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A Unified Local Convergence for Two-Step Newton-Type Methods with High Order of Convergence under Weak Conditions

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Abstract: We present a unified local convergence analysis for Newton-type methods in order to approximate a solution of a nonlinear equation. In earlier studies such as [1,2,5]-[36] hypotheses of at least the third derivative have been used to show convergence. Our local convergence is based on hypotheses up to the first derivative. This way, we expand the applicability of these methods. Moreover, the radius of convergence, the uniqueness ball and computable error bounds involving Lipschitz constants not given before are also provided in this study. Special cases and numerical examples are also given in this study.

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1 Introduction

This paper is devoted to the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to S$ is a twice differentiable nonlinear function, D is a convex subset of S where $S = \mathbb{C}$ or $S = \mathbb{R}$. Newton-like methods are famous for finding solution of (1.1), these methods are studied based on: semi-local and local convergence [3,4,23,27,28,30], [32]–[36].

The methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]-[36] require the evaluation of the second derivative F'' at each step, which in general is very expensive. To avoid this expensive computation, many authors have used higher order multipoint methods [1]-[36].

Newton's method is undoubtedly the most popular method for approximating a locally unique solution x^* provided that the initial point is close enough to the solution. In order to obtain a higher order of convergence Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy Halley and Ostrowski method [34]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [28] $EI = p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the method and m is the number of function evaluations.

It is well known that according to the Kung-Traub conjucture the convergence of any multipoint method without memory cannot exceed the upper bound 2^{m-1}

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[28] (called the optimal order). Hence the optimal order for a method with three function evaluations per step is 4. The corresponding efficiency index is $EI = 4^{\frac{1}{3}} = 1.58740...$ which is better than Newtons method which is $EI = 2^{\frac{1}{2}} = 1.414....$ Therefore, the study of new optimal methods of order four is important.

Many third order methods are special cases of the method defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - A(x_n)^{-1}F(x_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\alpha \in S$ is a parameter and function A is an approximation of F'. If $\alpha = 1$ we obtain:

- –Halley's method [14]: $A_0(x_n) = F'(x_n) - \frac{1}{2}\frac{F(x_n)F^{(2)}(x_n)}{F'(x_n)}$.
- –Arithmetic Mean Newton's method [3,4]: $A_1(x_n) = \frac{1}{2}(F'(y_n) + F'(x_n))$.
- –Midpoint Newton's method : $A_2(x_n) = F'(\frac{1}{2}(y_n + x_n))$.
- –Hasanov's VS Variant [25]: $A_3(x_n) = \frac{1}{6}(F'(y_n) + 4F'(\frac{y_n+x_n}{2}) + F'(x_n))$.
- –Nedzhibov V1 Variant [25]: $A_4(x_n) = \frac{1}{4}(F'(y_n) + 2F'(\frac{y_n+x_n}{2}) + F'(x_n))$.
- –Leap-froggin Newton's method or Newton-Secant method [18]: $A_5(x_n) = \frac{F'(x_n)(F(x_n)-F(y_n))}{F(x_n)}$.
- –Harmonic Mean Newton's method [18]: $A_6(x_n) = \frac{2F'(x_n)F'(y_n)}{F'(x_n)+F'(y_n)}$.
- –Geometric Mean Newton's method [18]:$A_7(x_n) = sign(F'(x_0))\sqrt{F'(x_n)F'(y_n)}$.
- –Noor's method [24]:$A_8(x_n) = \frac{1}{4}(F'(x_n) + 3F'(\frac{x_n+2y_n}{3}))$.
- –Newton's third order method with common Jacobian [3,4] is usually written as

$$\begin{aligned} y_n &= x_n - \frac{F(x_n)}{F'(x_n)} \\ x_{n+1} &= y_n - \frac{F(y_n)}{F'(x_n)}. \end{aligned} \tag{1.3}$$

Newton's method (1.3) is also a special case of method (1.2), if we define $A_9(x_n) = (I + \frac{F(x_n - F'(x_n)^{-1}F(x_n))}{F(x_n)})^{-1}F'(x_n)$.
- –Cauchy's method [28]:$A_{10}(x_n) = F'(x_n)\psi(x_n)^{-1}$ where

$$\psi(x_n) = \frac{2}{1 + \sqrt{1 - L(x_n)}}$$

and

$$L(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x).$$

Define functions

$$\begin{aligned} q(x) &= 2\frac{F(x - F'(x)^{-1}F(x))}{F(x)} \\ q_1(x) &= \frac{3}{2}(1 - F'(x)^{-1}F'(x - \frac{2}{3}F'(x)^{-1}F(x)) \\ q_2(x) &= F'(x)^{-1}F''(x - \frac{1}{3}F'(x)^{-1}F(x))F'(x)^{-1}F(x). \end{aligned}$$

- –Potra-Ptak method [29]: $A_{11}(x_n) = F(x_n)(1 + \frac{q(x_n)}{2})^{-1}$
- –Chebyshev method [1,3,4]:$A_{12}(x_n) = F'(x_n)(1 + \frac{L(x_n)}{2})^{-1}$
- –Newton Steffensen [13]: $A_{13}(x_n) = F'(x_n)(\frac{2}{2-q(x)})^{-1}$
- –Super Halley [2,3,4]: $A_{14}(x_n) = F'(x_n)(\frac{2-L(x_n)}{2(1-L(x_n))})^{-1}$
- –Higher than order three methods can also be brought in the form of method (1.2). Let us state some fourth order methods . The two-step Newton's method

$$\begin{aligned} y_n &= x_n - \frac{F(x_n)}{F'(x_n)} \\ x_{n+1} &= y_n - \frac{F(y_n)}{F'(y_n)}, \end{aligned} \tag{1.4}$$

is a special case of method (1.2), if we define $A_{15}(x_n) = (F'(x_n)^{-1} + \frac{F'(x_n - F'(x_n)^{-1}F(x_n))F(x_n - F'(x_n)^{-1}F(x_n))}{F(x_n)})^{-1}F'(x_n)$.
- –Cauchy-type method [28]: Replace function ψ in the definition of Cauchy's method by functions q_1 or q_2.
- –King's method [2]:$A_{16}(x_n) = F'(x_n)(1 + \frac{2g(x_n)+\beta q(x_n)^2}{4+2(\beta-2)q(x_n)})^{-1}$.
- –Jarratt's method [2,5]:$A_{17}(x_n) = F'(x_n)(a_1 + \frac{3a_2}{3-2q_3(x_n)} + \frac{3}{3\beta_1+\beta_2(3-2q_3(x_n))})^{-1}$ where $F'(x_n)^{-1}F'(x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)) = 1 - \frac{2}{3}q_3(x_n)$.
- –Ostrowsi's method [28]: $A_{18}(x_n) = F'(x_n)(\frac{q(x_n)-2}{2(q(x_n)-1)})^{-1}$.

Many other choices for function A are possible. Therefore it is important to study the local convergence of these methods using a common set of hypotheses. Notice also that the convergence analysis of these methods (for $m = 0, 1, 2, \cdots 18$) has been given using Taylor series expansion and hypotheses up to the third derivative of function F though only the first derivative of function F appears in these methods (for $m = 1, 2, \cdots, 18$). As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded second and third derivatives in X. In the present study we extend the applicability of these methods by using hypotheses up to the first derivative of function F and contractions. Moreover we avoid Taylor expansions and use instead Lipschitz parameters. This way we do not have to use higher order derivatives to show the convergence of these methods.

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2 Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Let $L_0 > 0, L > 0, M \geq 1$ and $\alpha \in S$ be given parameters. Let $U(v,\mu), \bar{U}(v,\mu)$ stand respectively for the open and closed balls in S with center $v \in S$ and of radius $\mu > 0$. We shall show the local convergence of method (1.2) under the following conditions ($\mathscr{C}$):

($\mathscr{C}_1$)$F : D \subset S \to S$ is a differentiable function; There exist:
($\mathscr{C}_2$)$x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(S,S)$;
($\mathscr{C}_3$)$L_0 > 0$ such that for each $x \in D$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*|;$$

Set $D_0 = D \cap U(x^*, \frac{1}{L_0})$.
($\mathscr{C}_4$)$L > 0$ such that for each $x, y \in D_0$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|;$$

($\mathscr{C}_5$)$M \geq 1$ such that for each $x \in D_0$

$$|F'(x^*)^{-1}F'(x)| \leq M;$$

($\mathscr{C}_6$)A continuous function $A : D_0 \to S$ and a continuous and non-decreasing function $\varphi^1 : [0, \rho_0) \to [0, 1)$ for some $\rho_0 \in [0, \frac{1}{L_0})$ such that for each $x \in D$

$$|F'(x^*)^{-1}(A(x) - F'(x^*))| \leq \varphi^1(|x - x^*|);$$

($\mathscr{C}_7$)A continuous and non-decreasing function $\varphi : [0, \frac{1}{L_0}) \to [0, +\infty)$ such that for each $x \in D$

$$|F'(x^*)^{-1}(\alpha A(x) - F'(x))| \leq \varphi(|x - x^*|);$$

($\mathscr{C}_8$)A minimal point $\rho \in [0, \frac{1}{L_0})$ such that

$$h_2(\rho) = 0,$$

where

$$h_2(t) = \frac{1}{2(1 - L_0 t)}(Lt + 2|1 - \alpha|M) + \frac{2M\varphi(t)}{1 - \varphi^1(t)}) - 1;$$

The following hold
($\mathscr{C}_9$)$M|1 - \alpha| < 1$;
($\mathscr{C}_{10}$)$\varphi^1(t) < 1$ for each $t \in [0, \rho)$; and
($\mathscr{C}_{11}$)$\bar{U}(x^*, r) \subseteq D$, where

$$r = \min\{r_1, \rho_0, \rho\} \tag{2.1}$$

and

$$r_1 = \frac{2(1 - |1 - \alpha|M)}{2L_0 + L}.$$

Then, we have that $g_1(r_1) = 1$,

$$0 \leq g_1(t) < 1 \text{ for each } t \in [0, r) \tag{2.2}$$

$$0 \leq g_2(t) < 1 \text{ for each } t \in [0, r) \tag{2.3}$$

$$r_1 < r_A = \frac{2}{2L_0 + L} < \frac{1}{L_0} \tag{2.4}$$

and

$$r < r_A, \tag{2.5}$$

where

$$g_1(t) = \frac{Lt + 2|1 - \alpha|M}{2(1 - L_0 t)},$$

and

$$g_2(t) = h_2(t) + 1.$$

Next, we present the local convergence analysis of method (1.2) using the preceding notations and conditions ($\mathscr{C}$).

THEOREM 21*Suppose that the conditions ($\mathscr{C}$) hold. Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold*

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{2.6}$$

and

$$|x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|. \tag{2.7}$$

Furthermore, if that there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, T)$.

Proof. We shall show estimates (2.6) and (2.7) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, ($\mathscr{C}_3$), (2.1) and (2.4) we have that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{2.8}$$

It follows from (2.8) and the Banach Lemma on invertible functions [3,4,19,20,22,23] that $F'(x_0)^{-1} \in L(S,S)$ and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0 r}. \tag{2.9}$$

Hence, y_0 is well defined by the first step of method (1.2) for $n = 0$. We can write by ($\mathscr{C}_1$) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*)(x_0 - x^*)d\theta. \tag{2.10}$$

We have that $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$. That is $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Using ($\mathscr{C}_5$) and (2.10) we get that

$$|F'(x^*)^{-1}F(x_0)| \leq M|x_0 - x^*|. \tag{2.11}$$

Then using the first substep of method (1.2) for $n=0$, ($\mathscr{C}_2$), ($\mathscr{C}_4$), (2.1), (2.2), (2.9 and (2.11), we get in turn that

$$\begin{aligned}|y_0-x^*| &\le |x_0-x^*-F'(x_0)^{-1}F(x_0)| \\ &\quad +|1-\alpha||F'(x_0)^{-1}F(x_0)| \\ &\le |F'(x_0)^{-1}F'(x^*)| \\ &\quad \times|\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*)) \\ &\quad -F'(x_0))(x_0-x^*)d\theta| \\ &\quad +|1-\alpha||F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)| \\ &\le \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)}+\frac{|1-\alpha|M|x_0-x^*|}{1-L_0|x_0-x^*|} \\ &= g_1(|x_0-x^*|)|x_0-x^*|<|x_0-x^*|<r, \qquad (2.12)\end{aligned}$$

which shows (2.6) for $n=0$ and $y_0\in U(x^*,r)$. It follows from ($\mathscr{C}_6$) that $A(x_0)^{-1}\in L(S,S)$ and

$$|A(x_0)^{-1}F'(x^*)|\le\frac{1}{1-\varphi^1(|x_0-x^*|)}. \qquad (2.13)$$

Hence, x_1 is well defined by the second substep of method (1.2) for $n=0$. Next, using the second substep of method (1.2) for $n=0$, ($\mathscr{C}_7$), (2.1), (2.2), (2.9), (2.12) and (2.13) we obtain in turn that

$$\begin{aligned}|x_1-x^*| &= |x_0-x^*-\alpha F'(x_0)^{-1}F(x_0) \\ &\quad +(\alpha F'(x_0)^{-1}-A(x_0)^{-1})F(x_0)| \\ &\le |y_0-x^*|+|F'(x_0)^{-1}F'(x^*)| \\ &\quad \times|F'(x^*)^{-1}(\alpha A(x_0)-F'(x_0))| \\ &\quad \times|A(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)| \\ &\le g_1(|x_0-x^*|)|x_0-x^*| \\ &\quad +\frac{\varphi(|x_0-x^*|)M|x_0-x^*|}{(1-L_0|x_0-x^*|)(1-\varphi^1(|x_0-x^*|))} \\ &= g_2(|x_0-x^*|)|x_0-x^*|<|x_0-x^*|<r, \qquad (2.14)\end{aligned}$$

which shows (2.7) for $n=0$ and $x_1\in U(x^*,r)$. By simply replacing x_0,y_0,x_1 by x_k,y_k,x_{k+1} in the preceding estimates we arrive at estimates (2.6) and (2.7). Then, it follows from the estimate $|x_{k+1}-x^*|<|x_k-x^*|<r$, we deduce that $x_{k+1}\in U(x^*,r)$ and $\lim_{k\to\infty}x_k=x^*$. To show the uniqueness part, let $Q=\int_0^1F'(y^*+\theta(x^*-y^*)d\theta$ for some $y^*\in\bar{U}(x^*,T)$ with $F(y^*)=0$. Using ($\mathscr{C}_3$) we get that

$$\begin{aligned}|F'(x^*)^{-1}(Q-F'(x^*))| &\le \int_0^1 L_0|y^*+\theta(x^*-y^*)-x^*|d\theta \\ &\le \int_0^1(1-\theta)|x^*-y^*|d\theta \\ &\le \frac{L_0}{2}R<1. \qquad (2.15)\end{aligned}$$

It follows from (2.15) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0=F(x^*)-F(y^*)=Q(x^*-y^*)$, we deduce that $x^*=y^*$. □

REMARK 22 *1.In view of ($\mathscr{C}_3$) and the estimate*

$$\begin{aligned}|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x)-F'(x^*))+I| \\ &\le 1+|F'(x^*)^{-1}(F'(x)-F'(x^*))| \\ &\le 1+L_0|x-x^*|\end{aligned}$$

condition ($\mathscr{C}_5$) can be dropped and M can be replaced by

$$M(t)=1+L_0t.$$

2.The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x)=P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*)=P(F(x^*))=P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x)=e^x-1$. Then, we can choose: $P(x)=x+1$.

3.The radius r_A was shown by us to be the convergence radius of Newton's method [2]-[4]

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n)\ \text{for each}\ n=0,1,2,\cdots \qquad (2.16)$$

under the conditions ($\mathscr{C}_1$)–($\mathscr{C}_4$). It follows from the definition of r that the convergence radius r of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.16). As already noted in [3,4] r_A is at least as large as the convergence ball given by Rheinboldt [26]

$$r_R=\frac{2}{3L}. \qquad (2.17)$$

In particular, for $L_0<L$ we have that

$$r_R<r$$

and

$$\frac{r_R}{r_A}\to\frac{1}{3}\ \text{as}\ \frac{L_0}{L}\to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [28].

4.It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 21 instead of the stronger conditions used in [1,5,12]-[27]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi=\ln\left(\frac{|x_{n+1}-x^*|}{|x_n-x^*|}\right)/\ln\left(\frac{|x_n-x^*|}{|x_{n-1}-x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1=\ln\left(\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}\right)/\ln\left(\frac{|x_n-x_{n-1}|}{|x_{n-1}-x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

5. *In view of* ($\mathcal{C}_6$) *and* ($\mathcal{C}_7$) *we can write*

$$\begin{aligned}&|F'(x^*)^{-1}(\alpha A(x)-F'(x))|\\ \leq\ &|\alpha||F'(x^*)^{-1}(A(x)-F'(x^*))|\\ &+|F'(x^*)^{-1}(F'(x)-F'(x^*))|+|1-\alpha|\\ \leq\ &|\alpha|\varphi^1(|x-x^*|)+L_0|x-x^*|+|1-\alpha|.\end{aligned}$$

Hence, function φ *can be defined by*

$$\varphi(t)=|\alpha|\varphi^1(t)+L_0t+|1-\alpha|. \tag{2.18}$$

However, notice that a direct estimation may lead to a better choice for φ *(see e.g. Section 3, Application 3.1 where* $\varphi^1=\varphi$*).*

6. *The proof of Theorem 21 was given in such a way that the results can be extended to hold if* $F: D\subset X\to Y$ *where* X *and* Y *are normed spaces with the norm denoted by* $\|.\|$ *(for the methods for which* $A^{-1}\in L(Y,X)$*). Simply replace* $|.|$ *by* $\|.\|$ *and the derivative of* F *by the Fréchet derivative of* F *in the proof of Theorem 21.*

3 Special cases and examples

In the next four special cases we shall choose for simplicity $\alpha=1$.

Application 31 *Let* $A_1(x_n)=\frac{1}{2}(F'(y_n)+F'(x_n))$*. Then, we have that*

$$\begin{aligned}&|F'(x^*)^{-1}(A_1(x_n)-F'(x^*))|\\ \leq\ &\frac{1}{2}(|F'(x^*)^{-1}(F'(x_n)-F'(x^*))| \qquad (3.1)\\ &+|F'(x^*)^{-1}(F'(y_n)-F'(x^*))|)\\ \leq\ &\frac{L_0}{2}(|x_n-x^*|+|y_n-x^*|)\\ \leq\ &\frac{L_0}{2}(1+g_1(|x_n-x^*|))|x_n-x^*|. \qquad (3.2)\end{aligned}$$

Hence by (3.2), we can choose

$$\varphi_3^1(t)=\frac{L_0}{2}(1+g_1(t))t. \tag{3.3}$$

Moreover, from the estimate

$$\begin{aligned}&|F'(x^*)^{-1}(A(x_n)-F'(x_n))|\\ =\ &\frac{1}{2}(|F'(x^*)^{-1}(F'(x_n)-F'(y_n))|\\ \leq\ &\frac{1}{2}(|F'(x^*)^{-1}(F'(x_n)-F'(x^*))|)\\ &+|F'(x^*)^{-1}(F'(y_n)-F'(x^*))|)\end{aligned}$$

and (3.2), we can choose

$$\varphi_3(t)=\varphi_3^1(t). \tag{3.4}$$

Application 32 *Let* $A_3(x_n)=\frac{1}{6}(F'(y_n)+4F'(\frac{x_n+y_n}{2})+F'(x_n))$*. Then, we can have the estimate*

$$\begin{aligned}&|F'(x^*)^{-1}(A_3(x_n)-F'(x^*))|\\ \leq\ &\frac{1}{6}(|F'(x^*)^{-1}(F'(y_n)-F'(x^*))|\\ &+\frac{4}{6}(|F'(x^*)^{-1}(F'(\frac{x_n+y_n}{2})-F'(x^*))|\\ &+\frac{1}{6}|F'(x^*)^{-1}(F'(y_n)-F'(x^*))|)\\ \leq\ &\frac{L_0}{6}|y_n-x^*|+\frac{4L_0}{6}\frac{1}{2}(|x_n-x^*|+|y_n-x^*|))\\ &+\frac{L_0}{6}|x_n-x^*|\\ =\ &\frac{L_0}{2}(|x_n-x^*|+|y_n-x^*|)\\ \leq\ &\frac{L_0}{2}(1+g_1(|x_n-x^*|))|x_n-x^*|.\end{aligned}$$

Hence, we can choose function φ_3^1 *as in (3.3) and by Remark 2.2, 5.*

$$\varphi_3(t)=\varphi_3^1(t)+L_0t. \tag{3.5}$$

In the next two applications we find the function g_2 *(i.e., the function* h_2*) in a more direct way.*

Application 33 *In the case of method (1.3) we get as in (2.12) and (2.14) that*

$$|y_n-x^*|\leq\frac{L|x_n-x^*|^2}{2(1-L_0|x_n-x^*|)}$$

and

$$\begin{aligned}|x_{n+1}-x^*|&\leq|y_n-x^*|+|F'(x_n)^{-1}F(y_n)|\\ &\leq\frac{L|x_n-x^*|^2}{2(1-L_0|x_n-x^*|)}+\frac{M|y_n-x^*|}{1-L_0|x_n-x^*|}.\end{aligned}$$

Therefore, we must choose

$$g_1(t)=\frac{Lt}{2(1-L_0t)},$$

$$g_2(t)=(1+\frac{M}{1-L_0t})g_1(t)$$

and

$$h_2(t)=g_2(t)-1.$$

In this case we have that

$$\rho_0=\frac{1}{L_0},\ r_1=r_A,\ \varphi_1(t)=0,$$

and

$$r=\rho=\frac{4}{4L_0+ML+L+\sqrt{(4L_0+ML+L)^2-8L_0(L+2L_0)}}.$$

Application 34 *In the case of method (1.4), since $\alpha = 1$ and $A(x) = F'(x)$, we can choose $\varphi^1(t) = L_0 t$ and $\varphi = 0$. Then, we have as in (2.12) that*

$$|y_n - x^*| \leq \frac{L|x_n - x^*|^2}{2(1 - L_0|x_n - x^*|)} = g_1(|x_n - x^*|)|x_n - x^*|$$

and

$$\begin{aligned}|x_{n+1} - x^*| &\leq \frac{L|y_n - x^*|^2}{2(1 - L_0|y_n - x^*|)} \\ &\leq \frac{Lg_1(|x_n - x^*|)^2|x_n - x^*|^2}{2(1 - L_0 g_1(|x_n - x^*|)|x_n - x^*|)} \\ &= g_2(|x_n - x^*|)|x_n - x^*|,\end{aligned}$$

where

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)},$$

$$g_2(t) = \frac{Lg_1(t)^2 t}{2(1 - L_0 g_1(t)t)},$$

$$h_2(t) = g_2(t) - 1$$

and

$$\rho = r = r_A.$$

EXAMPLE 35 *Let $D = (-\infty, +\infty)$. Define function f of D by*

$$f(x) = \sin(x). \tag{3.6}$$

Then we have for $x^* = 0$ that $L_0 = L = M = N = 1$. The parameters are given in Table 1

Table 1: Parameters of Applications A3.1, A3.2, A3.3 and A3.4

Applications/ parameters	3.1	3.2	3.3	3.4
r_1	0.6667	0.6667	0.6667	0.6667
r_A	0.6667	0.6667	0.6667	0.6667
r_R	0.6667	0.6667	0.6667	0.6667
ρ_0	0.7639	0.5000	1	0.5000
ρ	0.3790	0.7391	0.4226	0.6667
r	0.3790	0.6667	0.4226	0.6667
ξ_1	2.9980	3.1160	2.9583	9.6304
ξ	3.1592	3.2107	3.3091	10.3975

Table 1

EXAMPLE 36 *Let $D = [-1, 1]$. Define function f of D by*

$$f(x) = e^x - 1. \tag{3.7}$$

Using (3.7) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = N = e^{\frac{1}{L_0}}$. The parameters are given in Table 2

EXAMPLE 37 *Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073, M = 101.5578008$. The parameters are given in Table 3.*

Table 2: Parameters of Applications A3.1, A3.2, A3.3 and A3.4

Applications/ parameters	A3.1	A3.2	A3.3	A3.4
r_1	0.3827	0.3827	0.3827	0.3827
r_A	0.3827	0.3827	0.3827	0.3827
r_R	0.3725	0.3725	0.3725	0.3725
ρ_0	0.4415	0.3201	0.58200	0.4812
ρ	0.1966	0.0374	0.1983	0.3827
r	0.1966	0.0374	0.1983	0.3827
ξ_1	2.9894	0.0759	2.0003	3.9446
ξ	3.1416	1.0596	2.0568	4.3615

Table 2

Table 3: Parameters of Applications A3.1, A3.2, A3.3 and A3.4

Applications/ parameters	A3.1	A3.2	A3.3	A3.4
r_1	0.0045	0.0045	0.0045	0.0045
r_A	0.0045	0.0045	0.0045	0.0045
r_R	0.0045	0.0045	0.0045	0.0045
ρ_0	0.0052	0.0045	0.0068	0.0068
ρ	0.0048	0.0199	1.2804e-04	0.0045
r	0.0045	0.0199	1.2804e-04	0.0045
ξ_1	2.6092	1.4848	2.0415	1.0049
ξ	2.8152	6.7845	1.9863	0.9903

Table 3

4 Conclusion

Iterative methods for solving equations are very important, since closed form solutions are available only in special cases. The convergence order of these methods is usually determined using Taylors expansions requiring the existence of high order derivatives. Moreover, no computable radii or error bounds on $|x_n - x^*|$ are given. This way the applicability of the methods is limited. In the present study we study the local convergence analysis of two-step Newton-type methods using hypotheses only on the first derivative that actually appears in these methods. The convergence radii and the error bounds are given based on Lipschitz constants. The convergence order is found using COC or ACOC which do not require the computation of derivatives higher than one.

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