Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090523

Improving the Dynamics of Steffensen-type Methods

Sergio Amat[1,*], *Sonia Busquier*[1] *and Ángel Alberto Magreñán*[2]

[1] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Spain.
[2] Departamento de TFG/TFM, Universidad Internacional de La Rioja (UNIR), Logroño, La Rioja, Spain

Received: 24 Jan. 2015, Revised: 25 Apr. 2015, Accepted: 26 Apr. 2015
Published online: 1 Sep. 2015

Abstract: The dynamics of Steffesen-type methods, using a graphical tool for showing the basins of attraction, is presented. The study includes as particular cases, Steffesen-type modifications of the Newton, the two-steps, the Chebyshev, the Halley and the super–Halley iterative methods. The goal is to show that if we are interesting to preserve the convergence properties we must ensure that the derivatives are well approximated in all iterations.

Keywords: Complex dynamics, nonlinear equations, iterative methods, basins of attraction

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

One of the most important techniques to find the zeros of nonlinear equations is the use of iterative processes, starting from an initial approximation x_0, called pivot, successive approaches (until some predetermined convergence criterion is satisfied) x_n are computed.

Before presenting the iterative methods we are interested in, we shall recall some basic notions of complex dynamics. Let $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are complex polynomials with no common factors, be a rational map on the Riemann sphere. We say that z_0 is a *fixed point* of $R(z)$ if $R(z_0) = z_0$. For $z \in \overline{\mathbb{C}}$ we define its *orbit* as the set $\text{orb}(z) = \{z, R(z), R^2(z), \ldots, R^k(z), \ldots\}$, where R^k means the k–fold iterate of R. A *periodic point* of period n is a point z_0 such that $R^n(z_0) = z_0$ and $R^j(z_0) \neq z_0$ for $0 < j < n$. Observe that if $z_0 \in \overline{\mathbb{C}}$ is a periodic point of period $n \geq 1$, then z_0 is a fixed point of R^n. Also, recall that a fixed point z_0 is respectively *attracting, repelling* or *indifferent* in case $|R'(z_0)|$ is less than, greater than or equal to 1. A periodic point of period n is said to be attracting, repelling or indifferent if as a fixed point of $R^n(z)$ is respectively attracting, repelling or indifferent. A *superattracting fixed point* of $R(z)$ is a fixed point which is also a zero of the derivative $R'(z)$. A periodic point of period n is said to be a *superattracting periodic point* of $R(z)$ if, as a fixed point of $R^n(z)$, is superattracting.

Let ζ be an attracting fixed point of $R(z)$. The *basin of attraction* of ζ is the set $B(\zeta) = \{z \in \overline{\mathbb{C}} : R^n(z) \to \zeta \text{ as } n \to \infty\}$. The *immediate basin of attraction* of an attracting fixed point ζ of $R(z)$, denoted by $B^*(\zeta)$, is the connected component of $B(\zeta)$ containing ζ. Finally, if z_0 is an attracting periodic point of period n of $R(z)$, the *basin of attraction of the orbit* $\text{orb}(z_0)$ is the set $B(\text{orb}(z_0)) = \cup_{j=0}^{n-1} R^j(B(z_0))$, where $B(z_0)$ is the attraction basin of z_0 as a fixed point of R^n. The *Julia set* of a rational map $R(z)$, denoted by $\mathscr{J}(R)$, is the closure of the set of repelling periodic points. Its complement is the *Fatou set* $\mathscr{F}(R)$. If $R(z)$ has an attracting fixed point z_0, then the basin of attraction $B(z_0)$ is contained in the Fatou set and $\mathscr{J}(R) = \partial B(z_0)$. Therefore, the chaotic dynamics of $R(z)$ is contained in its Julia set.

To approximate nonlinear equations we can use iterative methods. Newton's method is the most widely used [3, 1, 17, 18, 21, 22, 24], although there are high-order variants that for certain problems can be more efficient [2, 6, 5, 15, 16, 19, 20]. If we want to maintain the order of the methods but without using derivatives we can consider Steffensen-type methods. But, if we are interesting to maintain the convergence properties we must ensure that the derivatives are well approximated in all iterations [23, 13, 14]. The use of some parameters α_n allow us to

* Corresponding author e-mail: sergio.amat@upct.es

achieve this goal [7,8,9]. This is our main motivation in this work.

On the other hand, higher order methods usually evaluate first and second derivatives. In this paper, we also study a modification of these classical third order iterative methods. The main advantage of the modified methods is that they do not need evaluate any derivative, but having the same order and the same convergence behavior of the original methods [4]. We use first and second order central divided differences and the user does not need to know explicitly any derivative.

The structure of this paper is as follows: in Section 2, we present the methods that we are interesting in. Using a graphical tool for showing the basins of attraction, we compare the Steffensen-type methods with the original methods using derivatives, in Section 3.

2 The iterative methods

We start with some classical iterative methods:

1.*Newton*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

2.*Two-steps*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}.$$

3.*Chebyshev*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n)\right)\frac{f(x_n)}{f'(x_n)},$$

4.*Halley*

$$x_{n+1} = x_n - \left(\frac{1}{1+\frac{1}{2}L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)},$$

5.*Super Halley*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1-L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)},$$

where

$$L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}.$$

We denote by $[\cdot,\cdot;f]$ and $[\cdot,\cdot,\cdot;f]$ the first and the second divided differences of the function f.

Our modify Steffensen-type methods associated to the above schemes write:

1.*Modify Steffensen*

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, x_n + \alpha_n f(x_n); f]},$$

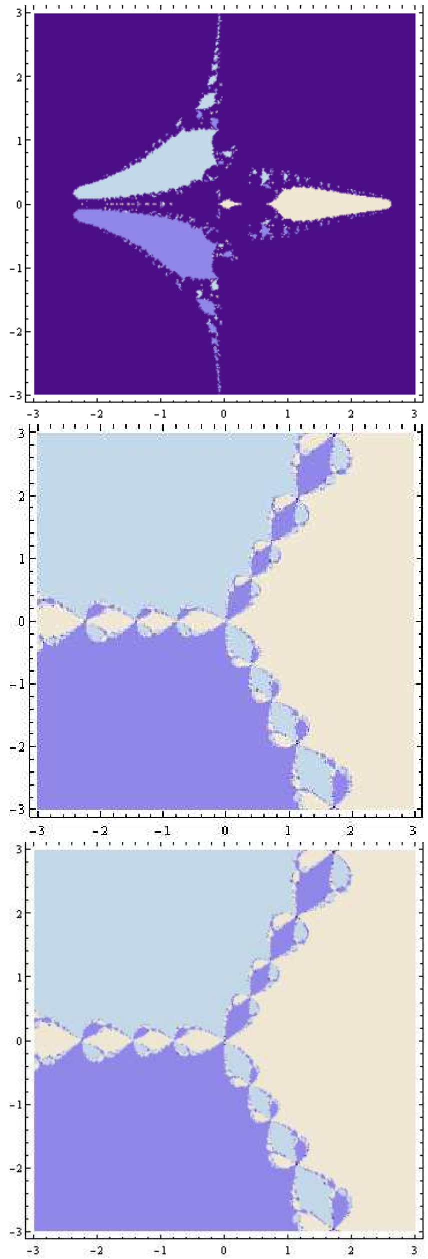


Fig. 1: Basins of attraction for $p(z) = z^3 - 1$. Left Steffensen's method, middle Newton's method and right modified Steffensen's method.

2.*Modify Steffensen-Two-steps*

$$y_n = x_n - \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]}.$$

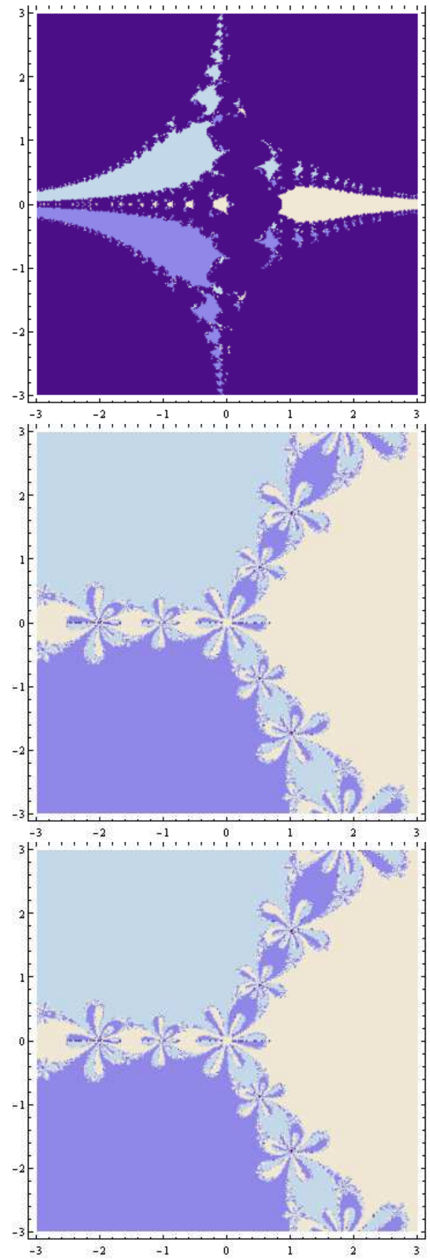


Fig. 2: Basins of attraction for $p(z) = z^3 - 1$. Left Two-Steps Steffensen's method, middle two-steps Newton's method and right modified two-step Steffensen's method.

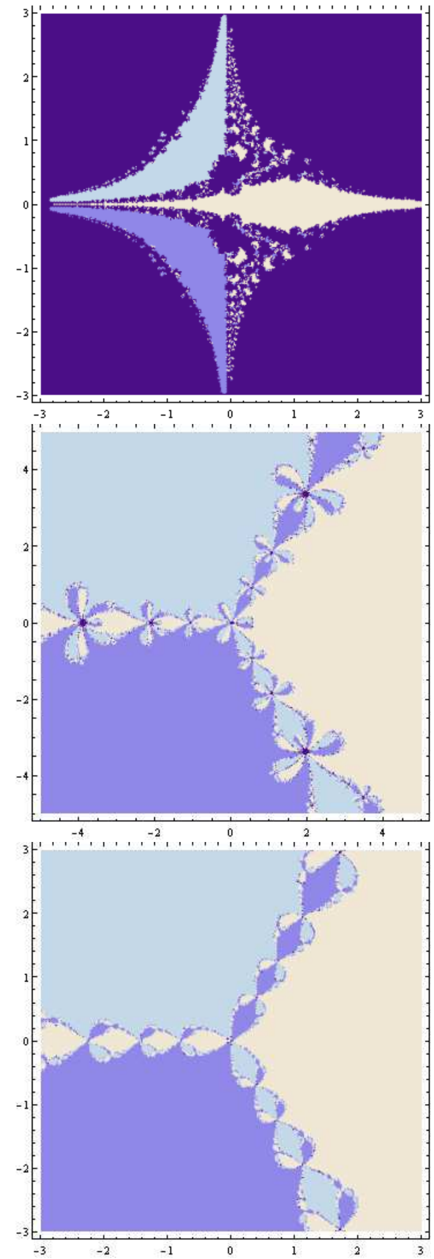


Fig. 3: Basins of attraction for $p(z) = z^3 - 1$. Left Chebyshev-Steffensen's method, middle Chebyshev's method and right modified Chebyshev-Steffensen's method.

3. *Modify Steffensen-Chebyshev*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\mathscr{L}_f(x_n)\right) \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

4. *Modify Steffensen-Halley*

$$x_{n+1} = x_n - \left(\frac{1}{1 + \frac{1}{2}\mathscr{L}_f(x_n)}\right) \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

5. *Modify Steffensen-Super Halley*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\mathscr{L}_f(x_n)}{1 - \mathscr{L}_f(x_n)}\right) \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

where

$$\mathscr{L}_f(x) = \frac{f(x)[x_n - \alpha_n f(x_n), x_n, x_n + \alpha_n f(x_n); f]}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]^2}.$$

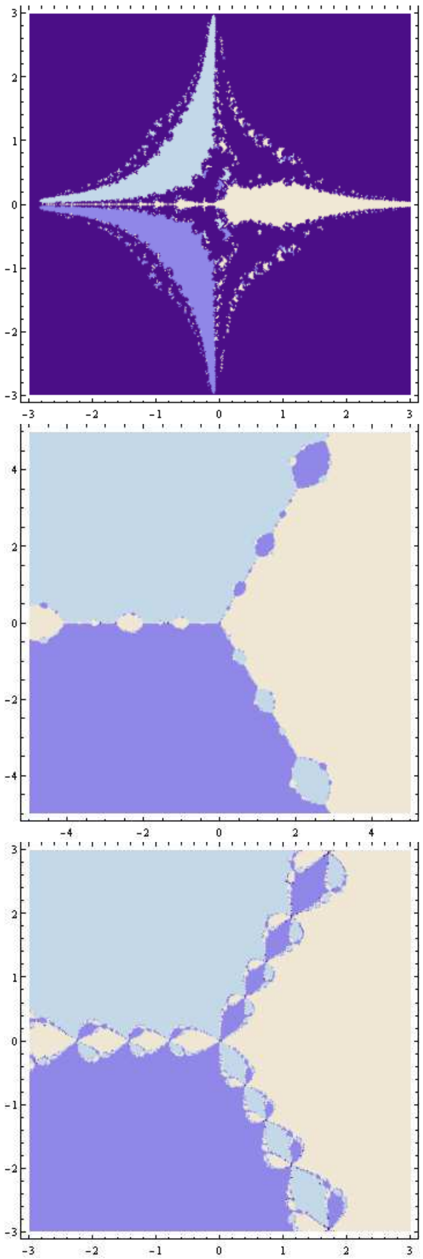


Fig. 4: Basins of attraction for $p(z) = z^3 - 1$. Left Halley-Steffensen's method, middle Halley's method and right modified Halley-Steffensen's method.

These methods depend, in each iteration, of some parameters α_n. These parameters are a control of the good approximation to the derivatives. In order to control the accuracy and stability in practice, the α_n can be computed such that

$$tol_c << \frac{tol_u}{2} \leq ||\alpha_n f(x_n)|| \leq tol_u,$$

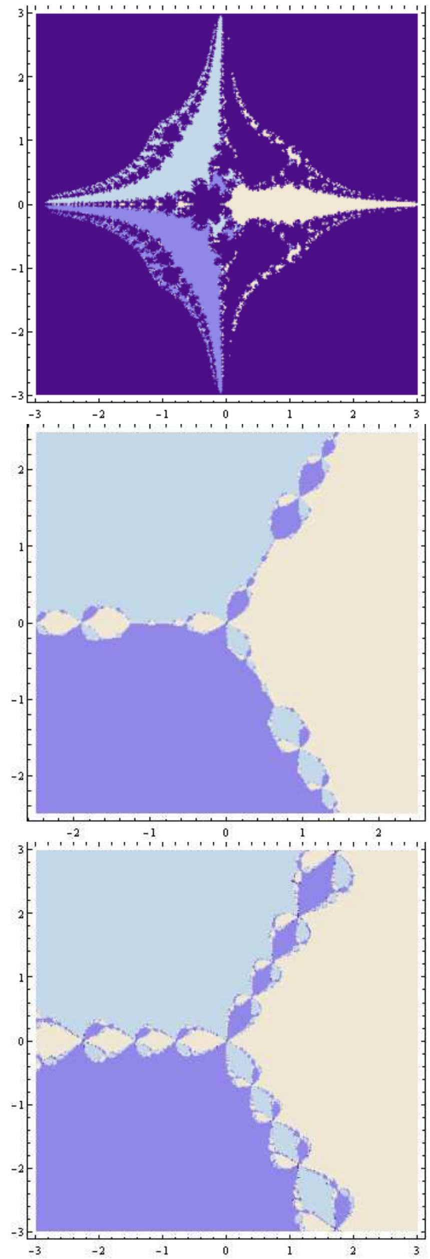


Fig. 5: Basins of attraction for $p(z) = z^3 - 1$. Left Halley-Steffensen, middle super-Halley and right modified super-Halley-Steffensen's method.

where tol_c is related with the computer precision and tol_u is a free parameter for the user.

The classical Steffensen-type methods use $\alpha_n = 1$.

3 A comparison of the basins of attraction

In this section we compare the dynamics of the above methods to introduce the benefits of using the parameters α_n. In the experiments we have taken $tol_u = 10^{-6}$.

We approximate the roots of polynomials. We use different colored painting regions of convergence of each root and dark violet is used for no convergence.

We include only the examples for $p(z) = z^3 - 1$ but similar conclusions are obtained for other examples.

The clear conclusion is that the good approximation of the derivatives (for instance using the parameters α_n) is crucial to remain the characteristic of the basins of attraction. The classical Steffensen-type methods ($\alpha_n = 1$) have smaller basins of attraction and great regions of no convergence.

See our works [10, 11, 12] for studies related with the dynamics of iterative methods.

4 Conclusions

We have presented by means of the basins of attraction, the dynamics of Steffesen-type methods, in which we have included some particular well-known cases such as Steffesen-type modifications of the Newton, the Two-steps, the Chebyshev, the Halley and the super–Halley iterative methods. We have shown that if the derivatives are well approximated in all iterations, using an adequate value of the parameter α_n, the convergence properties are preserved.

References

[1] I.Argyros, Advances in the Efficiency of Computational Methods abd Applications, Ed. World Science, Singapore, 2000.

[2] V. Alarcón, S. Amat, S. Busquier and F. Manzano, High order iterative schemes for quadratic equations. Numer. Algorithms 48 (2008), no. 4, 373-381.

[3] S. Amat, C. Bermúdez, S. Busquier, P. Leauthier and S.Plaza, On the dynamics of some Newton's type iterative functions. Int. J. Comput. Math. 86 (2009), no. 4, 631-639.

[4] S. Amat, C. Bermúdez, S. Busquier and D. Mestiri, A family of Halley-Chebyshev iterative schemes for non-Frćhet differentiable operators. J. Comput. Appl. Math. 228 (2009), no. 1, 486-493.

[5] S. Amat, C. Bermúdez, S. Busquier and S.Plaza, On a third-order Newton-type method free of bilinear operators. Numer. Linear Algebra Appl. 17 (2010), no. 4, 639-653.

[6] S. Amat and S. Busquier, Third-order iterative methods under Kantorovich conditions. J. Math. Anal. Appl. 336 (2007), no. 1, 243-261.

[7] S. Amat and S. Busquier, A two-step Steffensen's method under modified convergence conditions. J. Math. Anal. Appl. 324 (2006), no. 2, 1084-1092.

[8] S. Amat and S. Busquier, On a Steffensen's type method and its behavior for semismooth equations. Appl. Math. Comput. 177 (2006), no. 2, 819-823.

[9] S. Amat and S. Busquier, Convergence and numerical analysis of a family of two-step Steffensen's methods. Comput. Math. Appl. 49 (2005), no. 1, 13-22.

[10] S. Amat, S. Busquier and S.Plaza, On the dynamics of a family of third-order iterative functions. ANZIAM J. 48 (2007), no. 3, 343-359.

[11] S. Amat, S. Busquier and S.Plaza, Dynamics of a family of third-order iterative methods that do not require using second derivatives. Appl. Math. Comput. 154 (2004), no. 3, 735-746.

[12] S. Amat, S. Busquier and S.Plaza, Review of some iterative root-finding methods from a dynamical point of view. Sci. Ser. A Math. Sci. (N.S.) 10 (2004), 3-35.

[13] A. Cordero, J. L. Hueso, E. Martínez, and J. R. Torregrosa, A family of derivative-free methods with high order of convergence and its application to nonsmooth equations, Abstract and Applied Analysis, vol. 2012, Article ID 836901, 15 pages, 2012.

[14] F. Chicharro, A. Cordero, J.M. Gutiérrez and J.R. Torregrosa, Complex dynamics of derivative-free methods for nonlinear equations. Appl. Math. Comput. 219 (2013), no. 12, 7023-7035.

[15] J.A. Ezquerro, A modification of the Chebychev method, IMA J. Numer. Anal., 17, pp. 511-525, (1997).

[16] J.A. Ezquerro, J.M. Gutiérrez, M.A. Hernández and M.A. Salanova, Chebyshev-like methods and quadratic equations. Revue d'Analyse Numerique et theorie de l'Approximation, 28, 1, pp. 23-35, (1999).

[17] J.A. Ezquerro and M.A. Hernández, On an Application of Newton's Method to a Nonlinear Operators with *w*-Conditioned Second Derivative, BIT, 42, pp. 519-530, (2002).

[18] Á.A. Magreñán, Estudio de la dinámica del método de Newton amortiguado (PhD Thesis), Servicio de Publicaciones, Universidad de La Rioja, 2013. http://dialnet.unirioja.es/servlet/tesis?codigo=38821

[19] Á. A. Magreñán, Different anomalies in a Jarratt family of iterative root-finding methods, *Appl. Math. Comput.*, **233** (2014), 29–38.

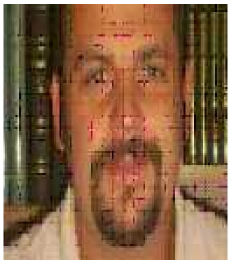
[20] Á. A. Magreñán, A. Cordero, J. M. Gutiérrez and J. R. Torregrosa, Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane, *Math. Comput. Simul.*, **105** (2014), 49–61.

[21] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Ed. New York: Academic Press, New York, 1970.

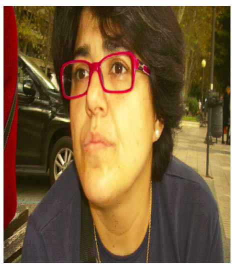
[22] F.A. Potra and V. Pták, *Nondiscrete Induction and Iterative Processes*, Research Notes in Mathematics, 103, Ed. Boston: Pitman, Boston, 1984.

[23] F. Soleimani, F. Soleymani and S. Shateyi, Some Iterative Methods Free from Derivatives and Their Basins of Attraction for Nonlinear Equations, Discrete Dynamics in Nature and Society Volume 2013, Article ID 301718, 10 pages.

[24] W. Werner, Newton-Like Methods for the Computation of Fixed Points, Comp. and Math. with Appls., 10, 1, pp. 77-86, (1984).



Sergio Amat received the M.S., B.S., and Ph.D. degrees in mathematics from Universidad de Valencia, Valencia, Spain, in 1996 and 2001, respectively. Currently, he is Full Professor in the Department of Applied Mathematics and Statistics, Universidad Politécnica de Cartagena, Cartagena, Spain. His research interests include nonlinear reconstructions, multiresolution and wavelets algorithms, as well as iterative schemes for nonlinear equations and numerical approximation of differential equations.



Sonia Busquier received the B.S. degree from Universidad de Valencia and her Ph.D. degree from Universidad Politécnica de Cartagena. Currently, she is Professor in the Department of Applied Mathematics and Statistics, Universidad Politécnica de Cartagena, Cartagena, Spain. Her research interests include nonlinear reconstructions, multiresolution and wavelets algorithms, as well as iterative schemes for nonlinear equations.

Á. A. Magreñan received the M.S., B.S., and Ph.D. degrees in mathematics from Universidad de La Rioja, Logroño, Spain, in 2008 and 2013, respectively. Currently, he is Collaborator Professor in the Department of TFG/TFM, Universidad Internacional de La Rioja, Logroño, Spain. His research interests include iterative schemes for nonlinear equations and numerical approximation of differential equations and real and complex dynamical study.