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Analysis of neural networks with ψ-type Caputo fractional derivative

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Abstract: This paper studies existence and uniqueness for a general class of neural networks with ψ-Caputo fractional derivative. Some sufficient conditions for existence of the solutions are established by applying standard fixed point method and the inequality technique. At last, we consider the uniform stability of the ψ-type Caputo fractional-order neural networks.

Keywords: Existence and uniqueness; neural networks; ψ-type Caputo fractional-order; fixed point.

1 Introduction

Fractional calculus (FC) dates from 300 years ago and deals with arbitrary (noninteger) order differentiation and integration. Even if it has a long history, it did not draw much attention from researchers due to its complexity and difficult application. Still, in the last decades, the theory of FC developed mainly as a pure theoretical field of mathematics and has been used in various fields as rheology, viscoelasticity, electrochemistry, diffusion processes, and so on; see, for example, [2,4,6,12,13] and the references therein. It is well known that compared with integer-order models, fractional-order calculus gives a more accurate instrument for the description of memory and hereditary properties of various processes. In recent times, Ricardo Almeida [1] introduced the so-referred to as ψ-fractional derivative with respect to another function.

The branch of neural networks, as other fields of science, has a long history of development with plenty of ups and downs. First model of artificial neurons, called the Threshold Logic Unit was introduced in 1943. In the last few decades, the subjective investigation of neural networks has gained much attention because of its strong applications in numerous fields such as signal and image processing, associative memories, combinatorial optimization and many others [3,5,7,9]. Though, such practical applications of neural networks are strongly dependent on the qualitative behaviors of neural networks. Taking these facts into account, the inclusion of the fractional-order calculus into a neural network model could better describe the dynamical behavior of the neurons, and many efforts have been made. Rakkiyappan et al. studied the existence and uniform stability analysis of fractional-order complex-valued neural networks with time delays in [14]. Dissipativity and stability analysis of fractional-order complex-valued neural networks with time delay was investigated by Velmurugan et al. in [17]. Wang et al. [18] analysed the stability of fractional-order neural networks with time delay. Recently, the dynamic analysis of fractional-order neural networks has received considerable attention, and some excellent results have been presented in [8,10,15,16,19].

Motivated by the above discussions, this paper considers a general class of neural networks with ψ-type Caputo fractional derivative of the form

$$\begin{cases} {}^{c}D^{\alpha;\psi}y_i = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + I_i, & t \geq 0, \\ y_i(0) = y_{i0}, \end{cases} \tag{1}$$

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where $\alpha \in (0,1)$, ${}^{c}D^{\alpha;\psi}$ is the ψ-type Caputo fractional derivative, n corresponds to the number of the units in a neural network, $y_i(t)$ stands for the state variable of the ith neuron at the time t, $f_j(y_j(t))$ denotes the measure of activation to its incoming potential of the unit jth at time t, $c_i > 0$ $(i = 1,...,n)$, I_i is the control input vector and a_{ij} represents the strength of the neuron interconnection within the network.

We arrange the rest of this paper as follows. In Section 2, we recall some preliminary concepts of fractional calculus related to our work. Section3 contains the main results.

2 Prerequisites

In this section, we outline some preliminary concepts of fractional calculus.

Definition 1.[1] *The ψ-type fractional integral of Riemann-Liouville of order $\alpha > 0$ of a function is defined as follows:*

$$I^{\alpha;\psi}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1}\, h(s)ds, \quad t > 0,$$

where the symbol $\Gamma(\cdot)$ stands for the Euler's gamma function.

Definition 2.[1] *The ψ-type Caputo fractional derivative of order α for a function h can be defined as*

$${}^{c}D^{\alpha;\psi}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{n-\alpha-1}\, h^{(n)}(s)ds,$$

where $t > 0$, $n-1 < \alpha < n$.

Remark.[1] The relationship between the ψ-type Riemann-Liouville derivative and the ψ-type Caputo derivative can be defined as

$${}^{c}D^{\alpha;\psi}h(t) = \quad {}^{R}D^{\alpha;\psi}h(t) - \sum_{k=0}^{n-1} \frac{h^{k}(0)}{k!}(\psi(t))^{k}.$$

The following lemma is needed to prove main results.

Lemma 1.[11] *Let D be a closed convex and nonempty subset of Banach space Y. Let ϕ_1, ϕ_2 be the operators such that*

1.$\phi_1 z_1 + \phi_2 z_2 \in D$ whenever $z_1, z_2 \in D$;
2.ϕ_1 is compact and continuous;
3.ϕ_2 is a contraction mapping.

Then, there exists $z_1 \in D$ such that $\phi_1 z_1 + \phi_2 z_1 = z_1$.

Definition 3.*The continuous function $y_i(t)$ is said to be a solution of the system (1) if the following condition*

$$y_i(t) = y_{i0} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1} \left[-c_i y_i(s) + \sum_{j=1}^{n} a_{ij} f_j(y_j(s)) + I_j \right] ds \tag{2}$$

is satisfied.

3 Main results

In this section, we will consider the existence and uniqueness of solution to system (1) by using fixed point method. We adopt the ideas from [16]. Now we state the hypotheses needed in the sequel.

(H1)The neuron activation functions f_i $(i = 1,2,...,n)$ satisfy the Lipschitz condition, i.e., there exist nonnegative constants l_i such that for any $u_1, v_1 \in R$

$$|f_i(u_1) - f_i(v_1)| \le l_i\,|u_1 - v_1| \le \max_{1\le i\le n} l_i\,|u_1 - v_1| = l_0\,|u_1 - v_1|.$$

(H2)For $i = 1,...,n$ there exists $M > 0$ such that for $u_a \in R$, $|f_i(u_a)| \le M$.

Theorem 1. *Under hypothesis (H1), system (1) has unique solution on $J := [0,T]$, if there exists a real number $p > 1$ such that*

$$\frac{c_0(\psi(T))^{\alpha} n^{\frac{1}{p}} + l_0(\psi(T))^{\alpha}\left(\sum_{i=1}^{n}\xi_i^p\right)^{\frac{1}{p}}}{\Gamma(\alpha+1)} < 1, \tag{3}$$

where $\xi = \left[\sum_{j=1}^{n}\left|a_{ij}\right|^{\frac{p}{(p-1)}}\right]^{\frac{(p-1)}{p}}$, $c_0 = \max_{1\le i\le n} c_i$.

Proof. Define $S : Y \to Y$ as

$$(Sy)(t) = \left((Sy_1)(t), (Sy_2)(t), \dots, (Sy_n)(t)\right)^T,$$

where

$$(Sy_i)(t) = y_{i0} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left[-c_i y_i(s) + \sum_{j=1}^{n} a_{ij} f_j(y_j(s)) + I_i\right] ds.$$

The proof is spilt into two steps as follows:

Claim 1.

Firstly, we prove $SB_r \subset B_r$, where $B_r = \{y \in Y : \|y\| \le r\}$.

$$r \ge \frac{\Gamma(\alpha+1)\left[\sum_{i=1}^{n}|y_{i0}|^p\right]^{\frac{1}{p}} + I_0(\psi(T))^{\alpha} n^{\frac{1}{p}} + f_0(\psi(T))^{\alpha}\left[\sum_{i=1}^{n} a_{i0}^p\right]^{\frac{1}{p}}}{\Gamma(\alpha+1) - c_0(\psi(T))^{\alpha} n^{\frac{1}{p}} - l_0(\psi(T))^{\alpha}\left(\sum_{i=1}^{n}\xi_i^p\right)^{\frac{1}{p}}},$$

where $I_0 = \max_{1\le i\le n}|I_i|$, $f_0 = \max_{1\le i\le n}\left|f_j(0)\right|$, $a_{i0} = \sum_{j=1}^{n}\left|a_{ij}\right|$.
Minkowski inequality gives that

$$\left[\sum_{i=1}^{n}(a_i + b_i + \dots + l_i)^p\right]^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \dots + \left(\sum_{i=1}^{n} l_i^p\right)^{\frac{1}{p}},$$

$a_i, b_i, \dots, l_i \ge 0$, $p > 1$, $i = 1, 2, \dots, n$.
Further,

$$\begin{aligned}
\|Sy\| &\le \sup_{t\in J}\left\{\sum_{i=1}^{n}\left|y_{i0} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}\left[-c_i y_i(s) + \sum_{j=1}^{n} a_{ij} f_j(y_j(s)) + I_i\right] ds\right|^p\right\}^{\frac{1}{p}} \\
&\le \left[\sum_{i=1}^{n}|y_{i0}|^p\right]^{\frac{1}{p}} + \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1} c_i\,|y_i(s)|\, ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}|I_i|\, ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}\left|a_{ij}\right| \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}\left|f_j(0)\right| ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}\left|a_{ij}\right| \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}\left|f_j(y_j(s)) - f_j(0)\right| ds\right)^p\right]^{\frac{1}{p}} \\
&\le M_0 + M_1 + M_2 + M_3 + M_4,
\end{aligned}$$

where

$$M_0 = \left[\sum_{i=1}^{n} |y_{i0}|^p\right]^{\frac{1}{p}},$$

$$M_1 = \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(s)-\psi(t))^{\alpha-1} c_i |y_i(s)|\,ds\right)^p\right]^{\frac{1}{p}},$$

$$M_2 = \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(s)-\psi(t))^{\alpha-1} |I_i|\,ds\right)^p\right]^{\frac{1}{p}},$$

$$M_3 = \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}|a_{ij}|\,\psi'(s)(\psi(s)-\psi(t))^{\alpha-1} |f_j(0)|\,ds\right)^p\right]^{\frac{1}{p}},$$

$$M_4 = \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}|a_{ij}|\,\psi'(s)(\psi(s)-\psi(t))^{\alpha-1} |f_j(y_j(s))-f_j(0)|\,ds\right)^p\right]^{\frac{1}{p}}.$$

Direct computation implies that

$$M_1 = \frac{c_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)} r,$$

$$M_2 = \frac{I_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)},$$

$$M_3 = \frac{f_0(\psi(T))^\alpha}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{n} a_{i0}^p\right]^{\frac{1}{p}},$$

$$\begin{aligned} M_4 &= \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}|a_{ij}|\,\psi'(s)(\psi(s)-\psi(t))^{\alpha-1} |f_j(y_j(s))-f_j(0)|\,ds\right)^p\right]^{\frac{1}{p}} \\ &\le \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=1}^{n}|a_{ij}|\,\psi'(s)(\psi(s)-\psi(t))^{\alpha-1} I_j |y_j(s)|\,ds\right)^p\right]^{\frac{1}{p}} \\ &\le \frac{I_0(\psi(T))^\alpha}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{n} \xi_i^p\right]^{\frac{1}{p}} r. \end{aligned}$$

The above second inequality concerning M_4 is obtained by using Holder inequality

$$\sum_{i=1}^{n} u_i v_i \le \left(\sum_{i=1}^{n} u_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} v_i^q\right)^{\frac{1}{q}},$$

where $u_i, v_i \ge 0, p,q > 0,\ \frac{1}{p}+\frac{1}{q}=1$.
Hence, we obtain

$$\begin{aligned} \|Sy\| &= \left[\sum_{i=1}^{n}|y_{i0}|^p\right]^{\frac{1}{p}} + \frac{I_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{f_0(\psi(T))^\alpha}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{n} a_{i0}^p\right]^{\frac{1}{p}} \\ &+ \left(\frac{c_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{l_0(\psi(T))^\alpha}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{n}\xi_i^p\right]^{\frac{1}{p}}\right) r \\ &\le r. \end{aligned}$$

Claim 2.

We prove that $S : Y \to Y$ is a contraction mapping.

Let $x, y \in Y$, similar to the above process, we have

$$\begin{aligned}
\|Sx - Sy\| = \sup_{t \in J} \Bigg\{ \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \int_0^t -c_i \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} [x_i(s) - y_i(s)] \\
+ \sum_{j=1}^{n} a_{ij} (f_j(x_j(s)) - f_j(y_j(s)) ds|^p \Bigg\}^{\frac{1}{p}} \\
\leq \sup_{t \in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(s) - \psi(t))^{\alpha-1} c_i |x_i(s) - y_i(s)| ds \right)^p \right]^{\frac{1}{p}} \\
+ \sup_{t \in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \sum_{j=1}^{n} \psi'(s) (\psi(s) - \psi(t))^{\alpha-1} |a_{ij} l_j| |x_i(s) - y_i(s)| ds \right)^p \right]^{\frac{1}{p}} \\
\leq \left(\frac{c_0 (\psi(T))^{\alpha} n^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{l_0 (\psi(T))^{\alpha} (\sum_{i=1}^{n} \xi_i^p)^{\frac{1}{p}}}{\Gamma(\alpha+1)} \right) \|x - y\| .
\end{aligned}$$

By (3), we conclude that S is a contraction mapping. The proof now can be finished by using the contraction mapping principle.

Theorem 2. *Under hypothesis (H2), the system* (1) *has at least one solution on J, if there exists a real number* $p > 1$ *such that*

$$\frac{c_0 (\psi(T))^{\alpha} n^{\frac{1}{p}}}{\Gamma(\alpha+1)} < 1. \tag{4}$$

Proof. Let

$$r \geq \frac{\Gamma(\alpha+1) [\sum_{i=1}^{n} |y_{i0}|^p]^{\frac{1}{p}} + I_0 (\psi(T))^{\alpha} n^{\frac{1}{p}} + M (\psi(T))^{\alpha} \left(\sum_{i=1}^{n} \xi_i^p\right)^{\frac{1}{p}}}{\Gamma(\alpha+1) - c_0 (\psi(T))^{\alpha} n^{\frac{1}{p}}}.$$

Now we can define on operators L and N on B_r ($B_r = \{y \in Y : \|y\| \leq r\}$) by

$$\begin{aligned}
(Ly)(t) = ((Ly_1)(t), (Ly_2)(t), ..., (Ly_n)(t))^T, \\
(Ny)(t) = ((Ny_1)(t), (Ny_2)(t), ..., (Ny_n)(t))^T,
\end{aligned}$$

where,

$$\begin{aligned}
(Ly_i)(t) = y_{i0} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(s) - \psi(t))^{\alpha-1} [-c_i y_i(s) + I_i] ds, \\
(Ny_i)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(s) - \psi(t))^{\alpha-1} \sum_{j=1}^{n} a_{ij} f_j(y_j(s)) ds.
\end{aligned}$$

Firstly, for $x, y \in B_r$, we have $Lx + Ny \in B_r$. Indeed, using Minkowski inequality gives that

$$
\begin{aligned}
&\|Lx+Ny\| \\
&\leq \sup_{t\in J}\left\{\sum_{j=1}^{n}\left|x_{i0}+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}\left[-c_i x_i(s)+\sum_{j=1}^{n}a_{ij}f_j(y_j(s))+I_i\right]ds\right|^p\right\}^{\frac{1}{p}} \\
&\leq \left[\sum_{i=1}^{n}|x_{i0}|^p\right]^{\frac{1}{p}}+\sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}c_i|x_i(s)|\,ds\right)^s\right]^{\frac{1}{p}} \\
&+\sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha+1)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}I_i ds\right)^p\right]^{\frac{1}{p}} \\
&+\sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t\sum_{j=1}^{n} \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}\left|a_{ij}f_j(y_j(s))\right|ds\right)^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{i=1}^{n}|x_{i0}|^p\right]^{\frac{1}{p}}+\frac{c_0(\psi(t))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)}+\frac{l_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)}+\frac{M(\psi(T))^\alpha(\sum_{i=1}^{n}\xi_i^p)^{\frac{1}{p}}}{\Gamma(\alpha+1)} \\
&\leq r.
\end{aligned}
$$

Hence, we conclude that $Lx + Ny \in B_r$.

Secondly, for any $x, y \in B_r$, we can get

$$
\begin{aligned}
\|Lx-Ly\| &\leq \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}c_i|x_i-y_i|\,ds\right)^p\right]^{\frac{1}{p}} \\
&\leq \frac{c_0(\psi(T))^\alpha n^{\frac{1}{p}}}{\Gamma(\alpha+1)}\|x-y\|.
\end{aligned}
$$

In view of (4), L is a contraction mapping.

Now, we show that N is continuous and compact.

Since f_j, $j = 1, 2, ..., n$, are continuous, it is clear that N is also continuous . Let $y \in B_r$, we obtain

$$
\begin{aligned}
\|(Ny)(t)\| &= \sup_{t\in J}\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^t\sum_{j=1}^{n} \psi'(s)\left(\psi(s)-\psi(t)\right)^{\alpha-1}a_{ij}\left|f_j(y_j(s))\right|ds\right)^p\right]^{\frac{1}{p}} \\
&\leq \frac{M(\psi(T))^\alpha\left(\sum_{i=1}^{n}\xi_i^p\right)^{\frac{1}{p}}}{\Gamma(\alpha+1)},
\end{aligned}
$$

which means that N is uniformly bounded on B_r.

Next, we can deduce that $(Ny)(t)$ is equicontinuous. In fact, for $y \in B_r$, $0 < t_2 < t_1$, we have

$$
\begin{aligned}
|(Ny)(t_1)-(Ny)(t_2)| \leq &\left[\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\sum_{i=1}^{n} \psi'(s)\left(\psi(s)-\psi(t_1)\right)^{\alpha-1}\left|a_{ij}f_j(y_j(s))\right|ds\right.\right. \\
&\left.\left.-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\sum_{j=1}^{n} \psi'(s)\left(\psi(s)-\psi(t_2)\right)^{\alpha-1}\left|a_{ij}f_j(y_j(s))\right|ds\right)^p\right]^{\frac{1}{p}} \\
\leq &\frac{M(\psi(T))^\alpha(\sum_{i=i}^{n}\xi_i^p)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\left|2(\psi(t_1)-\psi(t_2))^\alpha+(\psi(t_2))^\alpha-(\psi(t_1))^\alpha\right|.
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. So $N(B_r)$ is relatively compact. By the Arzela-Ascoli theorem, N is compact, L is a contraction mapping. Hence, Lemma 1 allows us to conclude that system (1) has at least one solution.

Remark.[16] The theoretical result on the existence and uniqueness of the nontrivial solution for fractional-order neural networks has not yet seen.

Theorem 3. *Under hypothesis (H1) and condition (3), the solution of system (1) is uniformly stable on J.*

Proof. Suppose that u_i and v_i are any two solutions of the system (1) with initial condition $u_i(0) = u_{i0}$, $v_i(0) = v_{i0}$ and

$$\left(\sum_{i=1}^{n} |u_{i0} - v_{i0}|^p\right)^{\frac{1}{p}} \leq r,$$

respectively. Thus, we can obtain

$$u_i(t) = u_{i0} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} \left[-c_i u_i(s) + \sum_{j=1}^{n} a_{ij} f_j(u_j(s)) + I_i\right] ds$$
$$v_i(t) = v_{i0} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} \left[-c_i v_i(s) + \sum_{j=1}^{n} a_{ij} f_j(v_j(s)) + I_i\right] ds,$$

which implies,

$$\begin{aligned}
\|u - v\| &= \sup_{t\in J} \left(\sum_{i=1}^{n} |u_i(t) - v_i(t)|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{n} |u_{i0} - v_{i0}|^p\right)^{\frac{1}{p}} \\
&+ \sup_{t\in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t c_i \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} |u_i(s) - v_i(s)|\, ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \sum_{j=1}^{n} |a_{ij}|\, \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} \left|f_j(u_j(s)) - f_j(v_j(s))\right| ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} c_i\, |u_i(s) - v_i(s)|\, ds\right)^p\right]^{\frac{1}{p}} \\
&+ \sup_{t\in J} \left[\sum_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(s) - \psi(t_1))^{\alpha-1} |a_{ij} I_j|\, |u_i(s) - v_i(s)|\, ds\right)^p\right]^{\frac{1}{p}} \\
&\leq r + \left(\frac{c_0(\psi(T))^{\alpha} n^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{I_0(\psi(T))^{\alpha} (\sum_{i=1}^{n} \xi_i^p)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\right) \|x - y\|.
\end{aligned}$$

So, we can get

$$\|u - v\| \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1) - c_0(\psi(T))^{\alpha} n^{\frac{1}{p}} - I_0(\psi(T))^{\alpha} \left(\sum_{i=1}^{n} \xi_i^p\right)^{\frac{1}{p}}} r.$$

Conflict of Interests

There is no conflict of interests by authors regarding the publication of this manuscript.

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