Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090536

Solving Volterra Integro-Differential Equation by the Second Derivative Methods

*Galina Mehdiyeva**, *Mehriban Imanova and Vagif Ibrahimov*

Department of Computational Mathematics, Baku State University, Str. Z.Khalilov 23, AZ 1148, Baku, Azerbaijan

Received: 4 Feb. 2015, Revised: 6 May 2015, Accepted: 7 May 2015
Published online: 1 Sep. 2015

Abstract: As is known, the solution of some problems of ecology, geophysics, nuclear physics, the study of some seasonal distribution of the disease and so on, reduced to solving of integro-differential equations of higher order. Note that solving of these equations can be reduced to solving system of integro-differential equations of the first order. However, special techniques adapted to solving of equations of higher order are usually effective. So here is investigated the numerical solution of integro-differential equations of second order. Prove that there are methods specially adapted to solving of integro-differential equations of second order, which are more accurate than the methods constructed to solving the system consisted from the integro-differential equations of first order or the system consisted from the integral and differential equations. For illustration this result constructed concrete methods, which are applied to solving model problem.

Keywords: Initial value problem, Volterra integro-differential equation of second order, second derivative multistep method, multistep hybrid method, degree and stability

1 Introduction

Consider to solving the following initial value problem:

$$\begin{aligned} &y''(x) = F(x, y(x), y'(x), \tau(x)),\\ &y(x_0) = y_0, \quad y'(x_0) = y_0', \end{aligned} \tag{1}$$

here $\tau(x)$ is the known function. Assume that the initial value problem (1) has a continuous unique solution defined on the segment $[x_0, X]$. To find an approximate solution of problem (1), the segment $[x_0, X]$ is divided into N equal parts by the step size $h > 0$, but the mesh point is defined as $x_i = x_0 + ih$ $(i = 0, 1, ..., N)$. In addition, we denote the approximate values of the solution of problem (1) by y_m $(m = 0, 1, ...)$ but by $y(x_m)$ the corresponding exact values of the solution of the problem (1).

It is obvious that by the change of variables, the problem (1) can be rewritten as follows:

$$z'(x) = F(x, y(x), z(x), \tau(x)), \quad z(x_0) = y_0', \tag{2}$$

$$y'(x) = z(x), \quad y(x_0) = y_0. \tag{3}$$

Thus solving of integro-differential equations of the second order reduced to solving of integro-differential equations of the first order. In this case the order of accuracy for the constructed methods for the finding the solution of the problem (1) to be the same as the order of accuracy of methods for constructing to solving of the problem (2) and (3). In order to construct a more accurate methods in [1], proposed using of the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}. \tag{4}$$

In this paper also considered application of some modification of method (4) to solving problem (1).

It is easy to see that the properties of the constructed methods for determined the solution of the problem (1) depends on the properties of the function $\tau(x)$. Consider a more wide-spread case when is the following is holds:

$$\tau(x) = \int_{x_0}^{x} K(x, s, y(s), y'(s)) ds. \tag{5}$$

Note that in [2] constructed an efficient method for solving integral equations of the type (5) based on the use of forward-jumping methods. By the application of different methods to solving of equation (5), we can

* Corresponding author e-mail: ibvag47@mail.ru

construct different methods for solving of the problem (1). Note that the problem (1) is an initial value problem for Volterra integro -differential equations of second order. Now, assume that the function $\tau(x)$ a solution of the following differential equation:

$$a_k(x)\tau(x-kh)+a_{k-1}(x)\tau(x-(k-1)h)+\ldots+$$

$$+a_1(x)\tau(x-h)+a_0(x)\tau(x)=b(x), \qquad (6)$$

here, the coefficients $a_i(x)$ $(i=0,1,...,k)$ and $b(x)$ are continuous functions defined in some closed area. Then by taking into account, the difference equation (6) in the problem (1), we obtain the initial value problem for a differential-algebraic equation. Know suppose that $\tau(x)$ is the solution of the following equation:

$$\sum_{i=0}^{k} a_i(x)\tau(x-ih)=\sum_{i=0}^{k} b_i(x)\tau'(x-ih). \qquad (7)$$

Then taking that in (1), we obtain the differential and difference problem. To find numerical solutions of integro-differential equations, Volterra supposed use quadrature methods (see [3, pp. 50-55], which has been successfully applied to solving of integro-differential equations and to present time.

It is known that to construct the quadrature method, usually the calculation of the integral in the problem (1) is replaced to the calculation of the integral sum, which is as follows (see, for example, [3]-[10]):

$$y''(x)=f(x,y(x),y'(x))+\sum_{i=0}^{n} a_iK(x,x_i,y_i,y_i')+R_n, \qquad (8)$$

$$y(x_0)=y_0,\ y'(x_0)=y_0',$$

where the coefficients of the quadrature formula a_i $(i=0,1,...,n)$ are real numbers ($x=x_0+nh$ – fixed point) and R_n is a remainder term of the quadrature formula. Thus, we receive that for solving initial value problem of ordinary differential equation, one can use the finite difference method (4) of the constant coefficients (see, for example, [11]-[23]).

Method (4) has a general and a specific form when $\gamma_i \neq 0$ $(i=0,1,2,...,k)$, which has been fully investigated by some authors and applied to solve ordinary differential equations of the first and second orders (see, for example, [11]-[30]).

Here, we apply method of type (4) to solve problem (1). Suppose that the function $K(x,s,y,z)$ is defined in a corresponding domain and there have continues derivatives up to p, inclusively.

For constructed a more accurate methods to solving of ordinary differential equations in [26] investigated the following method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}(\beta_i y'_{n+i}+\hat{\beta}_i y_{n+i+v_i})+$$

$$+h^2\sum_{i=0}^{k}(\gamma_i y''_{n+i}+\hat{\gamma}_i y''_{n+i+v_i}). \qquad (9)$$

And in [27] considered the application of the hybrid method of the type (9) to solving of integro-differential equations of first order, from which one can obtained method (4) as particular case. This method generalized many known methods applied to solving of ordinary differential equations of first and second orders. Therefore, we consider the application of method (9) to solving of the problem (1).

2 Research and application of the method (9) for solving of the problem (1)

As is known, if the method (9) for $v_i=0$ $(i=0,1,2,...,k)$ is converges, then its coefficients satisfies some bounders (see eg. [12], [25]). But now consider the transfer of these conditions on the coefficients of the method (9).

A. The values $\alpha_i, \beta_i, \hat{\beta}_i, \gamma_i, \hat{\gamma}_i, v_i$ $(i=0,1,2,...,k)$ are real numbers and $\alpha_k \neq 0$.

B. The polynomials

$$\rho(\lambda)\equiv\sum_{i=0}^{k}\alpha_i\lambda^i;\ \ \beta(\lambda)\equiv\sum_{i=0}^{k}\beta_i\lambda^i;\ \ \hat{\beta}(\lambda)\equiv\sum_{i=0}^{k}\hat{\beta}_i\lambda^{i+v_i};$$

$$\gamma(\lambda)\equiv\sum_{i=0}^{k}\gamma_i\lambda^i;\quad \hat{\gamma}(\lambda)\equiv\sum_{i=0}^{k}\hat{\gamma}_i\lambda^{i+v_i}$$

have no common factors different from constant.

C. If $\hat{\beta}_i=\beta_i=0$ $(i=0,1,...,k)$, then $\rho'(1)=0$, but $\rho''(1)\neq 0$. In this case $p\geq 1$ is holds. If $|\beta_0|+|\beta_1|+...+|\beta_k|+\left|\hat{\beta}_0\right|+\left|\hat{\beta}_1\right|+...+\left|\hat{\beta}_k\right|\neq 0$, then $\rho'(1)\neq 0$ and $p\geq 2$.

Here p is the degree of the method (9) and defined as:

Definition 1. For a sufficiently smooth function $y(x)$, method (9) has the degree $p>0$ if the following holds:

$$\sum_{i=0}^{k}\alpha_i y(x+ih)-h\sum_{i=0}^{k}(\beta_i y'_{n+i}+\hat{\beta}_i y'_{n+i+v_i})-$$

$$-h^2\sum_{i=0}^{k}(\gamma_i y''_{n+i}+\hat{\gamma}_i y'_{n+i+v_i})=O(h^{p+1}),\ h\to 0.$$

Validity of the condition A is obviously, since the solution of (1) $y(x)$ - is the real function. A condition $\alpha_k\neq 0$ due to the fact that the method (9) apply to the determination of the values of y_{n+k} and because its coefficient the quantity α_k must be different from zero. Naturally, if divided the equality (9) to α_k, then we can be put $\alpha_k=1$.

Consider to investigation of the condition B and suppose that the condition B is not holds. Then there exist the function $\varphi(\lambda)\neq$const, which is a common factor for polynomials $\rho(\lambda)$; $\beta(\lambda)$; $\hat{\beta}(\lambda)$; $\hat{\gamma}(\lambda)$ and $\gamma(\lambda)$.

Let's consider the following shift operator:

$$E^i y(x) = y(x+ih) \ (i = 0, 1, ..., k;\ E^0 y(x) = y(x)),$$

$$E^{i+\nu_i} y(x) = y(x+(i+\nu_i)h) \ (|\nu_i| < 1;\ \ i = 0, 1, ..., k).$$

After applying the shift operator to method (9), we have:

$$\rho(E)y_n = h(\beta(E)+\hat{\beta}(E))y'_n + h^2(\gamma(E)+\hat{\gamma}(E))y''_n, \quad (10)$$

but after using the suppose from the condition B and $\varphi(\lambda) \neq 0$ in the equality (10), receive the following:

$$\rho_1(E)y_n \ - \ h(\beta_1(E)+\hat{\beta}_1(E))y'_n - \\ - \ h^2(\gamma_1(E)+\hat{\gamma}_1(E))y''_n = 0, \quad (11)$$

where

$$\rho_1(\lambda) = \rho(\lambda)/\varphi(\lambda),\ \beta_1(\lambda) = \beta(\lambda)/\varphi(\lambda), \\ \hat{\beta}_1(\lambda) = \hat{\beta}(\lambda)/\varphi(\lambda),\ \gamma_1(\lambda) = \gamma(\lambda)/\varphi(\lambda), \\ \hat{\gamma}_1(\lambda) = \hat{\gamma}(\lambda)/\varphi(\lambda).$$

Obviously, if we denote the order of difference equation (11) by k_1, then we have $k_1 \leq k-1$. It is not difficult to understand that the difference equations (10) and (11) are equivalent, and because in order to difference equation (11) has a unique solution, it is necessary to $k_1 \leq k-1$ initial data. Therefore, for given the k_1 initial data, finite difference equation (10) of order k, will be has a unique solution. However, from the theory of difference equations it is known that if the number of initial data is less than the order of the linear finite difference equations with constant coefficients, then the number of solutions of this difference equation has not any solution. This fact implies that the sets of the roots of the polynomials $\rho(\lambda)$, $\sigma(\lambda)$, $\gamma(\lambda)$, $\hat{\vartheta}(\lambda)$ and $\hat{\gamma}(\lambda)$ are disjoint. Now consider to satisfying the condition C. For this aim passing the limit in (9) as $h \to 0$ then we have

$$\rho(1)y(x) = 0 \ \ (x = x_0 + nh). \quad (12)$$

This result implies that

$$\rho(1) = 0.$$

By taking condition (12) into account in equation (10), we obtain:

$$\rho_1(E)(y_{j+1}-y_j) \ - \ h\beta(E)y'_j - h\hat{\beta}(E)y'_j - \\ - \ h^2\gamma(E)y''_j - h^2\hat{\gamma}(E)y''_j = 0, \quad (13)$$

where $\rho_1(\lambda) = \rho(\lambda)/(\lambda-1)$.

By using Lagrange's theorem, we can write:

$$y_{j+1} - y_j = hy'_j + O(h^2).$$

By using this fact that in (13), we have:

$$(\rho_1(E) - \beta(E) \ - \ \hat{\beta}(E))y'_j - h\gamma(E)y''_j - \\ - \ h\hat{\gamma}(E)y''_j = O(h),\ \ h \to 0. \quad (14)$$

By passing the limit as $h \to 0$, we have:

$$\rho_1(1) = \beta(1) + \hat{\beta}(1). \quad (15)$$

Thus, it is a necessary condition for the convergence of method (9) is $\rho(1) = 0$. However, if $\beta(\lambda) \equiv 0$ and $\hat{\beta}(\lambda) \equiv 0$, then $\rho(1) = \rho'(1) = 0$ is a necessary condition for the convergence of method (9).

Consider the following expansions:

$$\rho(\lambda) = \rho(1) + \rho'(1)(\lambda-1) + \frac{1}{2}\rho''(1)(\lambda-1)^2 + \\ + O((\lambda-1)^3),$$

$$\beta(\lambda) = \beta(1) + \beta'(1)(\lambda-1) + O((\lambda-1)^2),$$

$$\gamma(\lambda) = \gamma(1) + \gamma'(1)(\lambda-1) + O((\lambda-1)^2),$$

$$y_{i+1} - y_i = hy'_i + \frac{h^2}{2}y''_i + O(h^3).$$

These expansions are subject to conditions (15), and due to the expansions in (10), we have:

$$\frac{1}{2}\rho''(1)\left(\frac{y_{j+2}-y_{j+1}}{h} - \frac{y_{j+1}-y_j}{h}\right) - (\beta'(1)+\hat{\beta}'(1)) \times \\ \times (y'_{j+1} - y'_j) - h(\gamma(1)+\hat{\gamma}(1))y''_j = O(h^2),\ \ h \to 0. \quad (16)$$

Summarising the asymptotic equality (16) in terms of j from 0 to n, we have:

$$(\rho''(1) - 2(\beta'(1)+\hat{\beta}'(1)))(y'_{n+1} - y'_0) = 2(\gamma(1)+\hat{\gamma}(1)) \times \\ \times \sum_{j=0}^{n} hy''_j - \frac{h}{2}\rho''(1)(y''_{n+1} - y''_0) + O(h),\ \ h \to 0.$$

By passing the limit as $h \to 0$, we obtain:

$$(\rho''(1) - 2(\beta'(1)+\hat{\beta}'(1)))(y'(x) - y'_0) = \\ = 2(\gamma(1)+\hat{\gamma}(1)) \int_{x_0}^{x} F(s,y(s),y'(s),\tau(s))ds.$$

From the problem (1) one can write

$$y'(x) = y'_0 + \int_{x_0}^{x} F(s,y(s),y'(s),\ \tau(s))ds.$$

By comparing these equations and utilising the fact that the solution of the problem (1) is unique, we have:

$$\rho''(1) = 2(\beta'(1)+\hat{\beta}'(1)) + 2(\gamma(1)+\hat{\gamma}(1)).$$

It follows that if $\hat{\gamma}(1)+\gamma(1)+\hat{\beta}'(1)+\beta'(1) = 0$, then $\rho''(1) = 0$. Thus, by using the substitution $z(x) = y'(x)$ in asymptotic equality (14), we have:

$$(\rho_1(E) - \beta(E) - \hat{\beta}(E))z_j -$$

$$-h(\hat{\gamma}(E)+\gamma(E))z'_j = O(h). \quad (17)$$

It is easy to see that from the condition $\rho'(1)=0$, consequent that $\lambda=1$ is a double root of the polynomial $\rho(\lambda)$. However, asymptotic relation (17) can be regarded as approximations of the difference method

$$\sum_{i=0}^{k-1}(\bar{\alpha}_i z_{n+i}+\hat{\alpha}_i z_{n+i+v_i}) = h\sum_{i=0}^{k}(\gamma_i z'_{n+i}+\hat{\gamma}_i z'_{n+i+v_i}),$$

which for $\beta(1)+\hat{\beta}(1)=0$ or $\rho'(1)=0$ is unstable. Therefore,

$$\beta(1)+\hat{\beta}(1)\neq 0.$$

Thus, we have proved that if method (9) is converge, then $\beta(1)+\hat{\beta}(1)\neq 0$. Now, we show that if method (9) converges, then $p\geq 2$. Indeed, if method (9) is converge, then

$$\rho(1)=0,\ \ \rho'(1)=\beta(1)+\hat{\beta}(1),$$

$$\frac{1}{2}\rho''(1)=\beta'(1)+\hat{\beta}'(1)+\gamma(1)+\hat{\gamma}(1)$$

from which it follows that $p\geq 2$.

Obviously, for the using of the method (9) need to construction of methods for calculating the values of the quantity $y'(x)$. For example, when applying the method (4) to solving of some problems for computing the values y'_j $(j=1,2,\dots)$ usually use method, which is determined by the following formula:

$$\sum_{i=0}^{k}\alpha_i y'_{n+i} = h\sum_{i=0}^{k}\beta_i y''_{n+i}, \quad (18)$$

This method fundamentally investigated by many authors (see eg. [11]-[30]). Take into account that the degree of stability of the type (18) satisfies the condition $p\leq 2\left[k/2\right]+2$. Here to construction a more accurate methods of the type (18) proposed to use hybrid methods. In one variant, these methods can be written as:

$$\sum_{i=0}^{k}\alpha_i y'_{n+i} = h\sum_{i=0}^{k}\beta_i y''_{n+i}+h\sum_{i=0}^{k}\hat{\beta}_i y''_{n+i+v_i} \quad (19)$$

$$(|v_i|<1,\ i=0,1,\dots,k).$$

It is known that there are stable methods of the type of (19) having the degree $p\geq 2k+2$. Thus we see that for the numerical solution of the problem (1) can used the method (19) and (4). It can be shown that the method constructing by this method has the degree $p_{\max}=2k+2$. In order to construct a more accurate method here supposed to use formula (9). Note that the method (4) can be applied to the following problem

$$y' = f(x,y)+\int_{x_0}^{x}K(x,s,y(s))ds, \quad (20)$$

$$y(x_0)=y_0,\quad x_0\leq s\leq x\leq X.$$

3 The application of second derivative multistep methods to solving problem (20)

Obviously, to apply method (4) to solve problem (20), we need to determine the values of the quantity y''_{n+k}. For using method (4), suppose that the quantities $y''_0, y''_1,\dots,y''_{k-1}$ are known. Then, $y''(x)$ can be determined from the initial value problem for the integro-differential equation of first order. In that case from the (20) we can write the following:

$$y''(x) = g(x,y,y')+a(x)+\int_{x_0}^{x}K'_x(x,s,y(s))ds, \quad (21)$$

where $g(x,y,y')=f'_x+f'_y\cdot y'$ but $a(x)=K(x,x,y(x))$.

Some authors reduce problem (20) to the following:

$$y' = f(x,y)+\vartheta(x),\ \ y(x_0)=y_0, \quad (22)$$

$$\vartheta(x)=\int_{x_0}^{x}K(x,s,y(s))ds. \quad (23)$$

If the function $\vartheta(x)$ is known, then problem (20) can be rewritten as problem (22). But problem (22) is the initial value problem for ordinary differential equations, which has been fundamentally investigated by many authors (see, for example, [11]-[30]). By continuing this method, problem (22) can also be reduced to the following:

$$y''=F(x,y,y'),\ \ y(x_0)=y_0,\ \ y'(x_0)=y'_0, \quad (24)$$

where $F(x,y,z)=g(x,y,z)+\vartheta'(x)$, but y'_0 is defined from an integro-differential equation of the first order.

Suppose that the quantity ϑ'_n is known. Then, to determine the quantity ϑ'_{n+1}, we propose the following:

$$\vartheta'_{n+1} = \vartheta'_n + K_{n+1}-K_n+h\vartheta''(\xi_{n+1})-ha'(\xi_{n+1})-$$
$$-\ hK'_x(\xi_{n+1},s,y(s))\big|_{s=\xi_{n+1}}+\int_{x_n}^{\xi_{n+1}}K'_x(x_n,s,y(s))ds+$$
$$+\int_{\xi_{n+1}}^{x_{n+1}}K'_x(x_{n+1},s,y(s))ds. \quad (25)$$

After simplifying equation (25), we have (see, for example, [22]):

$$\sum_{i=0}^{k}l_i\vartheta'_{n+i} = h\sum_{i=0}^{k}\beta_i a_{n+i}+$$
$$+\ h\sum_{i=0}^{k}\sum_{j=0}^{k}\gamma_i^{(j)}K'_x(x_{n+i},x_{n+j},y_{n+j}). \quad (26)$$

Thus, to use the method defined by formula (4), it is necessary to calculate the quantities y'_{n+k}. For this purpose, we suggest the following method:

$$\sum_{i=0}^{k} \alpha'_i y'_{n+i} = h \sum_{i=0}^{k} \beta'_i y''_{n+i}. \tag{27}$$

This method is identical to the classical k-step method with constant coefficients. It is known that if method (27) is stable, then the following holds (see [11]):

$$p \le 2\left[k/2\right] + 2,$$

where p is the degree and k is the order of method (27). Consequently, the degree of method (27) is bounded. To construct methods with a higher degree, scientists have used hybrid methods (see, for example, [7], [31]-[33]). For the construction of a stable method of type (27) with a high degree, we propose the following (see [29]):

$$\sum_{i=0}^{k} \alpha'_i y'_{n+i} = h \sum_{i=0}^{k} \beta'_i y''_{n+i} + h \sum_{i=0}^{k} \gamma'_i y''_{n+i+l_i} \tag{28}$$

$$(|l_i| < 1; i = 0, 1, 2, ..., k).$$

Note that to construct methods to determine the quantity y'_{n+k}, one can use problem (20). In this case, it is usually recommended to use ordinary multistep methods with constant coefficients. However, to construct a stable method with a high degree, it is necessary to use hybrid methods or other methods. For example, to use a method of type (28), one can construct a stable method with degree $p = 4k+2$ (see [26]).

We remark that to apply method (4) to solve problem (20), the first replace of problem (20) to the problem (24), and then, we define the solution of problem (20) by using the solution of problem (24). However, the degrees of the methods constructed in this way are bounded, as $p \le 2k+2$ (see, for example, [12], [25], or [23]). Therefore, to construct methods with a high degree, we recommend using hybrid methods.

For the simplicity let us consider the case when $k = 1$. Then one can be constructed the following method:

$$y'_{n+1} = y'_n + h(y''_{n+1/2-q} + y''_{n+1/2+q})/2 \ (q = \sqrt{3}/6). \tag{29}$$

This method is stable and has the degree $p = 4$. In this case the stable method with the degree $p = 6$ can be written as:

$$y'_{n+1} = y'_n + h(y''_{n+1} + y''_n)/12 + \\ + 5h(y''_{n+1/2-\alpha} + y''_{n+1/2+\alpha})/12 \ (\alpha = \sqrt{5}/10). \tag{30}$$

The application of these methods to solving problem (20) recommended algorithm constructed in [16].

In solving many practical problems, one is faced with the initial value problem for Volterra integro-differential equations with a degenerate kernel. Applying degenerate kernel to solving integro-differential equations by multistep method one can be find in [33]. For simplicity, consider the case $K(x,s,y) = a(x)b(s,y)$. Then, from (20) we have:

$$y' = f(x,y) + a(x) \int_{x_0}^{x} b(s, y(s))ds, \tag{31}$$

$$y(x_0) = y_0, \quad x_0 \le s \le x \le X.$$

By using the above mentioned method, one can derive the following from (31):

$$y'' = g(x,y,y') + a'(x)\vartheta(x) + a(x)\vartheta'(x), \tag{32}$$

$$y(x_0) = y_0, \quad y'(x_0) = y'_0,$$

$$\vartheta'(x) = b(x, y(x)), \quad \vartheta(x_0) = 0. \tag{33}$$

Thus, we change solving of the problem (1) by the solving of the problem (32) and the problem (33). It is known that finding the solution of the problem (32) and (33) is not difficult, because there are many methods for solving ordinary differential equations.

Remarks. Note that the application to solving of the problem (1) of the methods of the type (9) are more effective than the methods of the type (8) or type (4). Efficiency of this method contained its accuracy and its region of stability, etc. For example, from formula (9) can be obtained methods having the degree $p \le 10$. To this end, consider finding the values of the coefficient method (9) for the case, when $k = 1$, which are solutions of the following system of nonlinear algebraic equations:

$$\begin{aligned}
&\beta_1 + \beta_0 + \hat{\beta}_1 + \hat{\beta}_0 = 1,\\
&\gamma_1 + \gamma_0 + \hat{\gamma}_1 + \hat{\gamma}_0 + \beta_1 + l_1\hat{\beta}_1 + l_0\hat{\beta}_0 = 1/2,\\
&2(\gamma_1 + l_1\hat{\gamma}_1 + l_0\hat{\gamma}_0) + \beta_1 + l_1^2\hat{\beta}_1 + l_0^2\hat{\beta}_0 = 1/3,\\
&3(\gamma_1 + l_1^2\hat{\gamma}_1 + l_0^2\hat{\gamma}_0) + \beta_1 + l_1^3\hat{\beta}_1 + l_0^3\hat{\beta}_0 = 1/4,\\
&4(\gamma_1 + l_1^3\hat{\gamma}_1 + l_0^3\hat{\gamma}_0) + \beta_1 + l_1^4\hat{\beta}_1 + l_0^4\hat{\beta}_0 = 1/5,\\
&5(\gamma_1 + l_1^4\hat{\gamma}_1 + l_0^4\hat{\gamma}_0) + \beta_1 + l_1^5\hat{\beta}_1 + l_0^5\hat{\beta}_0 = 1/6,\\
&6(\gamma_1 + l_1^5\hat{\gamma}_1 + l_0^5\hat{\gamma}_0) + \beta_1 + l_1^6\hat{\beta}_1 + l_0^6\hat{\beta}_0 = 1/7,\\
&7(\gamma_1 + l_1^6\hat{\gamma}_1 + l_0^6\hat{\gamma}_0) + \beta_1 + l_1^7\hat{\beta}_1 + l_0^7\hat{\beta}_0 = 1/8,\\
&8(\gamma_1 + l_1^7\hat{\gamma}_1 + l_0^7\hat{\gamma}_0) + \beta_1 + l_1^8\hat{\beta}_1 + l_0^8\hat{\beta}_0 = 1/9,\\
&9(\gamma_1 + l_1^8\hat{\gamma}_1 + l_0^8\hat{\gamma}_0) + \beta_1 + l_1^9\hat{\beta}_1 + l_0^9\hat{\beta}_0 = 1/10.
\end{aligned} \tag{34}$$

Where $l_i = i + v_i \ (i = 0, 1, 2, ..., k)$.

Here one can be fined any methods of the above mentioned type in the following:

Variant I.

$$\begin{aligned}
&\beta_1 = 0.12215347920; \quad \beta_0 = 0.15666106539;\\
&\hat{\gamma}_1 = 0.00094817261; \quad \hat{\gamma}_0 = -0.00000719154;\\
&\gamma_1 = -0.00434946344; \gamma_0 = 0.00748767300;\\
&\hat{\beta}_1 = 0.34229328053; \quad \hat{\beta}_0 = 0.37889217490;\\
&l_1 = 0.71344424622; \quad l_0 = 0.34194466714.
\end{aligned} \tag{35}$$

Variant II.

$$\begin{array}{ll} \beta_1 = 0.18129750970; & \beta_0 = 0.29905999407; \\ \hat{\gamma}_0 = -0.00010121840; & \gamma_1 = -0.00947560581; \\ \gamma_0 = 0.03028505823; & \hat{\beta}_0 = 0.51964249631; \\ l_1 = 0.64494897427; & l_0 = 0.57328482896. \end{array} \quad (36)$$

To find the coefficients of methods (35) and (36) we used the system (34). When constructing method (35) in (34) the number of equations and unknowns coincide. A method for constructing (35) the number of unknowns is equal to 8, and the number of equations equals 10. In this case the system of (36) has several solutions one of them is a method (36). To illustrate the advantages of hybrid methods, we consider the application of the next method, which is identity with the method (29):

$$y_{n+1} = y_n + h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha})/2 \quad (\alpha = \frac{\sqrt{3}}{6}) \quad (37)$$

to solving of the following problems:

1. $y' = \int_0^x \cos s ds, \; 0 \le x \le 2, \; y(0) = -1.$

The exact solution for which is: $y(x) = -\cos x$.

2. $y' = \int_0^x xs \cos s^2 ds, \; 0 \le x \le 2, \; y(0) = -1/4.$

The exact solution can be written as following form: $y(x) = -\cos(x^2)/4$.

The obtained results, place in the following table.

Table 1 Results for Example 1 and 2.

Number of example	**Step size**	**Variable x**	**Error of the method (37)**
I	$h = 0,125$	0.125	0.23E-08
		1.00	0.17E-06
		2.00	0.52E-06
	$h = 0,25$	0.25	0.238E-07
		1.00	0.41E-06
		2.00	0.12E-05
II	$h = 0,125$	0.125	0.24E-09
		1.00	0.14E-05
		2.00	0.2E-04
	$h = 1/32$	0.03	0.37E-14
		1.00	0.42E-08
		2.00	0.79E-07

4 Perspective

Here are suggested some ways for the finding of the numerical solution of Volterra integro-differential equation of second order. Remark that there are some works devoted to the study of integro-differential equations of second order. Naturally, that the integro-differential equation of the second order can be studied as a separate object, or jointly with the integro-differential equation of the first order. With a simple comparison, it is shown that the study of integro-differential equation as a single object is the more promising. Another promising direction is the scheme which is using the hybrid method of the type (9) and (19). Therefore, the construction of algorithms for investigation of hybrid methods may be considered of current interest. We want remark that the advantages of hybrid method also has shown in [34] by application their to solving another problem. It is known that in solving some practical problems, we are encountered with problems whose solution is oscillation function (see for example [35]). For solving these problems one can be used forward-jumping methods (see for example [2])

Acknowledgment

The authors with express their thanks to academician Ali Abbasov for his suggestion that the investigate the computational aspects.

References

[1] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Recent Advances in Engineering Mechanics, Structures and Urban Planning, Cambridge, UK, 175-180 (2013).
[2] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Applied Numerical Mathematics **72**, 234-245 (2013).
[3] V. Volterra, Theory of functional and of integral and integro-differensial equations, Dover publications, Ing, New York,1959, (Russian).
[4] P.Linz, Journal of the Association for Computing Machinery **16**, 295-301 (1969).
[5] A. Feldstein, J.R. Sopka, Numerical methods for nonlinear Volterra integro differential equations, SIAM J. Numer. Anal. **11**, 826-846 (1974).
[6] H. Brunner, Mathematics of computation **42**, 95-109 (1984).
[7] A.A. Makroglou, Computing 3, **30**, 49-62 (1983).
[8] M.B. Bulatov, E.B. Chistakov, Dif. Equations **42**, 1218-1255, 2006, (Russian).
[9] O.S. Budnikova, M.V. Bulatov, Journal of Comput. Math. and Math. Phys. **52**, 829-839, 2012 (Russian).
[10] G. Mehdiyeva, V. Ibrahimov, M. Imanova, Proceedings of the World Academy of Science, engineering and Technology, Amsterdam, 349-352 (2010).
[11] G. Dahlquist, Math. Scand. **4**, 33-53 (1956).
[12] G. Dahlquist, **85**, Stockholm, K. Tekniska Högskolans Handlingar, 87 (1959).
[13] P. Henrici, Discrete variable methods in ordinary differential equation, Wiley, New York, 1962.
[14] J.C. Butcher, Numerical Methods for Ordinary differential equation, The University of Auckland, New Zeland, 2008.
[15] G.Yu. Mehdiyeva, V.R. Ibrahimov, On the research of multi-step methods with constant coefficients, Monograph, LAP LAMBERT, 2013.

[16] G.Yu. Mehdiyeva, V.R. Ibrahimov, M.N. Imanova, Proceedings of the IV international Conference "Problems of Cybernetics and Informatics", Baku, Azerbaijan, 107-110 (2012).
[17] G.Yu., Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Proceedings of the ICM, Al-Ain, 673-679 (2012).
[18] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Proceedings of the World Academy of Science, engineering and Technology, Paris, 240-243 (2011).
[19] E.A. Areo, R.A. Ademiluyi, P.O. Babatola, J.of Modern Math.and Statistics **2** (1), 1-6 (2008).
[20] C.S. Gear, SIAM, J. Numer. Anal. **2**, 69-86 (1965).
[21] J.C. Butcher, Journal of the ACM (JACM) **12**, 124-135 (1965).
[22] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, "Space and time - the coordinate system of human development," the collection of materials VIII International Scientific Conference, Kiev, London, 91-95 (2011).
[23] J. Kobza, Applikace Mathematicky **20**, 389-405 (1975).
[24] W.H. Enrite, SIAM, J. Numer. Anal. **2** , 321-332 (1974).
[25] V.R. Ibrahimov, Bulletin of Iranian Mathematical Society **20**, 1-28 (2002).
[26] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Mathematical methods for information science and economics, Montreux, Switzerland, 109-114 (2012).
[27] J.O. Ehigie, S.A. Okunuga, A.B. Sofoluwe, M.A. Akanbi, Archives of Applied Research **2**, 362-372 (2010).
[28] E. Hairier, S.P. Norsett, G. Wanner, Solving ordinary differential equations. Moscow, Mir, 1990 (Russian).
[29] H. Brunner, Math. Of Comput. **111**, 635-646 (1970).
[30] G.Y. Mehdiyeva, Ibrahimov V.R., M.N. Imanova, The Third international Conference "Problems of Cybernetics and Informatics", Baku, Azerbaijan, 255-258 (2010).
[31] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Models and methods of solving formal and applied scientific issues in phisico-mathematical technical and chemical research, London, 93-99 (2012).
[32] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, Applied Mathematical Sciences, HIKARI Ltd **7**, 4875-4890 (2013).
[33] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov, World Congress on Engineering, London, UK, 186-190 (2013).
[34] M.H. Moradi, M. Abedini, A combination of genetic algorithm and particle swarm optimization for optimal DG location and sizing in distribution systems, International Journal of Electrical Power & Energy Systems, **34**, 66-74 (2012).
[35] Taher S. Hassan, Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales, Applied Mathematics and Computation, **217**, 5285-5297 (2011).

Galina Mehdiyeva received her Doctor of science and PhD degree in Mathematics in Baku State University of Azerbaijan. She is chief of the department of Computational mathematics at Baku State University and investigating the application of approximate methods for solving operator equations. She constructs the numerical methods to solve ODE. In the last time she uses multistep methods to solving integral and integro-differential equations of Volterra type and published several papers in reputable international journals mathematical and engineering sciences. She is referee and editor of mathematical journals.

Mehriban Imanova is Associate Professor of Computational mathematics at Baku State University. She received the PhD degree in Mathematics at Baku State University of Azerbaijan. Her main research interests are: applied mathematics and informatics, numerical analysis, computational mathematics. In basically she investigates the numerical methods of solving integro-differential equations of Volterra type. For construction of effective methods she has used hybrid methods. She is participant of the many international conferences, which are held in the different countries.

Vagif Ibrahimov is Associate is Professor of department of Computational Mathematics of Baku State University of Azerbaijan. He received the degree doctor of sciences in Mathematics at the Institute of Cybernetics in Baku, and the PhD degree in the Computational center (now Institute of Computational Mathematics) in Georgia. He investigated multistep methods of Obreshkov type, and also he constructed numerous methods for solving ODE, Volterra integral equations and integro-differential equations of Volterra type. He received relations between order and degree for stable implicit and stable forward-jumping methods of Oreshkov type. He has constructed methods with the degree which use information about the solution of the investigated problem only at one mesh point. V.R. Ibrahimov gave some schemes for constructing hybrid methods with the higher degrees and extended stability region.