the third body with the particular interval of time (Figure-3). It was found that as the value of variation parameter β_2 increased, the number of trajectories increased in the same interval of time. These trajectories are not periodic. Figure 4 shows the Poincaré surfaces of sections at three different values of the variation parameter β_2, where we found the discrete type of paths which show the weak chaos. We also observed from here that as the value of β_2

Table 2: Nature of equilibrium points.

	Table-2		
Cases	Equilibrium points	Characteristic Roots	Nature
$\beta_1 = 0, \beta_2 = 1$	(-1.0041672, 0)	± 0.1615761	Unstable
$\alpha = 0.1, \lambda = 0.1$		$\pm 1.0082119\ i$	
$\mu = 0.01$	(0.8480733, 0)	± 2.9043323	Unstable
		$\pm 2.3168114\ i$	
	(1.1467739, 0)	± 2.1799466	Unstable
		$\pm 1.8749959\ i$	
	(0.4763014, $\pm$ 0.8764161)	$\pm 0.3034840\ i$	Stable
		$\pm 0.9562026\ i$	
$\beta_1 = 0.2, \beta_2 = 0.4$	(-0.6330009, 0)	± 0.0658829	Unstable
$\alpha = 0.1, \lambda = 0.1$		$0.1000000 \pm 0.9938685\ i$	
$\mu = 0.01$	(0.5358851, 0)	± 2.7876196	Unstable
		$0.0999999 \pm 2.3027300\ i$	
	(0.7243243, 0)	± 2.3175822	Unstable
		$0.1000000 \pm 1.8925532\ i$	
	(0.2976301, $\pm$ 0.5481408)	$0.0999999 \pm 0.9677089\ i$	Unstable
		$0.1000000 \pm 0.1258743\ i$	

increases, the weak chaos changes to the weakest chaos. The next figure 5 shows the basins of attraction, where the equilibrium points act as attractors. It was found that all the attractors extended to infinity. Finally in section 4, we have investigated the linear stability of the equilibrium points and observed that all the equilibrium points were unstable. However, in the classical case, the collinear equilibrium points were unstable while the triangular equilibrium points are stable [1]. Both the cases are presented in Table 2 which shows that variation of mass effect has tremendous effect on this model.

Conflict of Interest

The authors declare that they have no conflict of interest.

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