Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsapl/080305

Estimating the Unknown Parameters of Inverted Linear Exponential Distribution Based on Type-I Hybrid Censored Data

M. A. W. Mahmoud[1] *and H. M. M. Radwan*[2,*]

[1]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr city 11884, Cairo, Egypt
[2]Mathematics Department, Faculty of Science, Minia University, Minia, Egypt

Received: 8 Oct. 2020, Revised: 13 Apr. 2021, Accepted: 29 Apr. 2021
Published online: 1 Sep. 2021

Abstract: This paper deals with estimation of the unknown parameters of the inverted linear exponential distribution (ILED) based on Type-I hybrid censored (HC) data. The maximum likelihood estimation (MLE) and the related approximate confidence interval (ACI) are obtained. The general procedure for determining the bootstrap confidence interval (Boot-CI) is described. Furthermore Bayesian estimators of the unknown parameters based on squared error loss (SEL) function and linear exponential (LINEX) loss function are obtained. Using Lindley's approximation and Markov Chain Monte Carlo (MCMC) methods to approximate Bayesian estimators. A simulation study and one real data set are presented to illustrate the theoretical results.

Keywords: Bayesian estimation, Type-I HC, MCMC, Lindley approximation, Inverted linear exponential distributions

1 Introduction

In life time test, the HC scheme one of the most usage as a data which is a mixture of Type-I and Type-II censoring schemes which were analyzed by several authors like [1], [2] and [3]. It is first introduced by [4]. There are mainly two types called Type-I and Type-II HC schemes. In the Type-I HC scheme, the experiment is terminated when a pre-specified number r out of n items has failed or pre-fixed time T on test has been reached. That is, the experiment is terminated at the random time $\tau = \min(x_{r;n}, T)$. Type-I HC scheme had been studied by many authors for example, see, [5], [6], [7], [8], [9], [10] and [11]. Let $x_{1;n} < x_{2;n} < ... < x_{n;n}$ be the order statistics from a random sample of size n. Assume that r and T are known in advance and let k denote the number of $x_{i;n}$'s that are at most T. Then, under the Type-I HC scheme, we have one of the following two types of observations:

Case I:$x_{1;n} < x_{2;n} < ... < x_{r;n}$ if $x_{r;n} \leq T$ with $r \leq k \leq n$.
Case II:$x_{1;n} < x_{2;n} < ... < x_{k;n}$ if $x_{k;n} < T < x_{k+1;n}$ with $0 \leq k \leq r-1$.

The analysis of Type-I HC scheme from the two-parameters ILED is considered in this article. First, we consider the MLE of the unknown parameters. The observed Fisher information matrix (FIM) using the missing information principles, which have been used to obtain ACIs of the unknown parameters, are also evaluated. Furthermore, the procedure of the boot-CI is introduced. Second, Bayes estimators for the unknown parameters are considered. Lindley's approximation and MCMC method are considered to approximate the Bayes estimators. Also, the MCMC method is used to compute the highest posterior density (HPD) credible intervals. A simulation study to compare the performance of MLEs and the Bayes estimators are carried out. One real data set is used for illustrative purpose.

The rest of this article is organized as follows. Some statistical properties of ILED are studied in Section 2. The MLE, FIM and the ACI are introduced in Section 3. In Section 4, two CIs based on the bootstrapping are proposed. The Bayes estimator of unknown parameters is discussed in Section 5. Analysis of one real data set is introduced in Section 6. In Section 7, a simulation results for illustrating all the methods are discussed.

* Corresponding author e-mail: hmmradwan86@yahoo.com

2 Inverted Linear Exponential Distribution

The ILED was first proposed by [12]. [13] introduced the modified Inverse Rayleigh distribution with the same probability density function (PDF). Also, it can be introduced as a special case from the inverted generalized linear exponential distribution which was first proposed by [14]. The pdf and the cumulative distribution function (CDF) of the ILED are given by;

$$f(x;\lambda,\theta)=\frac{1}{x^2}(\lambda+\frac{\theta}{x})\exp[-\frac{1}{2x^2}(2\lambda x+\theta)]\ ,\lambda>0,\theta>0,x>0, \tag{1}$$

and

$$F(x;\lambda,\theta)=\exp[-\frac{1}{2x^2}(2\lambda x+\theta)]\ ,x>0. \tag{2}$$

The survival and hazard rate (HR) functions of the ILED are given by;

$$S(t;\lambda,\theta)=1-\exp[-\frac{1}{2t^2}(2\lambda t+\theta)]\ ,t>0, \tag{3}$$

and

$$h(t;\lambda,\theta)=\frac{\frac{1}{t^2}(\lambda+\frac{\theta}{t})\exp[-\frac{1}{2t^2}(2\lambda t+\theta)]}{1-\exp[-\frac{1}{2t^2}(2\lambda t+\theta)]}\ ,t>0. \tag{4}$$

From Equation (1), special distribution can be obtained:

1. For $\theta=0$, Equation (1) reduces to the pdf of the inverted exponential distribution [15].
2. For $\lambda=0$, Equation (1) reduces to the pdf of the inverted Rayleigh distribution [16].

Indeed, it is easy to show that the quantile t_q of the ILED can be obtained as

$$t_q=\frac{\theta}{-\lambda+\sqrt{\lambda^2-2\ \theta\ \ln q}}\ ,0<q<1, \tag{5}$$

and the median of the ILED is obtained when $q=0.5$ in Equation (5) as follow

$$Med=\frac{\theta}{-\lambda+\sqrt{\lambda^2+2\ \theta\ \ln 2}}. \tag{6}$$

2.1 HR function and mode

From Equation (4), one can show that

$$\lim_{t\to 0}h(t)=0, \tag{7}$$

and

$$\lim_{t\to\infty}h(t)=0. \tag{8}$$

Since $h(t)>0$ and from Equations (7) and (8), one can see that $h(t)$ is a non-monotonic function.This property makes the ILED widely applicable in several areas of life such as the active repair times for an airborne communication transceiver and the exceedances of flood peaks of the Wheaton River, see([12] and [13]). It is easy to show that the HR has a unimodal shape.

Theorem 4.1.

The HR function of ILED has a unimodal shape.

Proof.

Due to [17], $\eta(t)=\frac{-f'(t)}{f(t)}$ can be written as

$$\eta(t)=\frac{1}{t^3\ (\lambda\ t+\theta)}\Big(\ t^2\ (2\ \lambda\ t+3\ \theta)-(\lambda\ t+\theta)^2\Big), \tag{9}$$

where $f'(t)$ is the first derivative of f(t) with respect to t. The first derivative of $\eta(t)$ can be obtained as

$$\acute{\eta}(t) = \frac{1}{t^4\,(\lambda\,t+\theta)^2}\Big(-2\,\lambda^2\,t^4-(6\,\lambda\,\theta-2\,\lambda^3)\,t^3-(3\,\theta^2-7\,\lambda^2\,\theta)\,t^2+6\,\lambda\,\theta^2\,t+3\,\theta^3\Big). \tag{10}$$

Equating (10) by zero, we get

$-2\,\lambda^2\,t^4-(6\,\lambda\,\theta-2\,\lambda^3)\,t^3-(3\,\theta^2-7\,\lambda^2\,\theta)\,t^2+6\,\lambda\,\theta^2\,t+3\,\theta^3=0.$

By solving this equation by mathematica 11, the result is satisfied.
One can show that this distribution is a unimodal distribution, see [12].

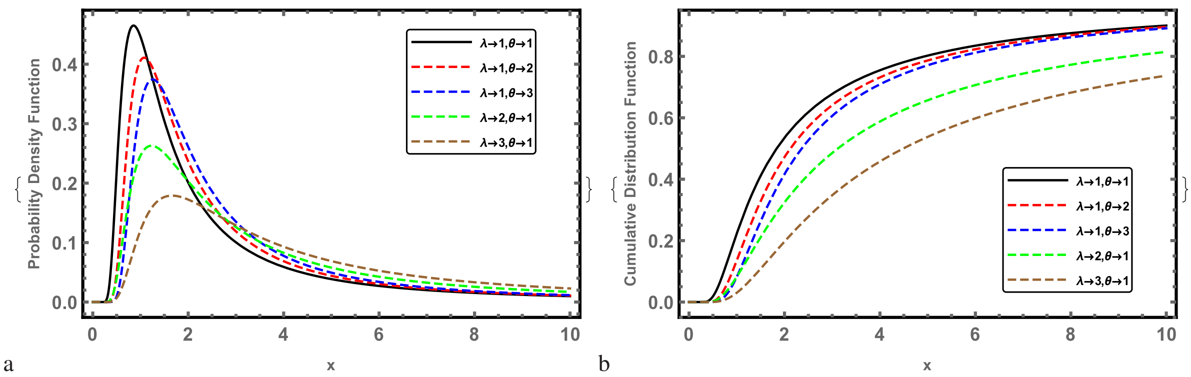

Fig. 1: a) The pdf of ILED with several parameters and b) The cdf of ILED with several parameters

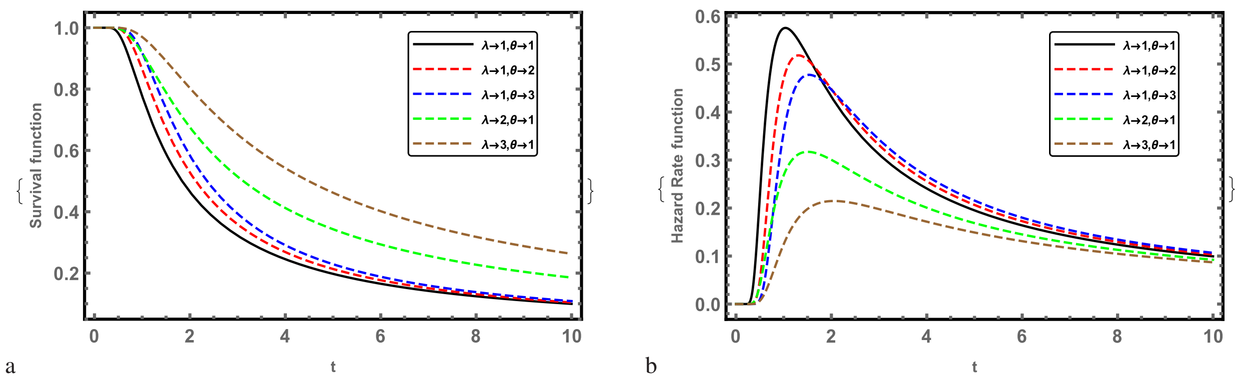

Fig. 2: a) The survival function of ILED with several parameters and b) The HR of ILED with several parameters

3 ML Estimators and FIM

MLE is probably the most widely used method of estimation in statistics. In this section, the MLE based on Type-I HC scheme from two parameters ILED of λ and θ is considered. Also we construct the ACIs of the parameters of ILED based on Type-I HC scheme.

3.1 Point estimation of parameters, survival and HR functions

Let $x_{1;n} < x_{2;n} < \ldots < x_{n;n}$ be the Type-I HC sample from ILED, then the likelihood function is given by one of the two cases:

Case I:$L_1(\lambda,\theta) = \frac{n!}{(n-r)!}\prod_1^r f(x_i)(1-F(x_r))^{(n-r)}$, where $\mathbf{x}_r = (x_1,\ldots,x_r)$ and $x_1 < \ldots < x_r \leq T$.
Case II:$L_2(\lambda,\theta) = \frac{n!}{(n-k)!}\prod_1^k f(x_i)(1-F(T))^{(n-k)}$, where $\mathbf{x}_k = (x_1,\ldots,x_k)$ and $x_1 < \ldots < x_k \leq T < x_{k+1}$.

We present likelihood functions for case I and case II as follows:

$$L(\lambda,\theta) = \frac{n!}{(n-R)!}\prod_1^R f(x_i)(1-F(c))^{(n-R)},$$

where

$$R = \begin{cases} r, & \text{for case I} \\ k, & \text{for case II} \end{cases}, and\ c = \begin{cases} x_r, & \text{for case I} \\ T, & \text{for case II} \end{cases}.$$

Then the likelihood functions for ILED can be written as:

$$L(\lambda,\theta) = \frac{n!}{(n-R)!}\prod_1^R \frac{1}{x_i^3}(\lambda x_i+\theta)(1-\exp[-\frac{1}{2c^2}(2\lambda c+\theta)])^{(n-R)}$$
$$\exp[-\sum_1^R \frac{1}{2x_i^2}(2\lambda x_i+\theta)]. \tag{11}$$

Taking the logarithm of equation(11), we obtain;

$$l(\lambda,\theta) \propto \sum_1^R \log[\lambda x_i+\theta] - \sum_1^R 3\ \log[x_i] - \sum_1^R \frac{1}{2x_i^2}(2\lambda x_i+\theta) + (n-R)$$
$$\log[1-\exp[-\frac{1}{2c^2}(2\lambda c+\theta)]]. \tag{12}$$

By taking the first derivative with respect to λ and θ from (12) and equating by zero, then we get the two normal equations as follows:

$$\sum_1^R \frac{x_i}{\hat{\theta}+\hat{\lambda}x_i} - \sum_1^R \frac{1}{x_i} + (n-R)\frac{\exp[-(\frac{\hat{\theta}}{2c^2}+\frac{\hat{\lambda}}{c})]}{c(1-\exp[-(\frac{\hat{\theta}}{2c^2}+\frac{\hat{\lambda}}{c})])} = 0 \tag{13}$$

and

$$\sum_1^R \frac{1}{\hat{\theta}+\hat{\lambda}x_i} - \sum_1^R \frac{1}{2x_i^2} + (n-R)\frac{\exp[-(\frac{\hat{\theta}}{2c^2}+\frac{\hat{\lambda}}{c})]}{2c^2(1-\exp[-(\frac{\hat{\theta}}{2c^2}+\frac{\hat{\lambda}}{c})])} = 0. \tag{14}$$

Since (13) and (14) cannot be solved analytically for $\hat{\lambda}$ and $\hat{\theta}$, some numerical methods must be employed. Now, to obtain the MLEs of S(t) and h(t), we replace λ and θ by the MLEs $\hat{\lambda}$ and $\hat{\theta}$ in (3) and (4). Hence;

$$\hat{S}_{ML}(t) = 1-\exp[-\frac{1}{2t^2}(2\hat{\lambda}t+\hat{\theta})]\ , t > 0, \tag{15}$$

and

$$\hat{h}_{ML}(t) = \frac{\frac{1}{t^2}(\hat{\lambda}+\frac{\hat{\theta}}{t})\exp[-\frac{1}{2t^2}(2\hat{\lambda}t+\hat{\theta})]}{1-\exp[-\frac{1}{2t^2}(2\hat{\lambda}t+\hat{\theta})]}\ , t > 0. \tag{16}$$

3.2 Approximate confidence interval

From the log-likelihood equation (12), we get

$$\frac{\partial^2 l(\lambda,\theta)}{\partial\lambda^2}=\sum_1^R-\frac{x_i^2}{(\theta+\lambda x_i)^2}-(n-R)\frac{\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})]}{c^2(1-\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})])}-(n-R)\frac{\exp[-(\frac{\theta}{c^2}+\frac{2\lambda}{c})]}{c^2(1-\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})])^2},\tag{17}$$

$$\frac{\partial^2 l(\lambda,\theta)}{\partial\theta^2}=\sum_1^R-\frac{1}{(\theta+\lambda x_i)^2}-(n-R)\frac{\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})]}{4c^4(1-\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})])}-(n-R)\frac{\exp[-(\frac{\theta}{c^2}+\frac{2\lambda}{c})]}{4c^4(1-\exp[-(\frac{\theta}{,}2c^2+\frac{\lambda}{c})])^2}\tag{18}$$

and

$$\frac{\partial^2 l(\lambda,\theta)}{\partial\lambda\partial\theta}=\sum_1^R-\frac{x_i}{(\theta+\lambda x_i)^2}-(n-R)\frac{\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})]}{2c^3(1-\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})])}-(n-R)\frac{\exp[-(\frac{\theta}{c^2}+\frac{2\lambda}{c})]}{2c^3(1-\exp[-(\frac{\theta}{2c^2}+\frac{\lambda}{c})])^2}.\tag{19}$$

Then, the asymptotic variance-covariance matrix of the estimators of the parameters λ and θ is obtained by inverting the FIM (given by taking the expectation of equations (17), (18) and (19)) in which elements are negatives. In the present situation, it seems appropriate to approximate the expected values by their MLEs. Accordingly, the approximate variance-covariance matrix is given as [see, [18]];

$$\begin{pmatrix}\hat{\sigma}_{\lambda\lambda} & \hat{\sigma}_{\lambda\theta}\\ \hat{\sigma}_{\lambda\theta} & \hat{\sigma}_{\theta\theta}\end{pmatrix}=\begin{pmatrix}-\frac{\partial^2 l(\lambda,\theta)}{\partial\lambda^2} & -\frac{\partial^2 l(\lambda,\theta)}{\partial\lambda\partial\theta}\\ -\frac{\partial^2 l(\lambda,\theta)}{\partial\lambda\partial\theta} & -\frac{\partial^2 l(\lambda,\theta)}{\partial\theta^2}\end{pmatrix}^{-1}_{(\hat{\lambda},\hat{\theta})}.\tag{20}$$

The ACIs for the parameters λ and θ are, respectively given as:

$$\hat{\lambda}\pm z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\lambda\lambda}}\text{ and }\hat{\theta}\pm z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\theta\theta}},$$

where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

4 Bootstrap Confidence Intervals

In this section, we introduce the following two parametric boot-CIs for λ and θ, as follow:

1. The percentile bootstrap (Boot-p) proposed by [19], and
2. The bootstrap-t method (Boot-t) proposed by [20].

There are many articles was proposed boot-CI, see [21].

4.1 Boot-p method

1. From the original sample $\mathbf{x}=x_{1;n},x_{2;n},\ldots,x_{R;n}$, compute the MLEs of $\hat{\lambda}$ and $\hat{\theta}$ from (13) and (14).
2. Get a bootstrap sample $\mathbf{x}^*=x^*_{1;n},x^*_{2;n},\ldots,x^*_{R;n}$ by resampling with replacement. Obtain the bootstrap estimate of λ and θ say, $\hat{\lambda}^*$ and $\hat{\theta}^*$ using the bootstrap sample as in step 1.
3. Repeat step 2 B times representing B bootstrap MLEs of λ and θ based on B different bootstrap samples.
4. Arrange all $\hat{\lambda}^*$'s and $\hat{\theta}^*$'s, in order to obtain the bootstrap sample $\varphi_l^{[1]},\ldots,\varphi_l^{[B]},l=1,2$, where $\varphi_1=\hat{\lambda}^*,\varphi_2=\hat{\theta}^*$.
5. Let $G(z)=P(\varphi_l\leq z)$ be the cdf of φ_l. Define $\varphi_{lBoot-p}=G^{-1}(z)$ for given z. The approximate bootstrap $100(1-\alpha)$ CI of φ_l is given by $[\varphi_{lBoot-p}(\frac{\alpha}{2}),\varphi_{lBoot-p}(1-\frac{\alpha}{2})]$.

4.2 Boot-t method

1. From the original sample $\mathbf{x} = x_{1;n}, x_{2;n}, \ldots, x_{R;n}$, compute the MLEs of $\hat{\lambda}$ and $\hat{\theta}$ from (13) and (14).
2. Get a bootstrap sample $\mathbf{x}^* = x^*_{1;n}, x^*_{2;n}, \ldots, x^*_{R;n}$ by resampling with replacement. Obtain the bootstrap estimate of λ and θ say, $\hat{\lambda}^*$ and $\hat{\theta}^*$ using the bootstrap sample as in step 1.
3. Construct the following statistics $T_1^* = \frac{\hat{\lambda}^* - \hat{\lambda}}{\sqrt{var(\hat{\lambda}^*)}}$ and $T_2^* = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{var(\hat{\theta}^*)}}$, where $var(\hat{\lambda}^*)$ and $var(\hat{\theta}^*)$ are obtained using the FIM.
4. Repeat step 2 and 3 B times.
5. Let $G(z) = P(T_l^* \leq z), l = 1,2,$ be the cdf of T_l^*.Define $\hat{\lambda}_{Boot-t} = \hat{\lambda} + \sqrt{var(\hat{\lambda})}G^{-1}(z)$ and $\hat{\theta}_{Boot-t} = \hat{\theta} + \sqrt{var(\hat{\theta})}G^{-1}(z)$ for given z.The approximate bootstrap $100(1-\alpha)$ CI of λ and θ is given by $[\lambda_{Boot-t}(\frac{\alpha}{2}), \lambda_{Boot-t}(1-\frac{\alpha}{2})]$. and $[\theta_{Boot-t}(\frac{\alpha}{2}), \theta_{Boot-t}(1-\frac{\alpha}{2})]$.

5 Bayes Estimates

In this section, similarly as in [22] it is assumed that λ and θ have the following independent gamma priors:

$$\begin{cases} \pi_1(\lambda) \propto \lambda^{a_1-1}\exp[-b_1\lambda], \\ \pi_2(\theta) \propto \theta^{a_2-1}\exp[-b_2\theta]. \end{cases} \tag{21}$$

All parameters a_1, b_1, a_2, b_2 are chosen to be known and non-negative. The joint prior distribution of λ and θ is given by

$$\pi(\lambda,\theta) \propto \pi_1(\lambda)\pi_2(\theta) \propto \lambda^{a_1-1}\theta^{a_2-1}\exp[-b_1\lambda]\exp[-b_2\theta]. \tag{22}$$

Now, the posterior density function is given by:

$$\pi^*(\lambda,\theta|\mathbf{x}) \propto \frac{1}{K}L(\lambda,\theta)\pi(\lambda,\theta) = \frac{1}{K}(1-\exp[-\frac{1}{2c^2}(2\lambda c+\theta)])^{(n-R)}$$
$$\prod_1^R \frac{1}{x_i^3}(\lambda x_i+\theta)\theta^{a_2-1}\exp[-b_1\lambda - b_2\theta]\lambda^{a_1-1}\exp[-\sum_1^R \frac{1}{2x_i^2}(2\lambda x_i+\theta)]. \tag{23}$$

where $K = \int_0^\infty \int_0^\infty L(\lambda,\theta)\pi(\lambda,\theta)d\lambda d\theta$. For any function $u(\lambda,\theta)$ of λ and θ, the Bayes estimates is given by

$$\hat{u}(\lambda,\theta) = \frac{1}{K}\int_0^\infty \int_0^\infty u(\lambda,\theta)L(\lambda,\theta)\pi(\lambda,\theta)d\lambda d\theta, \tag{24}$$

We observe that the equation (24) cannot be solved explicitly, so two different procedures are introduced (Lindley approximation and MCMC method).

5.1 Lindley's approximation

[23] introduced a manner to approximate the ratio of two integrals such as in (24). Now we can write (24) as:

$$E(u|\mathbf{x}) = \frac{\int_0^\infty \int_0^\infty u(\lambda,\theta)e^{\rho(\lambda,\theta)+l(\lambda,\theta)}d\lambda d\theta}{\int_0^\infty \int_0^\infty \exp[\rho(\lambda,\theta)+l(\lambda,\theta)]d\lambda d\theta}, \tag{25}$$

where $\rho(\lambda,\theta) = \log[\pi(\lambda,\theta)]$ and $l(\lambda,\theta)$ is given by (12). Based on Lindley approximation $E(u|\mathbf{x})$ can be approximated as:

$$E(u|\mathbf{x}) = u(\hat{\lambda},\hat{\theta}) + \frac{1}{2}[(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\lambda\theta} + 2\hat{u}_\theta\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\theta} + (\hat{u}_{\lambda\theta} + 2\hat{u}_\lambda\hat{\rho}_\theta)\hat{\sigma}_{\lambda\theta} + (\hat{u}_{\theta\theta} + 2\hat{u}_\theta\hat{\rho}_\theta)\hat{\sigma}_{\theta\theta}] + \frac{1}{2}[(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{u}_\theta\hat{\sigma}_{\lambda\theta})(\hat{l}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{l}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}) + (\hat{u}_\lambda\hat{\sigma}_{\theta\lambda} + \hat{u}_\theta\hat{\sigma}_{\theta\theta})(\hat{l}_{\theta\lambda\lambda}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta})],$$

where $\hat{\lambda}$ and $\hat{\theta}$ are the MLEs of λ and θ and $u_{\lambda\lambda}$ is the second derivative of $u(\lambda,\theta)$ w.r.t. λ. Also $\hat{u}_{\lambda\lambda}$ is the value of $u_{\lambda\lambda}$ at $(\hat{\lambda},\hat{\theta})$ and $\hat{\sigma}_{\lambda\lambda}$ is given by (20).
Now, we use two loss functions to determine the Bayes estimates based on Lindley's approximation.

5.1.1 Bayes estimate based on Lindley's approximation under LINEX loss function

1.For estimating λ, set $u(\lambda,\theta)=\exp[-a\lambda]$, hence

$$\hat{\lambda}_{BL}=-\frac{1}{a}\log[E(\exp[-a\lambda]|\mathbf{x})]. \tag{26}$$

2.For estimating θ, set $u(\lambda,\theta)=\exp[-a\theta]$, hence

$$\hat{\theta}_{BL}=-\frac{1}{a}\log[E(\exp[-a\theta]|\mathbf{x})]. \tag{27}$$

3.For estimating $h(\lambda,\theta)$, set $u(\lambda,\theta)=\exp[-ah(\lambda,\theta)]$, hence

$$\hat{h}_{BL}=-\frac{1}{a}\log[E(\exp[-ah(\lambda,\theta)]|\mathbf{x})]. \tag{28}$$

4.For estimating $S(\lambda,\theta)$, set $u(\lambda,\theta)=\exp[-aS(\lambda,\theta)]$, hence

$$\hat{S}_{BL}=-\frac{1}{a}\log[E(\exp[-aS(\lambda,\theta)]|\mathbf{x})]. \tag{29}$$

5.1.2 Bayes estimate based on Lindley's approximation under SEL function

1.For estimating λ, set $u(\lambda,\theta)=\lambda$, hence

$$\hat{\lambda}_{BL}=E(\lambda|\mathbf{x}). \tag{30}$$

2.For estimating θ, set $u(\lambda,\theta)=\theta$, hence

$$\hat{\theta}_{BL}=E(\theta|\mathbf{x}). \tag{31}$$

3.For estimating $h(\lambda,\theta)$, set $u(\lambda,\theta)=h(\lambda,\theta)$, hence

$$\hat{h}_{BL}=E(h(\lambda,\theta)|\mathbf{x}). \tag{32}$$

4.For estimating $S(\lambda,\theta)$, set $u(\lambda,\theta)=S(\lambda,\theta)$, hence

$$\hat{S}_{BL}=E(S(\lambda,\theta)|\mathbf{x}). \tag{33}$$

Now, we introduce the importance sampling technique.

5.2 Importance sampling

The MCMC with the importance sampling technique to draw samples from the posterior density function is proposed; and in turn computes the Bayes estimates, and also construct HPD credible interval. This procedure was proposed in several articles, see, [7], [24], [25] and [26]. HPD credible interval is also proposed by [27]. From Equation (23) the posterior density function of λ and θ can be written as:

$$\pi^*(\lambda,\theta|\mathbf{x})=g_\lambda(a_1^*,b_1^*)g_\theta(a_2^*,b_2^*)g_3(\lambda,\theta),$$

where $g_\lambda(a_1^*,b_1^*)$ is a Gamma density function with the shape and scale parameters $a_1^*=a_1+R$ and $b_1^*=b_1+\sum_1^R\frac{1}{x_i}$ and $g_\theta(a_2^*,b_2^*)$ is a Gamma density function with the shape and scale parameters $a_2^*=a_2+R$ and $b_2^*=b_2+\sum_1^R\frac{1}{2x_i^2}$. Moreover,

$$g_3(\lambda,\theta)=[(1-\exp[-\tfrac{1}{2c^2}(2\lambda c+\theta)])^{(n-R)}\prod_1^R\tfrac{1}{\lambda\theta x_i^3}(\lambda x_i+\theta)],$$

which is a function of λ and θ. Now, the following scheme is introduced to generate (λ,θ) from the posterior density function and compute the Bayes estimates and the corresponding credible intervals.

1.Set the initial values of λ and θ say λ_0 and θ_0.
2.Set j=1.
3.Generate λ_j from $g_\lambda(a_1^*,b_1^*)\sim gamma(a_1+R,b_1+\sum_1^R\frac{1}{x_i})$.

4.Generate θ_j from $g_\theta(a_2^*, b_2^*) \sim gamma(a_2 + R, b_2 + \sum_1^R \frac{1}{2x_i^2})$.
5.Compute $S(t)$ and $h(t)$, replacing λ and θ by λ_j and θ_j in equations (3) and (4).
6.Compute $g_3(\lambda_1, \theta_1)$.
7.Repeat steps 1-5 N times and obtain $(\lambda_1, \theta_1), \ldots, (\lambda_N, \theta_N)$ and $g_3(\lambda_j, \theta_j), j = 1, \ldots, N$.
8.The Bayes estimates of $u(\lambda, \theta)$ can be approximated as:

$$\hat{u}_{MC} \approx \frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}. \tag{34}$$

9.The idea developed in [27] is used to construct the HPD credible interval. The procedure to evaluate the HPD credible interval of the unknown parameter is given in several articles see, [9] and [25].

Now, two loss functions to determine the Bayes estimates based on MCMC method from (34) are used.

5.2.1 Bayes estimate based on MCMC method under LINEX loss function

1.For estimating λ, set $u(\lambda_j, \theta_j) = \exp[-a\lambda_j]$, hence

$$\hat{\lambda}_{BG} = -\frac{1}{a}\log[\frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}]. \tag{35}$$

2.For estimating θ, set $u(\lambda_j, \theta_j) = \exp[-a\theta_j]$, hence

$$\hat{\theta}_{BL} = -\frac{1}{a}\log[\frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}]. \tag{36}$$

3.For estimating $h(\lambda, \theta)$, set $u(\lambda_j, \theta_j) = \exp[-ah(\lambda_j, \theta_j)]$, hence

$$\hat{h}_{BL} = -\frac{1}{a}\log[\frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}]. \tag{37}$$

4.For estimating $S(\lambda, \theta)$, set $u(\lambda_j, \theta_j) = \exp[-aS(\lambda_j, \theta_j)]$, hence

$$\hat{S}_{BL} = -\frac{1}{a}\log[\frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}]. \tag{38}$$

5.2.2 Bayes estimate based on MCMC method under SEL function

1.For estimate of λ, $u(\lambda_j, \theta_j) = \lambda_j$,

$$\hat{\lambda}_{BL} = \frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}. \tag{39}$$

2.For estimating θ, set $u(\lambda_j, \theta_j) = \theta_j$, hence

$$\hat{\theta}_{BL} = \frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}. \tag{40}$$

3.For estimating $h(\lambda, \theta)$, set $u(\lambda_j, \theta_j) = h(\lambda_j, \theta_j)$, hence

$$\hat{h}_{BL} = \frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}. \tag{41}$$

4.For estimating of $S(\lambda, \theta)$, set $u(\lambda_j, \theta_j) = S(\lambda_j, \theta_j)$, hence

$$\hat{S}_{BL} = \frac{\frac{1}{N}\sum_1^N u(\lambda_j, \theta_j) g_3(\lambda_j, \theta_j)}{\frac{1}{N}\sum_1^N g_3(\lambda_j, \theta_j)}. \tag{42}$$

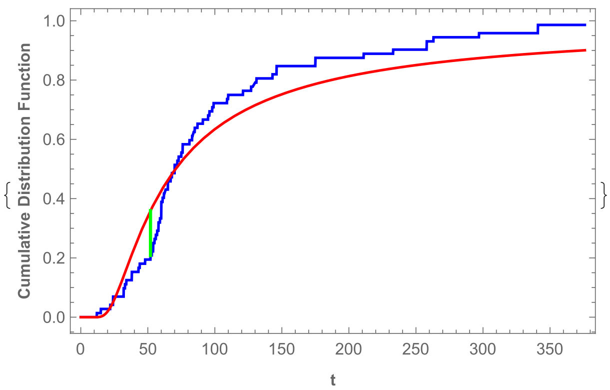

Fig. 3: Empirical and fitted distribution function for the completed data set.

Table 1: Estimates of λ and θ for different methods under the non-informative priors

R	T	Estimate	MLE	Boot-p	Boot-t	Lindley (SEL)	Lindley (LINEX)			MCMC (SEL)	MCMC (LINEX)		
							0.0001	1	-1		0.0001	1	-1
48	0.88	λ	0.4617	0.3964	0.4809	0.5013	0.5013	0.4942	0.5079	0.2679	0.2679	0.2676	0.2681
		θ	0.1098	0.1774	0.1252	0.0823	0.0823	0.0796	0.085	0.176	0.176	0.1759	0.1761
	0.95	λ	0.4723	0.4041	0.4109	0.5151	0.5151	0.5082	0.5214	0.2682	0.2682	0.268	0.2685
		θ	0.1037	0.173	0.1292	0.0739	0.0739	0.0715	0.0766	0.1762	0.1762	0.1761	0.1763
55	1.15	λ	0.4322	0.3672	0.3718	0.4632	0.4632	0.4563	0.4697	0.2888	0.2888	0.2887	0.2889
		θ	0.1275	0.1985	0.1542	0.1055	0.1055	0.1027	0.1085	0.1979	0.1979	0.1979	0.198
	1.51	λ	0.4424	0.3726	0.3812	0.4759	0.4759	0.4692	0.4822	0.2889	0.2889	0.2888	0.289
		θ	0.1213	0.1999	0.1453	0.0975	0.0975	0.0949	0.1003	0.198	0.198	0.1979	0.198

6 Real Data Analysis

The data set, which was originally reported by [28], is the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. This data is analyzed by several articles see, [29] and [30]. We have created four artificially HC data sets from the survival times data set (after we divided each data point by 100), using the following censoring schemes:

Scheme 1:R = 48, T = 0.88.
Scheme 2:R = 48, T = 0.95.
Scheme 3:R = 55, T = 1.15.
Scheme 4:R = 55, T = 1.51.

First, we would like to check whether the ILED fits this data or not. The calculated value of the Kolomogorov-Smirnov test is 0.1509 for the ILED and this value is smaller than their corresponding values expected at 5% significance level, which is 0.160278 at $n = 72$. We have just plotted the empirical distribution function, and the fitted distribution function in figure 3. Observe that the ILED can be a good model for this data.

For computing the MLEs, use the numerical methods and also compute the 95% ACIs using the observed FIM. The SEL function and LINEX loss function are considered to compute the Bayes estimates in all cases. For comparison purposes, the informative and the non-informative priors were assumed. The Bayes estimates in all cases are obtained by using importance samples of size N = 20000. In all the cases $\lambda = 0.37$, $\theta = 0.1668$ and $a = -1,\ 0.0001,\ 1$ are considered. First, the non-informative priors are considered.

Second, we would like to see the effect of the hyper parameters on the sensitivity of the Bayesian estimation and the HPD credible interval, so, the following two different hyper parameters are obtained by the same method in [31]:

1. For the prior variance $(V_p) = 0.1$, $a_1 = 1.71231, a_2 = 0.8628, b_1 = 4.138, b_2 = 2.937$,
2. For the prior variance $(V_p) = 4$, $a_1 = 0.04281, a_2 = 0.02157, b_1 = 0.10345, b_2 = 0.07343$,

Table 2: The 95% ACI, bootstrap CI and the HPD credible interval under the non-informative priors

R	T		ACI	Boot-p CI	Boot-t CI	HPD
48	0.88	λ	(0.2195,0.704)	(0.0831,0.5842)	(0.4773,0.4862)	(0.6405,0.7405)
		θ	(0,0.2637)	(0.007,0.5472)	(0.1177,0.139)	(0.4207,0.4846)
	0.95	λ	(0.232,0.7126)	(0.0721,0.5815)	(0.3942,0.4229)	(0.6399,0.7405)
		θ	(0,0.2549)	(0.0065,0.5759)	(0.118,0.1491)	(0.4203,0.4864)
55	1.15	λ	(0.1965,0.668)	(0.0645,0.5578)	(0.356,0.3826)	(0.7211,0.8479)
		θ	(0,0.2825)	(0.012,0.5951)	(0.1413,0.1804)	(0.4942,0.5811)
	1.51	λ	(0.2084,0.6764)	(0.0701,0.5745)	(0.3661,0.3912)	(0.7208,0.8479)
		θ	(0,0.2737)	(0.0142,0.6209)	(0.1341,0.1648)	(0.494,0.5811)

Table 3: Estimates of λ and θ for different methods under informative priors

R	T	(V_p)		Lindley (SEL)	Lindley (LINEX)			MCMC (SEL)	HPD	MCMC (LINEX)		
					0.0001	1	-1			0.0001	1	-1
48	0.88	0.1	λ	0.456	0.456	0.4485	0.4637	0.2691	(0.6336,0.7299)	0.2691	0.2688	0.2693
			θ	0.116	0.116	0.1129	0.119	0.1767	(0.4189,0.4835)	0.1767	0.1766	0.1768
	0.95		λ	0.4664	0.4664	0.4589	0.4839	0.2694	(0.6323,0.7299)	0.2694	0.2692	0.2696
			θ	0.1099	0.1099	0.1069	0.1128	0.1769	(0.4179,0.4835)	0.1769	0.1768	0.177
55	1.15	0.1	λ	0.4313	0.4314	0.4241	0.4385	0.289	(0.7121,0.8339)	0.289	0.2889	2891
			θ	0.1308	0.1308	0.1277	0.1339	0.1981	(0.4913,0.5765)	0.1981	0.1981	0.1982
	1.51		λ	0.441	0.441	0.4339	0.4481	0.2891	(0.7121,0.8339)	0.2891	0.289	0.2892
			θ	0.1248	0.1248	0.1217	0.1278	0.1982	(0.4913,0.5765)	0.1982	0.1981	0.1982
48	0.88	4	λ	0.5002	0.5002	0.4931	0.5068	0.2679	(0.6406,0.7402)	0.2679	0.2676	0.2681
			θ	0.0831	0.0831	0.0805	0.0859	0.1759	(0.4208,0.4863)	0.1759	0.1758	0.1761
	0.95		λ	0.5139	0.5139	0.507	0.5202	0.2682	(0.6397,0.7402)	0.2682	0.2679	0.2684
			θ	0.0748	0.0748	0.0724	0.0775	0.1762	(0.4203,0.4863)	0.1762	0.1761	0.1763
55	1.15	4	λ	0.4624	0.4624	0.4555	0.469	0.2888	(0.7209,0.8476)	0.2888	0.2887	0.2889
			θ	0.1062	0.1062	0.1033	0.1091	0.1979	(0.4941,0.581)	0.1979	0.1979	0.198
	1.51		λ	0.4751	0.4751	0.4683	0.4814	0.2889	(0.7206,0.8476)	0.2889	0.2888	0.289
			θ	0.0982	0.0982	0.0955	0.101	0.198	(0.4939,0.581)	0.198	0.1979	0.198

7 Numerical Experiments

In this section, we carry out a simulation study to compare the performance of MLEs and Bayes estimators. We estimate the unknown parameters using the MLE, bootstrap estimate, Bayes estimators obtained by Lindley approximations and MCMC technique. The performances of different estimators with mean square errors (MSE) are compared.

7.1 Case I for Type-I HC scheme

The comparison between the estimates is taking place according to the following steps:

1. For given the hyper parameters a_1, b_1, a_2, b_2, generate random values of λ and θ from the gamma distributions.
2. For given values of n (and r) with the initial value of λ and θ given in step (1), we generate random samples from the inverse CDF of a distribution and then ordered them.
3. The MLEs of λ and θ are then obtained by solving numerically the two nonlinear equations (13) and (14) with R = r.
4. The MLEs of the hazard function and the survival function are obtained from the equation (15) and (16) with t=0.4.
5. The Bayes estimates of λ, θ, hazard function and the survival function are computed by using Lindley's approximation forms under SEL function, given by (30)- (33) and under LINEX loss function, given by (26)- (29).
6. The Bayes estimates of λ, θ, hazard function and the survival function are computed by applying the Monte Carlo integration technique with 11000 observations under SEL function, given by (39)- (42) and under LINEX loss function, given by (35)- (38).
7. The quantities $(\hat{\hat{\vartheta}} - \vartheta)^2$ are computed where $\hat{\vartheta}$ stands for an estimate of ϑ (MLE or Boot or Bayes).

Table 4: Estimates of λ and the corresponding mean square error(in the bracket) for case I

n	r	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	16	0.4945	0.6454	0.6477	0.6172	0.6191
		(0.16693)	(0.02849)	(0.02881)	(0.2647)	(0.27764)
	20	0.5269	0.6323	0.6341	0.6713	0.6732
		(0.11315)	(0.02276)	(0.02295)	(0.20532)	(0.19728)
80	40	0.5631	0.6112	0.612	0.6223	0.6241
		(0.0509)	(0.01157)	(0.0116)	(0.05646)	(0.05642)
	45	0.5729	0.6082	0.6089	0.6263	0.6279
		(0.04624)	(0.01112)	(0.01113)	(0.0482)	(0.04821)

Table 5: Estimates of θ and the corresponding mean square error(in the bracket) for case I

n	r	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	16	0.9262	0.7606	0.6651	0.7898	0.798
		(0.42432)	(0.08344)	(0.08565)	(0.52957)	(0.53977)
	20	0.8704	0.7846	0.7881	0.7333	0.7392
		(0.29659)	(0.7834)	(0.08007)	(0.40718)	(0.44032)
80	40	0.7544	0.7135	0.715	0.6861	0.6894
		(0.09901)	(0.03027)	(0.03058)	(0.09863)	(0.09954)
	45	0.7613	0.7379	0.7392	0.7087	0.712
		(0.08891)	(0.0285)	(0.0288)	(0.09397)	(0.09494)

Table 6: Estimates of $h(t)$ and the corresponding mean square error(in the bracket) for case I

n	r	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	16	0.3462	0.3843	0.3858	0.3797	0.3812
		(0.04344)	(0.03319)	(0.03342)	(0.0316)	(0.03182)
	20	0.3544	0.3707	0.372	0.3784	0.3797
		(0.03907)	(0.03261)	(0.0328)	(0.03125)	(0.03148)
80	40	0.3822	0.392	0.3926	0.397	0.3977
		(0.01795)	(0.01573)	(0.01578)	(0.01433)	(0.01483)
	45	0.3674	0.3748	0.3753	0.3796	0.3803
		(0.01666)	(0.01577)	(0.01581)	(0.01458)	(0.01461)

Table 7: Estimates of $S(t)$ and the corresponding mean square error(in the bracket) for case I

n	r	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	16	0.9742	0.9688	0.9688	0.9639	0.9639
		(0.00045)	(0.00042)	(0.00042)	(0.00079)	(0.00079)
	20	0.9738	0.9712	0.9712	0.964	0.964
		(0.00041)	(0.00037)	(0.00037)	(0.00076)	(0.00076)
80	40	0.9729	0.9713	0.9713	0.9682	0.9682
		(0.00019)	(0.00016)	(0.00016)	(0.00024)	(0.00024)
	45	0.9744	0.9733	0.9733	0.9702	0.9702
		(0.00017)	(0.00015)	(0.00015)	(0.00021)	(0.00021)

Steps 1-7 were repeated at least 1000 times for informative prior and for different sample sizes n and r at T=5. In all cases $a_1 = 1, a_2 = 0.2, b_1 = 0.5, b_2 = 0.1$, $\lambda = 0.613811$ and $\theta = 0.674349$. The mean square error of the estimates were estimated by:

$$MSE(\hat{\vartheta}) = \sum_1^{1000} \frac{(\hat{\hat{\vartheta}} - \vartheta)^2}{1000}$$

Table 8: Estimates of λ and the corresponding mean square error(in the bracket) for case II

n	T	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	2	0.4219	0.6258	0.627	0.5471	0.541
		(0.11518)	(0.01817)	(0.0183)	(0.24488)	(0.44665)
	3	0.6469	0.6854	0.6868	0.7963	0.7981
		(0.05022)	(0.02255)	(0.02285)	(0.144)	(0.13717)
80	4.5	0.576	0.7685	0.7687	0.6179	0.619
		(0.02826)	(0.03313)	(0.03321)	(0.02796)	(0.02802)
	5	0.5952	0.7676	0.7679	0.6348	0.6359
		(0.02511)	(0.03261)	(0.03269)	(0.02838)	(0.02848)

Table 9: Estimates of θ and the corresponding mean square error(in the bracket) for case II

n	T	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	2	1.0571	0.9233	0.9262	0.9104	0.9142
		(0.48474)	(0.16112)	(0.16364)	(0.62039)	(0.59468)
	3	0.7754	0.8922	0.895	0.6283	0.6321
		(0.20916)	(0.13734)	(0.13961)	(0.30258)	(0.37306)
80	4.5	0.7606	1.0802	1.0807	0.7171	0.7196
		(0.06616)	(0.19923)	(0.19964)	(0.06511)	(0.06565)
	5	0.728	1.069	1.0694	0.6875	0.6899
		(0.05688)	(0.19093)	(0.1913)	(0.06415)	(0.06454)

Table 10: Estimates of $h(t)$ and the corresponding mean square error(in the bracket) for case II

n	T	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	2	0.3242	0.2931	0.2936	0.3656	0.367
		(0.04623)	(0.0407)	(0.04073)	(0.03893)	(0.03918)
	3	0.3433	0.2748	0.2752	0.3622	0.3634
		(0.03662)	(0.04008)	(0.04009)	(0.02439)	(0.02454)
80	4.5	0.3642	0.1246	0.1246	0.3816	0.3822
		(0.01526)	(0.07261)	(0.07261)	(0.01351)	(0.01355)
	5	0.3748	0.1281	0.1281	0.3905	0.3911
		(0.0152)	(0.07086)	(0.07086)	(0.01409)	(0.01414)

Table 11: Estimates of $S(t)$ and the corresponding mean square error(in the bracket) for case II

n	T	MLEs	MCMC(SEL)	MCMC(LINEX)	Lindley(SEL)	Lindley(LINEX)
30	2	0.9776	0.9796	0.9796	0.9674	0.9674
		(0.0004)	(0.00032)	(0.00032)	(0.00108)	(0.00106)
	3	0.9736	0.981	0.981	0.9643	0.9643
		(0.00039)	(0.00028)	(0.00028)	(0.0007)	(0.00071)
80	4.5	0.975	0.9938	0.9938	0.971	0.971
		(0.00015)	(0.00041)	(0.00041)	(0.00018)	(0.00018)
	5	0.9737	0.9935	0.9935	0.9696	0.9696
		(0.00016)	(0.00041)	(0.00041)	(0.0002)	(0.0002)

7.2 Case II for Type-I HC scheme

The same steps in Case-I are considered with replacing steps 2 and 3 by:

1. For given values of n (and T) with the initial value of λ and θ given in step (1), we generate random samples from the inverse CDF of a distribution and then ordered them.
2. The MLEs of λ and θ are then obtained by solving numerically the two nonlinear equations (13) and (14) with R = k.

8 Conclusion

In this paper, Bayes estimation of the unknown parameters of the ILED, when the data are collected under the Type-I HC scheme, are considered. Gamma priors are used for both the unknown parameters to calculate Bayes estimators under the assumptions of SEL and LINEX loss functions. We found that when both parameters are unknown, the Bayes estimates cannot be obtained in explicit form. So, Lindley approximations and the MCMC technique are used to compute the approximate Bayes estimates.

Under the real data analysis, it is observed that:

1. It is easy to observe from Table 2 that the Boot-t and HPD credible intervals of λ and θ are better than the corresponding boot-p and ACIs in terms of average confidence lengths obtained.
2. It is clear from Tables 1, 2 and 3 that the Bayes estimates and the HPD credible intervals of λ and θ are relatively insensitive to the specification of the hyper parameters (a_1, a_2, b_1, b_2).

Furthermore, we observe the following from the simulation study:

1. Tables (4-7) showed that the mean square errors decrease at almost by increasing r when n and T are kept fixed .
2. From Tables (8-11) we noted that the mean square errors decrease at almost by increasing T with n and r are kept fixed.
3. In general, the mean squared error values of all estimates decreases as n increases
4. The Bayes estimates of the two parameters λ and θ using MCMC method are generally better than their MLEs and Bayes estimates using Lindley approximation based on the mean square error for all cases.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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