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Bayesian Estimation and Prediction for Inverse Weibull Distribution under Generalized Order Statistics

Mostafa M. Mohie El-Din[1], *Mohamed S. Kotb*[1] *and Haidy A. Newer*[2,*]

[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo 11884, Egypt
[2]Department of Mathematics, Faculty of Education, Ain-Shams University, Roxi, Cairo 11757, Egypt

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Abstract: This paper deals with the problem of Bayesian prediction for a past ordered observations when the r ordered observations remaining drawn from inverse Weibull (IW) distribution based on dual generalized order statistics. The predictive survival function in the one sample case can not be obtained in closed form, so Markov Chain Monte Carlo (MCMC) samples used to compute the approximate predictive survival function. Estimation for the two parameters, reliability and hazard functions of the IW distribution are obtained in two cases: squared error loss (SEL) and asymmetric loss functions (LINEX). A real data set and simulation data are used to illustrate the theoretical results.

Keywords: Bayesian prediction, Dual generalized order statistics, Markov Chain Monte Carlo, Order statistics, Inverse Weibull distribution

1 Introduction

The idea of dual generalized order statistics (dgos) has been introduced by Burkschat et al. [1] and Bairamov and Tanil [2] as a unified approach to several models of ordered random variables. The connection between generalized order statistics (gos) was introduced by Kamps [3] and the dgos are also established. Let F be an absolutely continuous cumulative distribution function (cdf) with density function (pdf) f with parameters $n \in N$, $\widetilde{m} = (m_1, ..., m_{n-1}) \in R_{n-1}$, $k \geq 1$, be given constants such that for all $1 \leq i \leq n-1, \gamma_i = k+n-i+M_i > 0$, where $M_i = \sum_{j=i}^{n-1} m_j$. The random variables $X_i \equiv X_{i,n,m,k}$, $i = 1, \ldots, n$ are said to be dgos from an absolutely continuous distribution function if their joint density function in the form

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(F(x_j))^{m_j} f(x_j)\right)(F(x_n))^{k-1} f(x_n),$$
$$F^{-1}(0) < x_n < x_{n-1} < \ldots < x_2 < x_1 < F^{-1}(1). \quad (1)$$

The joint density function of first r dgos $X_1, \ldots, X_r$ is given by

$$f_{1,2,\ldots,r}(\mathbf{x}) = c_{r-1}\left(\prod_{j=1}^{r-1}(F(x_j))^{m_j} f(x_j)\right)(F(x_r))^{\gamma_r-1} f(x_r), \quad (2)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ and $r \leq n$.

gos, dgos and prediction problems have been discussed extensively by many authors including Aboeleneen [4], Abushal [5], Jaheen et al. [6,7], Kamps and Cramer [8], Mohie El-Din et al. [9] and Raqab [10]. Some articles on Bayesian and non-Bayesian estimation are found in Al-Hussaini and Ahmad [11], Geisser [12,13], Mohammadi et al. [14], Panaitescu et al. [15], Soliman et al. [16], Sultan et al. [17], Zellner [18] and, among others.

* Corresponding author e-mail: sfalcon8000@gmail.com

A random variable (r.v) X is said to have the IW distribution was introduced by Keller [19] if its pdf is given by

$$f(x;\alpha,\lambda)=\alpha\lambda x^{-(\alpha+1)}\exp\left(-\lambda x^{-\alpha}\right),\quad x\geq 0,\quad \lambda,\alpha>0, \tag{3}$$

and cdf is

$$F(x)=\exp\left(-\lambda x^{-\alpha}\right),\quad x\geq 0,\quad \lambda,\alpha>0. \tag{4}$$

The reliability function $R(t)$, and hazard function $H(t)$ at time t are given by

$$R(t)=1-F(t)=1-\exp\left(-\lambda t^{-\alpha}\right),\quad t>0,$$

and

$$H(t)=\frac{f(t)}{R(t)}=\frac{\alpha\lambda t^{-(\alpha+1)}}{e^{\lambda t^{-\alpha}}-1},\quad t>0, \qquad .$$

The IW distribution plays an important role in many applications, see Nelson [20]. Jiag et al. [21] have discussed some useful measures for the IW distribution. Some articles on IW distribution are found in Calabria and Pulcini [22], Maswadah [23], Dumonceaux and Antle [24], Johnson et al. [25] and Murthy et al. [26], among others.

***Loss function*:** The minimal loss occur at $\psi^*=\psi(\alpha,\lambda)$ is

$$L(\Delta)\propto\exp(c\Delta)-c\Delta-1,\quad c\neq 0, \tag{5}$$

where $\Delta=(\psi^*-\psi)$, ψ^* is an estimate of ψ. If $c>0$, the overestimation is more serious than underestimation, and vice-versa) and if c close to zero, the LINEX loss is approximately squared error loss. The posterior expectation of the LINEX loss function is defined by

$$E[L(\psi^*-\psi)]\propto\exp(c\psi^*)E_{\psi}[\exp(-c\psi)]-c(\psi^*-E_{\psi}(\psi))-1, \tag{6}$$

where $E_{\psi}(.)$ is the posterior expectation of the posterior density of ψ. The Bayes estimator ψ^*_{BL} under the LINEX loss function can be obtained by minimizing (6) to be

$$\psi^*_{BL}=-\frac{1}{c}\ln\left\{E_{\psi}[\exp(-c\psi)]\right\}, \tag{7}$$

where the expectation $E_{\psi}[\exp(-c\psi)]$ exists and finite.

2 Bayes Prediction

Let $X_r<X_{r-1}<\ldots<X_1$ be the ordered observations remaining drawn from IW distribution based on dgos when the $(n-r)$ smallest observations have been censored. In the following, we present Bayesian prediction in one sample scheme, when the two parameters α and λ are unknown.

2.1 Prior and posterior distributions

We now take up the problem of the Bayes prediction interval, for the past observation $X_n<X_{n-1}<\ldots<X_{r+1}$. Accordingly we find that the likelihood function of the observed data from the IW distribution is

$$L(\alpha,\lambda|\mathbf{x})\propto(\alpha\lambda)^r\left(\prod_{j=1}^{r}x_j\right)^{-(\alpha+1)}\exp\left(-\lambda\xi(\alpha)\right), \tag{8}$$

where $\xi(\alpha)=\sum_{j=1}^{r-1}(m_j+1)x_j^{-\alpha}+\gamma_r x_r^{-\alpha}$.

Under the assumption that both parameters λ and α are unknown, a bivariate prior which was suggested by Al-Hussaini and Jaheen [27] is given by

$$\begin{aligned}\pi(\lambda,\alpha;\delta)&=\pi_1(\lambda)\pi_2(\alpha|\lambda)\\&\propto\lambda^{a+c-1}\alpha^{c-1}\exp\left(-\lambda\left[b+\alpha d\right]\right),\quad a,b,c,d>0,\end{aligned} \tag{9}$$

where $\delta=(a,b,c,d)$ and $\pi(\lambda,\alpha;\delta)$ is composed of $\pi_1(\lambda)$ as a gamma distribution with parameters a and b, and conditional prior $\pi_2(\alpha|\lambda)$ as a gamma distribution with parameters c and $d\lambda$.

Using (8) and (9), the posterior density of λ and α is

$$\pi^*(\alpha,\lambda|\mathbf{x}) = \frac{L(\alpha,\lambda|\mathbf{x})\pi(\lambda,\alpha;\delta)}{\int_0^\infty\int_0^\infty L(\alpha,\lambda|\mathbf{x})\pi(\lambda,\alpha;\delta)}$$
$$= R^{-1}\lambda^a(\lambda\alpha)^{r+c-1}\exp\left(-\lambda\left[\xi(\alpha)+b\right]-\alpha\left[\sum_{j=1}^r \ln x_j + d\lambda\right]\right), \tag{10}$$

where

$$R=\int_0^\infty \frac{\Gamma(r+a+c)\alpha^{r+c-1}}{[\xi(\alpha)+d\alpha+b]^{r+a+c}}\exp\left(-\alpha\sum_{j=1}^r \ln x_j\right)d\alpha. \tag{11}$$

2.2 Bayesian prediction intervals

Let $Y_s = X_{r+s}$, $s=1,2,\dots,n-r$ denote the past ordered observations. The conditional density function of Y_s given X_r is given by

$$f_{s|r,n,m,k}(y_s|x_r) = \frac{c_{r+s-1}}{(s-1)!c_{r-1}}(F(x_r))^{m-\gamma_r+1}(F(y_s))^{\gamma_{r+s}-1}$$
$$\times\left(h_m(F(y_s))-h_m(F(x_r))\right)^{s-1}f(y_s),\quad y_s<x_r, \tag{12}$$

where $g_m(z)=h_m(z)-h_m(1)$, $0<z<1$,

$$h_m(z)=\begin{cases}-z^{m+1}/(m+1), & m\neq-1,\\ -\ln z, & m=-1.\end{cases} \tag{13}$$

When $m\neq -1$, using (3) and (4) in (12), we obtain

$$f(y_s|\alpha,\lambda,x_r) = \frac{c_{r+s-1}}{(s-1)!c_{r-1}}\exp\left(-\lambda\left[(m-\gamma_r+1)x_r^{-\alpha}+\gamma_{r+s}y_s^{-\alpha}\right]\right)$$
$$\times\left(\frac{e^{-\lambda(m+1)x_r^{-\alpha}}}{m+1}-\frac{e^{-\lambda(m+1)y_s^{-\alpha}}}{m+1}\right)^{s-1}\lambda\alpha y_s^{-(\alpha+1)}$$
$$= c_{r,s}\sum_{i=0}^{s-1}\frac{(-1)^i\lambda\alpha}{(s-i-1)!i!}\exp\left(-\lambda\gamma_{r+s-i}\left(y_s^{-\alpha}-x_r^{-\alpha}\right)\right)y_s^{-(\alpha+1)}, \tag{14}$$

where $c_{r,s}=\left(\prod_{i=r+1}^s\gamma_i\right)/(m+1)^{s-1}$ and

$$\left(e^{-\lambda(m+1)x_r^{-\alpha}}-e^{-\lambda(m+1)y_s^{-\alpha}}\right)^{s-1}=\sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i e^{-\lambda(m+1)(s-i-1)x_r^{-\alpha}}e^{-\lambda(m+1)iy_s^{-\alpha}}.$$

Using (10) and (14), the predictive density function of Y_s given X_r dgos is given by

$$f(y_s|x_r) = \int_0^\infty\int_0^\infty f(y_s|\alpha,\lambda,x_r)\pi^*(\alpha,\lambda|\mathbf{x})\,\mathrm{d}\lambda\,\mathrm{d}\alpha$$
$$=\frac{c_{r,s}}{R}\sum_{i=0}^{s-1}\frac{(-1)^i}{(s-i-1)!i!}\int_0^\infty\int_0^\infty\lambda^a(\lambda\alpha)^{r+c}y_s^{-(\alpha+1)}$$
$$\times\exp\left(-\lambda\left[\phi(\alpha)+\gamma_{r+s-i}y_s^{-\alpha}+b\right]-\alpha\left[\sum_{j=1}^r\ln x_j+d\lambda\right]\right)d\alpha d\lambda, \tag{15}$$

where $\phi(\alpha)=\xi(\alpha)-\gamma_{r+s-j}x_r^{-\alpha}$.

Hence, the predictive survival function for the sth past dgos is given by

$$P[Y_s>\nu|x_r] = \int_\nu^{y_r} f(u_s|y_r)\,\mathrm{d}u_s = 1-\int_0^\nu f(u_s|y_r)\,\mathrm{d}u_s$$
$$=1-\frac{c_{r,s}}{R}\sum_{i=0}^{s-1}\frac{(-1)^i}{(s-i-1)!i!\gamma_{r+s-i}}\int_0^\infty\int_0^\infty\lambda^a(\lambda\alpha)^{r+c-1}$$
$$\times\exp\left(-\lambda\left[\phi(\alpha)+b\right]-\alpha\left[\sum_{j=1}^r\ln x_j+d\lambda\right]\right)$$
$$\times\exp\left(-\lambda\gamma_{r+s-i}\nu^{-\alpha}\right)d\alpha d\lambda. \tag{16}$$

When $m_1 = \ldots = m_{n-1} = m = -1$ and $k = 1$, we have

$$f(y_s|\alpha,\lambda,x_r) = \frac{\alpha\lambda^s}{(s-1)!} y_s^{-(\alpha+1)} e^{-\lambda(y_s^{-\alpha}-x_r^{-\alpha})}(y_s^{-\alpha}-x_r^{-\alpha})^{s-1}$$
$$= \sum_{i=0}^{s-1} \frac{(-1)^i\lambda^s\alpha}{(s-i-1)!i!} \exp\left(-\lambda\left(y_s^{-\alpha}-x_r^{-\alpha}\right)\right) y_s^{-\alpha(s-i)-1} x_r^{-\alpha i}, \tag{17}$$

where $(y_s^{-\alpha}-x_r^{-\alpha})^{s-1} = \sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i x_r^{-\alpha i} y_s^{-\alpha(s-i-1)}$.

Using (10) and (17), the predictive density function of Y_s given X_r dgos is given by

$$f(y_s|x_r) = \int_0^\infty\int_0^\infty f(y_s|\alpha,\lambda,x_r)\pi^*(\alpha,\lambda|\mathbf{x})\,\mathrm{d}\lambda\,\mathrm{d}\alpha$$
$$= \sum_{i=0}^{s-1}\frac{(-1)^i}{R(s-i-1)!i!}\int_0^\infty\int_0^\infty \lambda^{a+s-1}(\lambda\alpha)^{r+c} y_s^{-\alpha(s-i)-1} x_r^{-\alpha i}$$
$$\times \exp\left(-\lambda\left[y_s^{-\alpha}+b\right]-\alpha\left[\sum_{j=1}^r \ln x_j + d\lambda\right]\right) d\alpha\, d\lambda. \tag{18}$$

Hence, the predictive survival function for the sth past dgos is given by

$$P[Y_s > \nu|x_r] = 1-\frac{1}{R}\sum_{i=0}^{s-1}\frac{(-1)^i}{(s-i-1)!i!}\int_0^\infty\int_0^\infty \lambda^{a+s-1}(\lambda\alpha)^{r+c}x^{-i\alpha}$$
$$\times \exp\left(-\lambda b-\alpha\left[\sum_{j=1}^r \ln x_j + d\lambda\right]\right)\psi(\nu,\alpha)\,d\lambda\, d\alpha, \tag{19}$$

where $\psi(\nu,\alpha) = \int_{\nu^{-\alpha}}^\infty y_s^{s-i-1}\exp(-\lambda y_s)\,dy_s$.

It does not seem to be possible to compute the integration in (16) and (19) analytically, so we using the MCMC technique to compute this integration and constructing the Bayesian prediction interval. We use the following Algorithm 1 to generate a Gibbs sample from the posterior density function (10) of λ and α:

Algorithm 1: we use the following procedure:

1.Start with an initial values α_0.
2.Set $i = 1$.
3.Generate λ_i from Gamma$(r+a+c, \xi(\alpha_{i-1})+b)$.
4.Generate α_i from Gamma$\left(r+c, d\lambda_i + \sum_{j=1}^r \ln x_j\right)$.
5.Compute λ_i and α_i.
6.Set $i = i+1$.
7.If $i \le N$ then go to step 3 else stop end if.

By using Algorithm 1, (16) and (19), the lower and upper $100\tau\%$ prediction bounds for Y_s can be obtained by solving the following equation with respect to ν

$$1-\frac{c_{r,s}}{R_1}\sum_{i=0}^{s-1}\frac{(-1)^i\gamma_{r+s-i}^{-1}}{(s-i-1)!i!N}\sum_{\ell=1}^N \lambda_\ell^a(\lambda_\ell\alpha_\ell)^{r+c-1}\exp\left(-\lambda_\ell\gamma_{r+s-i}\nu^{-\alpha_\ell}\right)$$
$$\times \exp\left(-\lambda_\ell\left[\phi(\alpha_\ell)+b\right]-\alpha_\ell\left[\sum_{j=1}^r \ln x_j + d\lambda_\ell\right]\right) = \begin{cases}(1+\tau)/2, \\ (1-\tau)/2.\end{cases}, \tag{20}$$

where

$$R_1 = \sum_{\ell=1}^N \frac{\Gamma(r+a+c)\alpha^{r+c-1}}{[\xi(\alpha_\ell)+d\alpha_\ell+b]^{r+a+c}}\exp\left(-\alpha_\ell\sum_{j=1}^r \ln x_j\right),$$

and for record values

$$1-\frac{1}{R_1}\sum_{i=0}^{s-1}\frac{(-1)^i}{(s-i-1)!i!N}\sum_{\ell=1}^{N}\lambda_\ell^{a+s}(\lambda_\ell\alpha_\ell)^{r+c-1}x^{-i\alpha_\ell}$$
$$\times\exp\left(-\lambda_\ell b-\alpha_\ell\left[\sum_{j=1}^{r}\ln x_j+d\lambda_\ell\right]\right)\psi(\nu,\alpha_\ell)=\begin{cases}(1+\tau)/2,\\(1-\tau)/2.\end{cases}. \quad (21)$$

3 Bayes Estimation

In this section, Bayesian estimators under SEL and LINEX functions for the parameters, $R(t)$ and $H(t)$ of IW distribution are derived based on dgos. Suppose that $X_1,X_2,\ldots,X_n$ are $n(>1)$ dgos from the pdf (3). Using (3) and (4) in (2), the likelihood function from IW distribution is given by

$$L(\alpha,\lambda|\underline{\mathbf{x}})=k\left(\prod_{j=1}^{n-1}\gamma_j\right)(\alpha\lambda)^n\prod_{j=1}^{n}x_j^{-(\alpha+1)}\exp\{-\lambda z(\alpha)\}, \quad (22)$$

where $\underline{\mathbf{x}}=(x_1,x_2,\ldots,x_n)$ and $z(\alpha)=kx_n^{-\alpha}+\sum_{j=1}^{n-1}(m_j+1)x_j^{-\alpha}$.

The joint posterior density function of α and λ can be written as

$$\pi^*(\alpha,\lambda|\underline{\mathbf{x}})=A_{1,0}^{-1}\lambda^a(\lambda\alpha)^{n+c-1}\exp\left(-\lambda[z(\alpha)+b]-\alpha\left[\sum_{j=1}^{n}\ln x_j+d\lambda\right]\right), \quad (23)$$

where $A_{1,0}=A_{1,0}^{0,0}$ and

$$A_{\ell_1,\ell_2}^{\nu_1,\nu_2}=\int_0^\infty\frac{\Gamma(n+a+c)\alpha^{n+c-\ell_1}}{(z(\alpha)+d\alpha+b+\nu_1)^{n+a+c+\ell_2}}\exp\left(-\alpha\left(\sum_{j=1}^{n}\ln x_j+\nu_2\right)\right)\mathrm{d}\alpha. \quad (24)$$

Therefore, the CBE of α and λ under a SEL function is

$$\begin{aligned}\widehat{\lambda}_{BS}&=E(\lambda|\underline{\mathbf{x}})\\&=\int_0^\infty\int_0^\infty\lambda\pi^*(\alpha,\lambda|\underline{\mathbf{x}})\mathrm{d}\lambda\mathrm{d}\alpha\\&=A_{1,0}^{-1}\int_0^\infty\frac{\Gamma(n+a+c+1)\alpha^{n+c-1}}{(z(\alpha)+d\alpha+b)^{n+a+c+1}}\exp\left(-\alpha\left(\sum_{j=1}^{n}\ln x_j\right)\right)\mathrm{d}\alpha\\&=\frac{A_{1,1}}{A_{1,0}}\end{aligned} \quad (25)$$

and

$$\begin{aligned}\widehat{\alpha}_{BS}&=A_{1,0}^{-1}\int_0^\infty\frac{\Gamma(n+a+c)\alpha^{n+c}}{(z(\alpha)+d\alpha+b)^{n+a+c}}\exp\left(-\alpha\left(\sum_{j=1}^{n}\ln x_j\right)\right)\mathrm{d}\alpha\\&=\frac{A_{0,0}}{A_{1,0}}.\end{aligned} \quad (26)$$

The Bayes estimators of $R(t)$ and $H(t)$ under a SEL function are given, respectively, by

$$\begin{aligned}\widehat{R}_{BS}(t)&=1-A_{1,0}^{-1}\int_0^\infty\frac{\Gamma(n+a+c)\alpha^{n+c-1}}{(z(\alpha)+d\alpha+b+t^{-\alpha})^{n+a+c}}\exp\left(-\alpha\left(\sum_{j=1}^{n}\ln x_j\right)\right)\mathrm{d}\alpha\\&=1-\frac{A_{1,0}^{t^{-\alpha},0}}{A_{1,0}}\end{aligned} \quad (27)$$

and

$$\widehat{H}_{BS}(t) = \sum_{i=0}^{\infty} t^{-1} A_{1,0}^{-1} \int_0^{\infty} \frac{\Gamma(n+a+c+1)\alpha^{n+c}}{(z(\alpha)+d\alpha+b+it^{-\alpha})^{n+a+c}} \exp\left(-\alpha\left(\sum_{j=1}^{n} \ln x_j + \ln(t)\right)\right) d\alpha$$
$$= \sum_{i=0}^{\infty} \frac{A_{0,1}^{it^{-\alpha},\ln(t)}}{tA_{1,0}}, \tag{28}$$

and the CBE of α and λ under the LINEX loss function is

$$\widehat{\lambda}_{BL} = \frac{-1}{c_1} \ln\left(E\left(e^{-c\lambda}|\underline{\mathbf{x}}\right)\right)$$
$$= \frac{-1}{c_1} \ln\left(A_{1,0}^{-1} \int_0^{\infty} \frac{\Gamma(n+a+c)\alpha^{n+c-1}}{(z(\alpha)+d\alpha+b+c_1)^{n+a+c}} \exp\left(-\alpha\left(\sum_{j=1}^{n} \ln x_j\right)\right) d\alpha\right)$$
$$= \frac{-1}{c_1} \ln\left(\frac{A_{1,0}^{c_1,0}}{A_{1,0}}\right), \tag{29}$$

and

$$\widehat{\alpha}_{BL} = \frac{-1}{c_1} \ln\left(A_{1,0}^{-1} \int_0^{\infty} \frac{\Gamma(n+a+c)\alpha^{n+c-1}}{(z(\alpha)+d\alpha+b)^{n+a+c}} \exp\left(-\alpha\left(\sum_{j=1}^{n} \ln x_j + c_1\right)\right) d\alpha\right)$$
$$= \frac{-1}{c_1} \ln\left(\frac{A_{1,0}^{0,c_1}}{A_{1,0}}\right), \tag{30}$$

Similarly, the Bayes estimators of $R(t)$ and $H(t)$ under the LINEX loss function are given, respectively, by

$$\widehat{R}_{BL}(t) = \frac{-1}{c_1} \ln\left(e^{-c} A_{1,0}^{-1} \int_0^{\infty} \sum_{i=0}^{\infty} \frac{c^i}{i!} \frac{\Gamma(n+a+c)\alpha^{n+c-1}}{(z(\alpha)+d\alpha+b+it^{-\alpha})^{n+a+c}} \exp\left(-\alpha\left(\sum_{j=1}^{n} \ln x_j\right)\right) d\alpha\right)$$
$$= \frac{-1}{c_1} \ln\left(\sum_{i=0}^{\infty} \frac{c^i}{i!} \frac{A_{1,0}^{it^{-\alpha},0}}{e^c A_{1,0}}\right), \tag{31}$$

and

$$\widehat{H}_{BL}(t) = \frac{-1}{c_1} \ln[A_{1,0}^{-1} \int_0^{\infty} \sum_{\nu=0}^{\infty} \frac{(-c)^{\nu}}{\nu! t^{\nu}} \sum_{i=0}^{\infty} \binom{\nu+i-1}{i} \frac{\Gamma(n+a+c+\nu)\alpha^{n+c+\nu-1}}{(z(\alpha)+d\alpha+b+it^{-\alpha})^{n+a+c}}$$
$$\times \exp\left(-\alpha\left(\sum_{j=1}^{n} \ln x_j + \nu \ln(t)\right)\right) d\alpha]$$
$$= \frac{-1}{c_1} \ln\left(\sum_{\nu=0}^{\infty} \frac{(-c)^{\nu}}{\nu! t^{\nu}} \sum_{i=0}^{\infty} \binom{\nu+i-1}{i} \frac{A_{\nu-1,\nu}^{it^{-\alpha},\nu\ln(t)}}{A_{1,0}}\right). \tag{32}$$

It is impossible to compute analytically (25)-(32) in this case an alternative method is applied which is MCMC algorithm.

Algorithm 2: we use the following procedure:

1.Start with an initial values α_0.
2.Set $i = 1$.
3.Generate λ_i from Gamma$(n+a+c, z(\alpha_{i-1})+b)$.
4.Generate α_i from Gamma$\left(n+c, d\lambda_i + \sum_{j=1}^{n} \ln x_j\right)$.
5.Compute λ_i and α_i.
6.Set $i = i+1$.
7.If $i \leq N$ then go to step 3 else stop end if.

From Algorithm 2, we obtain $\alpha_1,\alpha_2,\ldots,\alpha_N$. Then the approximate value of (24) is given by

$$A^{\nu_1,\nu_2}_{\ell_1,\ell_2}=\frac{1}{N}\sum_{i=1}^{N}\frac{\Gamma(n+a+c)\alpha_i^{n+c-\ell_1}}{(z(\alpha_i)+d\alpha_i+b+\nu_1)^{n+a+c+\ell_2}}\exp\left(-\alpha_i\left(\sum_{j=1}^{n}\ln x_j+\nu_2\right)\right). \quad (33)$$

By choosing suitable parameters in our results, we can obtain our results in several models of ordered r.v's. In the following numerical example, we take lower ordinary order statistics and lower record values as a special cases of dgos.

4 Illustrative example

To illustrate our results presented in the preceding sections, we present a numerical study for the IW distribution when both parameters are unknown.

Example 1. To illustrate the prediction results for the IW distribution, we use a numerical technique according to the following steps:

1. By select the parameter values $(a,b,c,d)=(2,1.5,4,1.070)$, we generate the values $\lambda=1.338$ from $\pi_1(\lambda)$ and $\alpha=3.742$ from $\pi_2(\alpha|\lambda)$.
2. We generate $n=19$ OS from the IW pdf in (3), using the transformation $X_i=\left(\frac{-1}{\lambda}\ln(U_i)\right)^{-1/\alpha}$ where U_i from $U(0,1)$. The generated sample (*) of the IW is

5.108	4.441	2.369	2.176	1.873	1.451	1.387	1.326	1.308	1.237
1.200	1.154	1.082	0.990	0.960	0.936	0.875	0.788	0.680	

Therefore, eight lower record values as follows
2.369, 1.451, 1.326, 1.200, 0.990, 0.960, 0.875, 0.680
3. Using Algorithm 1 to obtain $(\lambda_1,\alpha_1),(\lambda_2,\alpha_2),\ldots,(\lambda_N,\alpha_N)$ with $N=15000$.
4. Using our results in (20) and (21), the lower and upper 95% one-sample Bayesian prediction intervals for X_{r+s}, $s=1,2,3$ $(r=16)$ and for the next record values $X_{L(n+s)}$, $s=1,2,3$ $(n=8)$ are obtained and presented in Table 1.
5. By using Algorithm 2 of Gibbs sampling procedure and (33), the different Bayes estimates of λ, α, $R(t)$ and $H(t)$ are computed through (25)-(32), the results are displayed in Table 2.

Table 1: The lower, the upper and the width of the 95% prediction intervals.

	order statistics			Recorded values		
s	Lower	Upper	Width	Lower	Upper	Width
1	0.762	0.951	0.189	0.548	0.679	0.131
2	0.664	0.980	0.316	0.499	0.690	0.191
3	0.545	0.909	0.364	0.461	0.660	0.199

Table 2: Estimates of α, λ, $R(t)$ and $H(t)$ with $t=1$.

	order statistics			Recorded values		
	$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{BS}$	$(\cdot)_{BL}$	
Par.		$c=-5$	$c=20$		$c=-0.1$	$c=1$
λ	0.338	0.338	0.337	4.133	4.160	3.817
α	2.914	4.081	2.602	0.902	0.908	0.900
$R(t)$	0.287	0.287	0.286	0.984	0.984	0.983
$H(t)$	2.480	2.543	1.935	0.061	0.061	0.060

Example 2. The real data set is considered to illustrate the prediction and the estimation techniques. We consider the data $(**)$ given by Nelson consisting of time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are

72.89	36.71	33.91	32.52	31.75	12.06	8.27	8.01	7.35	6.50
4.85	4.67	4.15	3.16	2.78	1.31	0.96	0.78	0.19	

Using Kaplan-Meier [28] estimator (KME) for detecting that IW distribution is more suitable for the real data set to the hazard function behavior. Also, it is known as the product-limit estimator, of a survival function (SF) is defined as

$$\overline{F}_n(t)=\prod_{i:t_{(i)}\le t}\left\{1-\frac{\delta_i}{n-i+1}\right\},\quad t>0, \quad (34)$$

where $t_{(i)}s$ is the ordered survival times and $\delta_{(i)}s$ is their corresponding censoring indicators. Figure 1 show that the empirical and P-P plot of KME versus fitted survival function of IW distribution provide a reasonable fit to the real data.

Now, we suppose that the following eight lower record values from real data set

$$72.89, 33.91, 31.75, 12.06, 7.35, 4.67, 3.16, 0.19$$

By using the same values of hyperparameters in Example 1, the lower and upper 95% Bayesian prediction intervals for the next record values $X_{L(9)}$ are 0.048 and 0.188, respectively. Based on the complete sample $(**)$ the Bayes estimates $((\cdot)_{BS}, (\cdot)_{BL})$ of λ, α, $R(t)$ and $H(t)$ are computed and displayed in Table 3.

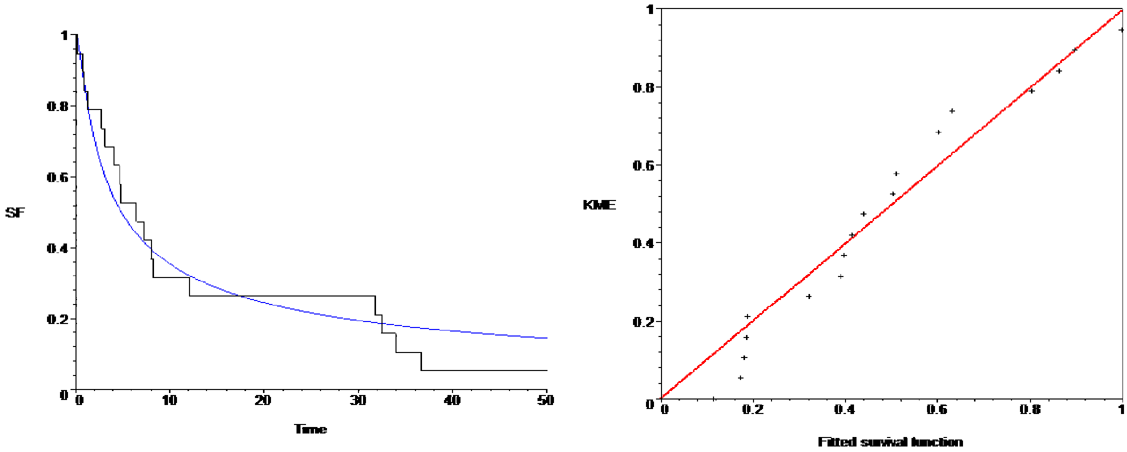

Fig. 1: The empirical and P-P plot of KME versus fitted survival function of IW distribution.

Table 3: Estimates of α, λ, $R(t)$ and $H(t)$ with $t = 1$.

	order statistics			Recorded values		
	$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{BS}$	$(\cdot)_{BL}$	
Par.		$c = -5$	$c = 20$		$c = -0.1$	$c = 1$
λ	0.913	0.974	0.677	4.626	5.167	4.597
α	0.670	0.711	0.502	0.404	0.405	0.402
$R(t)$	0.599	0.599	0.598	0.990	0.990	0.990
$H(t)$	0.410	0.410	0.317	0.019	0.019	0.019

5 Conclusion

In this paper, Bayesian prediction and Bayesian estimation for the two parameters, reliability and hazard functions of the IW distribution based on dgos are considered. A Gibbs sampling procedure is used to draw MCMC samples which are then used to compute the approximate the Bayes prediction and estimation. From Table 1, we notice that the lengths of the prediction intervals are increasing in s. Table 2 show that the Bayes estimates related to LINEX have the smallest $|\vartheta - \hat{\vartheta}|$, for each estimate of the vector of $\vartheta = (\lambda, \alpha)$, $R(t)$ and $H(t)$, as compared with quadratic Bayes estimates.

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