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New Extended Riemann-Liouville Fractional Derivative Operator and some of its Applications

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Abstract: The main object of this paper is to introduce new extension of the Riemann- Liouville fractional derivative operator and investigate some of its properties. Some applications for this operator such as integral transforms and generating function are obtained.

Keywords: Beta function, Riemann-Liouville fractional derivative, hypergeometric functions, integral transform, generating function.

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1 Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering (see [1,11,13,14,23]). In recent years, due to the above-mentioned motivation, certain extended fractional derivative operators associated with special functions have been actively investigated. Many authors have introduced certain extended fractional derivative operators (see, e.g., [4,10,16]).

Very recently, Al-Gonah and Mohammed [2] introduced a new extension of extended Gamma and Beta functions and studied some properties of these functions. The extended Gamma and Beta functions are defined by [2, p.259 (2.1) (2.2)]:

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \int_0^\infty t^{x-1}\, E_{\alpha,\beta}^{\gamma}\left(-t-\frac{p}{t}\right) dt, \qquad (1.1)$$

$(Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(p) > 0, Re(x) > 0)$,

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right) dt, \qquad (1.2)$$

$(Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(p) > 0, Re(x) > 0, R(y) > 0)$,

where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the generalized Mittag-Leffler function defined by [18, p.7 (1.3)]:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \qquad (1.3)$$

$(\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0)$.

It clears that

$$\Gamma(\beta) E_{1,\beta}^{\gamma}(z) = {}_1F_1(\gamma; \beta; z), \qquad (1.4a)$$

$$E_{\alpha,1}^{1}(z) = E_\alpha(z), \qquad (1.4b)$$

$$E_{1,1}^{1}(z) = e^z, \qquad (1.4c)$$

where $E_\alpha(z)$ is the classical Mittag-Leffler function given in [8].

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From equations (1.1) and (1.2), we note that

$$\Gamma_p^{(1,\beta,\gamma)}(x) = \frac{1}{\Gamma(\beta)}\Gamma_p^{(\gamma,\beta)}(x), \quad (1.5a)$$

$$B_p^{(1,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\beta)}B_p^{(\gamma,\beta)}(x,y), \quad (1.5b)$$

$$\Gamma_p^{(1,1,1)}(x) = \Gamma_p(x), \quad (1.5c)$$

$$B_p^{(1,1,1)}(x,y) = B_p(x,y), \quad (1.5d)$$

where $\Gamma_p^{(\gamma,\beta)}(x)$, $B_p^{(\gamma,\beta)}(x,y)$ and $\Gamma_p(x)$, $B_p(x,y)$ denoted the various forms of generalized Gamma and Beta functions given in (see Özergin [15]) and (see Chaudhry *et al.* [5]) respectively.

Also, we not that

$$B_p^{(\alpha,1,1)}(x,y) = B_\alpha^p(x,y), \quad (1.6)$$

where $B_\alpha^p(x,y)$ denotes the new extended Beta function given (see Shadab *et al.* [19]).

Along with, generalized Beta function (1.2), Al-Gonah and Mohammed [3]. introduced and studied a family of the following potentially useful generalized Gauss hypergeometric function defined as follows:

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (1.7)$$

$$(Re(p) \geq 0; |z| < 1, Re(c) > Re(b) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0),$$

and the generalized confluent hypergeometric function

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \sum_{n=0}^{\infty}\frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (1.8)$$

$$(Re(p) \geq 0; Re(c) > Re(b) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0).$$

We observe that

$$F_p^{(1,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta)}F_p^{(\gamma,\beta)}(a,b;c;z), \quad (1.9a)$$

$$F_p^{(\alpha,1,1)}(a,b;c;z) = F_{p,\alpha}(a,b;c;z), \quad (1.9b)$$

$$F_p^{(1,1,1)}(a,b;c;z) = F_p(a,b;c;z), \quad (1.9c)$$

$$F_0^{(\alpha,1,\gamma)}(a,b;c;z) = {}_2F_1(a,b;c;z), \quad (1.9d)$$

and

$$\Phi_p^{(1,\beta,\gamma)}(b;c;z) = \frac{1}{\Gamma(\beta)}\Phi_p^{(\gamma,\beta)}(b;c;z), \quad (1.10a)$$

$$\Phi_p^{(\alpha,1,1)}(b;c;z) = \Phi_{p,\alpha}(b;c;z), \quad (1.10b)$$

$$\Phi_p^{(1,1,1)}(b;c;z) = \Phi_p(b;c;z), \quad (1.10c)$$

$$\Phi_0^{(\alpha,1,\gamma)}(b;c;z) = {}_1F_1(b;c;z) = \Phi(b;c;z), \quad (1.10d)$$

where $F_p^{(\gamma,\beta)}(a,b;c;z)$ and $\Phi_p^{(\gamma,\beta)}(b;c;z)$ the extended Gauss hypergeometric and confluent hypergeometric functions defined by Özergin *et al.* [17], $F_{p,\alpha}(a,b;c;z)$ and $\Phi_{p,\alpha}(b;c;z)$ the extended Gauss hypergeometric and confluent hypergeometric functions defined by Shadab *et al.* [19], $F_p(a,b;c;z)$ and $\Phi_p(b;c;z)$ the extended Gauss hypergeometric and confluent hypergeometric functions defined by Chaudhry *et al.* [6], and ${}_2F_1(a,b;c;z)$ and $\Phi(b;c;z)$ the classical Gauss hypergeometric and confluent hypergeometric functions (see [9]).

Motivated by the various extensions of the fractional derivative operators which have recently been considered by many authors, here we aim to introduce a new extended Riemann-Liouville fractional derivative and operator study its properties and applications.

2 Extended Riemann-Liouville Fractional Derivative Operator

In this section, we consider the extended Riemann-Liouville type fractional derivative operator and then determine the extended fractional derivatives of some elementary functions. For this purpose, we begin by recalling the classical Riemann-Liouville fractional derivative of f(z) of order μ defined by [8]:

$$D_z^\mu\{f(z)\} = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)\,(z-t)^{-\mu-1}dt, (Re(\mu) < 0), \quad (2.1)$$

where the integration path is a line from 0 to z in the complex t-plane. When Re (μ) ≤ 0 , let m∈ℕ be the smallest integer greater than Re(μ) and so m – 1 ≤ Re(μ) < m. Then the Riemann-Liouville fractional derivative of f(z) of order μ is defined by:

$$D_z^\mu\{f(z)\} = \frac{d^m}{dz^m}D_z^{\mu-m}\{f(z)\},$$
$$= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\mu)}\int_0^z f(t)\,(z-t)^{m-\mu-1}dt\right\}. \quad (2.2)$$

The fractional integral and derivative operators involving various special functions have found significant importance and applications in various areas, for example, mathematical physics as well as mathematical analysis. In recent years, many authors have developed various extended fractional derivative formulas of Riemann-Liouville type. Here, we present some new extended Riemann-Liouville type fractional derivative formulas.

Definition 2.1. The extended Riemann-Liouville fractional derivative of f(z) of order μ is defined by:

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{f(z)\}=\frac{1}{\Gamma(-\mu)}\int_0^z f(t)\,(z-t)^{-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt, \tag{2.3}$$

$(Re(\mu)<0, Re(p)>0\ Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0)$.

When $Re(\mu)\le 0$, let $m\in\mathbb{N}$ be the smallest integer greater than μ and so $m-1\le Re(\mu)<m$. Then the extended Riemann-Liouville fractional derivative of $f(z)$ of order μ is defined by:

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{f(z)\}=\frac{d^m}{dz^m}D_{z;p}^{\alpha,\beta,\gamma,\mu-m}\{f(z)\},$$

$$=\frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\mu)}\int_0^z f(t)\,(z-t)^{m-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt\right\}. \tag{2.4}$$

Remark 2.1. The special case of equations (2.3) and (2.4) when $p=0$ and $\beta=1$ becomes the classical Riemann-Liouville fractional derivative. The special case of equations (2.3) and (2.4) when $\alpha=\beta=\gamma=1$ is seen to reduce to the known one given in [16].

Theorem 2.1. *Let $Re(\lambda)>-1$, $Re(\mu)<0$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\}=\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}z^{\lambda-\mu}. \tag{2.5}$$

Proof. Using equation (2.3) and then putting $t=uz$, we get

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\}=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda\,(1-u)^{-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \tag{2.6}$$

which on using equation (1.2), yields the desired result.

Remark 2.2. Using the following known relation [2]:

$$B_p^{(\alpha,\beta,\gamma)}(x,y)=\sum_{k=0}^{\infty}B_p^{(\alpha,\beta,\gamma)}(x+k,y+1), \tag{2.7}$$

in the R.H.S. of equation (2.5), we get the following result:

Corollary 2.1. *Let $Re(\lambda)>-1$, $Re(\mu)<0$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\}=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}B(\lambda+1,1-\mu)\,F_p^{(\alpha,\beta,\gamma)}(1,\lambda+1;2-\mu+\lambda;1). \tag{2.8}$$

Theorem 2.2. *Let $Re(\lambda)>0, Re(\xi)>0, Re(\mu)<0$ and $|z|<1$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\xi}\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_p^{(\alpha,\beta,\gamma)}(\xi,\lambda;\mu;z). \tag{2.9}$$

Proof. Applying equation (2.3) in the L.H.S. of the above equation to the function $z^{\lambda-1}(1-z)^{-\xi}$, we have

$$D_{z;p}^{\alpha,\beta,\gamma,\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\xi}\}=\frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\xi}\,(z-t)^{\mu-\lambda-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt. \tag{2.10}$$

Putting $t=uz$ in the above equation and using some simplification, we get

$$D_{z;p}^{\alpha,\beta,\gamma,\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\xi}\}=\frac{z^{\mu-1}\Gamma(\lambda)}{\Gamma(\mu)}\frac{1}{B(\lambda,\mu-\lambda)}\int_0^1 u^{\lambda-1}\,(1-u)^{\mu-\lambda-1}(1-zu)^{-\xi}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du. \tag{2.11}$$

Now using the following relation (see [3]):

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\,E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \tag{2.12}$$

in the above equation, we get the desired result.

Theorem 2.3. *Let $Re(\mu)<0$ and $|z|<1$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{e^z\}=\frac{z^{-\mu}}{\Gamma(1-\mu)}\Phi_p^{(\alpha,\beta,\gamma)}(1;1-\mu;z), \tag{2.13}$$

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{e^z\}=\frac{z^{-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}B_p^{(\alpha,\beta,\gamma)}(n+1,-\mu)\frac{z^n}{n!}, \tag{2.14}$$

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{{}_2F_1(a,b;c;z)\}=\frac{z^{-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}B_p^{(\alpha,\beta,\gamma)}(n+1,-\mu)\frac{z^n}{n!}. \tag{2.15}$$

Proof. Applying equation (2.3) in the L.H.S. of equation (2.13) to the function e^z, we have

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{e^z\} = \frac{1}{\Gamma(-\mu)}\int_0^z e^t\,(z-t)^{-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt. \quad (2.16)$$

Putting $t = uz$ in the above equation and using some simplification, we get

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{e^z\} = \frac{z^{-\mu}}{\Gamma(1-\mu)}\int_0^1 u^{1-1}\,(1-u)^{-\mu-1}e^{zu}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \quad (2.17)$$

Using the following relation (see [3]):

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}e^{zu}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \quad (2.18)$$

in the above equation, we get the desired result (2.13).

Similarly, following the same procedure leading to result (2.13), we can get the results (2.14) and (2.15) and thus the proof of Theorem 2.3 is completed.

Theorem 2.4. *Let $f(z)$ be an analytic function in the disc $|z| < p$ and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda-1}f(z)\} = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B_p^{(\alpha,\beta,\gamma)}(\lambda+n,-\mu)\,z^n, \quad (2.19)$$

provided that $Re(\lambda) > 0, Re(\mu) < 0$ and $|z| < p$.

Proof. We have

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda-1}f(z)\} = D_{z;p}^{\alpha,\beta,\gamma,\mu}\left\{z^{\lambda-1}\sum_{n=0}^{\infty} a_n z^n\right\},$$

$$= \sum_{n=0}^{\infty} a_n D_{z;p}^{\alpha,\beta,\gamma,\mu}\, z^{\lambda+n-1}. \quad (2.20)$$

Using equation (2.5) in the above equation, we obtain the desired result,

Remark 2.3. Using the following known relation [2]:

$$B_p^{(\alpha,\beta,\gamma)}(x,1-y) = \sum_{k=0}^{\infty}\frac{(y)_k}{k!}B_p^{(\alpha,\beta,\gamma)}(x+k,1), \quad (2.21)$$

in the R.H.S. of equation (2.19), we get the following result:

Corollary 2.2. *Let $f(z)$ be an analytic function in the disc $|z| < p$ and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda-1}f(z)\} = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B(\lambda+n,1)F_p^{(\alpha,\beta,\gamma)}(1-\mu,\lambda+n;\lambda+n+1;1)\,z^n. \quad (2.22)$$

Theorem 2.5. *Let $Re(p) \geq 0, Re(\lambda) > 0$ and $m-1 < Re(\mu) < m$ where $m \in \mathbb{N}$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\} = \frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,m-\mu)}{\Gamma(m-\mu)}\frac{\Gamma(\lambda+m-\mu+1)}{\Gamma(\lambda-\mu+1)}z^{\lambda-\mu}. \quad (2.23)$$

Proof. From equations (2.3) and (2.4), we obtain

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\} = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1}\,t^{\lambda}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt\right\}. \quad (2.24)$$

Setting $t = uz$ in equation (2.24), we get

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\} = \left(\frac{d^m}{dz^m}z^{m+\lambda-\mu}\right)\frac{1}{\Gamma(m-\mu)}\int_0^1 u^{\lambda}\,(1-u)^{m-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du. \quad (2.25)$$

Now using equation (1.2) and the following formula:

$$\frac{d^k}{dz^k}z^n = \frac{n!}{(n-k)!}z^{n-k}, \quad (2.26)$$

in equation (2.25), we get the desired result.

Theorem 2.6. *Let $f(z)$ be an analytic function in the disc $|z| < p$ and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^n\}. \quad (2.27)$$

Proof. Applying equation (2.4) and using the series expansion of the function $f(z)$, we have

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{f(z)\} = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)\sum_{n=0}^{\infty} a_n t^n \, dt\right\}. \quad (2.28)$$

Since the power series $\sum_{n=0}^{\infty} a_n z^n u^n$ is uniformly converges in the disk $|z| < p$ for $0 \le u \le 1$ and the integral $\int_0^z \left|u^{\lambda-1}(1-u)^{-\mu-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)\right| du$ in convergent provided that $Re(\lambda) > 0, Re(\mu) < 0$ and $Re(p) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,$ we can change the order of integration and summation, and obtain

$$D_{z;p}^{\alpha,\beta,\gamma,\mu}\{f(z)\} = \sum_{n=0}^{\infty} a_n \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right) t^n dt\right\}.$$

$$= \sum_{n=0}^{\infty} a_n D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^n\}. \quad (2.29)$$

3 Integral Transforms

In this section, we derive some integral transforms for the new extended Riemann- Liouville fractional integrals and fractional derivatives of the potential function $f(z) = z^v, v > 0$ by applying certain integral transforms (like Beta transform, Laplace transform, Mellin transform, Varma transform and Wittaker transform).

Theorem 3.1. *The following Beta transform formula holds true*:

$$B\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : l, m\right\} = \frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)} B(l+\lambda-\mu, m). \quad (3.1)$$

Proof. We know that the Beta transform is defined as (see [20]):

$$B\{f(u) : a, b\} = \int_0^1 u^{a-1}(1-u)^{b-1} f(u)\, du. \quad (3.2)$$

Then, we have

$$B\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : l, m\right\} = \int_0^1 z^{l-1}(1-z)^{m-1} D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\}\, dz. \quad (3.3)$$

Using equation (2.5) in the R.H.S. of the above equation, we obtain

$$B\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : l, m\right\} = \frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\int_0^1 z^{l+\lambda-\mu-1}(1-z)^{m-1}\, dz, \quad (3.4)$$

which on using the following relation [22]:

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad (3.5)$$

yields the desired result.

Theorem 3.2. *The following Laplace transform formula holds true*:

$$L\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : s\right\} = \frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)} \frac{\Gamma(\lambda-\mu+1)}{s^{\lambda-\mu+1}}. \quad (3.6)$$

Proof. We know that the Laplace transform of $f(z)$ is defined as [20]:

$$L\{f(z) : s\} = \int_0^{\infty} e^{-sz} f(z)\, dz. \quad (3.7)$$

Then, we have

$$L\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : s\right\} = \int_0^{\infty} e^{-sz} D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\}\, dz. \quad (3.8)$$

Using equation (2.5) in the R.H.S. of the above equation, we obtain

$$L\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : s\right\} = \frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\int_0^{\infty} e^{-sz} z^{\lambda-\mu}\, dz, \quad (3.9)$$

which on using Gamma integral in the R.H.S., yields the desired result.

Theorem 3.3. *The following Mellin transform formula holds true*:

$$\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^\lambda\} : s\right\} = \frac{\Gamma^{(\alpha,\beta,\gamma)}(s)}{\Gamma(-\mu)} z^{\lambda-\mu} B(\lambda+s+1, s-\mu). \quad (3.10)$$

Proof. Using the definition of Mellin transform [20], we get

$$\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}:s\right\}=\int_0^{\infty}p^{s-1}D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}\,dp. \quad (3.11)$$

Denoting the L.H.S. of equation (3.11) by Δ and using equation (2.3) in the R.H.S., we obtain

$$\Delta=\int_0^{\infty}p^{s-1}\left(\frac{1}{\Gamma(-\mu)}\int_0^{z}t^{\lambda}(z-t)^{-\mu-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pz^2}{t(z-t)}\right)dt\right)dp. \quad (3.12)$$

Setting $t=uz$ in equation (3.12), we get

$$\Delta=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^{1}u^{\lambda}(-u)^{-\mu-1}\left[\int_0^{\infty}p^{s-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)dp\right]du. \quad (3.13)$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region on to itself) $t=\frac{p}{u(1-u)}$ in equation (3.13), we get

$$\Delta=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^{1}u^{\lambda+s}(-u)^{-\mu+s-1}\,du\int_0^{\infty}t^{s-1}E_{\alpha,\beta}^{\gamma}(-t)\,dt, \quad (3.14)$$

which on using equation (3.5) and using equation (1.1) (for $p=0$) in the R.H.S. of the above equation, yields the desired result.

Theorem 3.4. *The following Varma transform formula holds true*:

$$V\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}:k,m;s\right\}$$
$$=s^{\mu-\lambda-1}\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\frac{\Gamma(\lambda-\mu+2m+1)\,\Gamma(\lambda-\mu+1)}{\Gamma\left(\frac{3}{2}+m+\lambda-k-\mu\right)}. \quad (3.15)$$

Proof. We know that the Varma transform of $f(z)$ is defined by [12,p.55(2.38)]:

$$V(f,k,m;s)$$
$$=\int_0^{\infty}(sz)^{m-\frac{1}{2}}\exp\left(-\frac{1}{2}sz\right)W_{k,m}(sz)f(z)\,dz,\quad (Re(s)>0), \quad (3.16)$$

where $W_{k,m}(z)$ is the Whittaker function defined by [7,p.50(1.7)]:

$$W_{k,m}(z)=\frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)}M_{k,m}(z)$$
$$+\frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)}M_{k,-m}(z), \quad (3.17)$$

where $M_{k,m}(z)$ is the other Whittaker function see [22,p.39(23)]:

$$M_{k,m}(z)=z^{m+\frac{1}{2}}e^{-\frac{z}{2}}{}_1F_1\left(\frac{1}{2}-k+m;2m+1;z\right). \quad (3.18)$$

Now applying equation (3.16) in equation (3.15), we have

$$V\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}:k,m;s\right\}$$
$$=\int_0^{\infty}(sz)^{m-\frac{1}{2}}\exp\left(-\frac{1}{2}sz\right)W_{k,m}(sz)\,D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}\,dz. \quad (3.19)$$

Using equation (2.5) in the R.H.S. of equation (3.19), we obtain

$$V\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}:k,m;s\right\}$$
$$=s^{m-\frac{1}{2}}\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\int_0^{\infty}z^{m+\lambda-\mu+\frac{1}{2}-1}\exp\left(-\frac{1}{2}sz\right)W_{k,m}(sz)\,dz. \quad (3.20)$$

Now using the following integral formula (see Mathai *et al.* [12,p.56(2.41)]):

$$\int_0^{\infty}z^{\rho-1}\exp\left(-\frac{1}{2}sz\right)W_{k,m}(sz)\,dz$$
$$=s^{-\rho}\frac{\Gamma\left(\frac{1}{2}+\rho+m\right)\Gamma\left(\frac{1}{2}+\rho-m\right)}{\Gamma(1-k+\rho)}, \quad (3.21)$$
$$\left(Re(s)>0,Re(\rho\pm m)>-\frac{1}{2}\right),$$

in the R.H.S. of equation (3.20) and after little simplification, we get the desired result.

Theorem 3.5. *The following Whittaker transform formula holds true*:

$$\int_0^{\infty}z^{q-1}e^{\frac{-\delta z}{2}}W_{\kappa,\nu}(\delta z)\,D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^{\lambda}\}\,dz$$
$$=s^{\mu-\lambda-q}\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\frac{\Gamma\left(\frac{1}{2}+\nu+q+\lambda-\mu\right)\Gamma\left(\frac{1}{2}-\nu+q+\lambda-\mu\right)}{\Gamma(1-\kappa+q+\lambda-\mu)}. \quad (3.22)$$

Proof. Denoting the L.H.S. of equation (3.22) by Δ and using equation (2.5) in the R.H.S., we obtain

$$\Delta=\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\int_0^{\infty}z^{q+\lambda-\mu-1}e^{\frac{-\delta z}{2}}W_{\kappa,\nu}(\delta z)\,dz. \quad (3.23)$$

Setting $\delta z=u$ in the above equation, we get

$$\Delta=\delta^{\mu-\lambda-q}\frac{B_p^{(\alpha,\beta,\gamma)}(\lambda+1,-\mu)}{\Gamma(-\mu)}\int_0^{\infty}u^{q+\lambda-\mu-1}e^{\frac{-u}{2}}W_{\kappa,\nu}(u)\,du. \quad (3.24)$$

Now using the following integral formula involving the Whittaker function [7, p.57 (2.24)]:

$$\int_0^\infty z^{\upsilon-1}e^{-\frac{z}{2}}W_{\kappa,\nu}(z)\,dz$$
$$=\frac{\Gamma\left(\frac{1}{2}+\nu+\upsilon\right)\Gamma\left(\frac{1}{2}-\nu+\upsilon\right)}{\Gamma(1-\kappa+\upsilon)},\left(Re(\upsilon\pm\nu)>-\frac{1}{2}\right), \qquad (3.25)$$

in the R.H.S. of equation (3.24) and after little simplification, we get the desired result.

Theorem.3.6. *Let* $Re(p)\geq 0, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(a)>0$ *and* $|z|<1$, *then*

$$\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{(1-z)^{-a}\}:s\right\}$$
$$=\frac{\Gamma^{(\alpha,\beta,\gamma)}(s)B(s+1,s-\mu)}{z^{\mu}\Gamma(-\mu)}\,{}_2F_1(a,s+1;2s-\mu+1;z). \quad (3.26)$$

Proof. Using the following binomial series expansion [22]:

$$(1-t)^{-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}t^n, \qquad (|t|<1), \qquad (3.27)$$

and definition (3.11) in the L.H.S. of equation (3.26), we have

$$\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{(1-z)^{-a}\}:s\right\}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{z^n\}:s\right\}. \quad (3.28)$$

Now using equation (3.10) in the R.H.S. of the above equation, we get

$$\mathcal{M}\left\{D_{z;p}^{\alpha,\beta,\gamma,\mu}\{(1-z)^{-a}\}:s\right\}$$
$$=\frac{\Gamma^{(\alpha,\beta,\gamma)}(s)}{\Gamma(-\mu)}z^{-\mu}\sum_{n=0}^{\infty}(a)_n\,B(n+s+1,s-u)\frac{z^n}{n!}. \qquad (3.29)$$

Using the following formula [6, p.590 (1.5)]:

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (3.30)$$
$$(|z|<1, Re(c)>Re(b)>0),$$

in the R.H.S. of equation (3.29) and after little simplification, we get the desired result.

4 Generating Function

In this section, we obtain linear generating relation for the extended hypergeometric function $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ by following the method described in [22]. We have the following theorem:

Theorem 4.1. *For the extended hypergeometric function, we have*

$$\sum_{n=0}^{\infty}\frac{(a)_n}{n!}F_p^{(\alpha,\beta,\gamma)}(a+n,b;c;z)\,t^n$$
$$=(1-t)^{-a}F_p^{(\alpha,\beta,\gamma)}\left(a,b;c;\frac{z}{1-t}\right), \qquad (4.1)$$

$(|z|\min\{1,|1-t|\}; Re(c)>Re(b)>0; Re(a), Re(\alpha), Re(\beta), Re(\gamma)>0)$.

Proof. Considering the elementary identity

$$[(1-z)-t]^{-a}=(1-t)^{-a}\left(1-\frac{z}{1-t}\right)^{-a}, \qquad (4.2)$$

and expanding the L.H.S., we have

$$\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(1-z)^{-a}\left(\frac{t}{1-z}\right)^n$$
$$=(1-t)^{-a}\left(1-\frac{z}{1-t}\right)^{-a}, \quad (|t|<|1-z|). \qquad (4.3)$$

Now, multiplying both sides of the above equality by z^{b-1} and applying the extended Riemann-Liouville fractional derivative $D_{z;p}^{\alpha,\beta,\gamma,b-c}\{f(z)\}$ on both sides, we can write

$$D_{z;p}^{\alpha,\beta,\gamma,b-c}\left\{\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(1-z)^{-a}\left(\frac{t}{1-z}\right)^n z^{b-1}\right\}.$$
$$=(1-t)^{-a}D_{z;p}^{\alpha,\beta,\gamma,b-c}\left\{z^{b-1}\left(1-\frac{z}{1-t}\right)^{-a}\right\}. \qquad (4.4)$$

Interchanging the order in the L.H.S. of the above equation, which is valid for $Re(a)>0$ and $|t|<|1-z|$, we get

$$\sum_{n=0}^{\infty}\frac{(a)_n}{n!}D_{z;p}^{\alpha,\beta,\gamma,b-c}\{z^{b-1}(1-z)^{-a-n}\}\,t^n$$
$$=(1-t)^{-a}D_{z;p}^{\alpha,\beta,\gamma,b-c}\left\{z^{b-1}\left(1-\frac{z}{1-t}\right)^{-a}\right\}, \qquad (4.5)$$

which on using result (2.9), we get the desired result.

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