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Symmetry Solutions and Conserved Quantities of an Extended (1+3)-Dimensional Kadomtsev-Petviashvili-Like Equation

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Abstract: In this paper, we study an extended (1+3)-dimensional Kadomtsev-Petviashvili-like equation. Lie symmetry reductions of the equation were performed and direct integration technique was adopted. Kudryashov's approach was utilized to generate a closed-form solution of the equation. Besides, we employ power series approach to secure a solution of the underlying equation. Solutions found for the underlying equation include hyperbolic function as well as series solutions of the equation. In conclusion, we construct conserved quantities of the aforementioned equation by invoking Ibragimov's theorem.

Keywords: Extended (1+3)-dimensional Kadomtsev-Petviashvili-like equation; Lie point symmetries; exact solutions; power series; Kudryashov's method; conserved quantities.

1 Introduction

Investigation of nonlinear wave phenomenon has made a huge impact in many natural sciences comprising biology, mathematics, and precisely in diverse branches of physics, including condensed matter physics, nonlinear optics, chemical physics, plasma physics, solid-state physics as well as fluid dynamics. The study of nonlinear partial differential equations is a very viable and active area of research with regards to theoretical physics, applied mathematics with various engineering fields [3]- [12]. In particular, considerable interest in searching for exact travelling wave solutions of the differential equations that delineate some significant physical as well as dynamic processes have been one of the main concerns and focal points to researchers.

Nonetheless, there is no general well-structured theory that can be used to obtain exact solutions of nonlinear partial differential equations. Nonetheless, in later times, there has been the emergence of various sound and efficient techniques to find exact solutions to nonlinear partial differential equations. Some of these techniques are bifurcation technique [13], Adomian decomposition approach [14], homotopy perturbation technique [15], mapping and extended mapping technique [16], extended homoclinic test approach [17], tanh-coth approach [18], $\exp(-\Phi(\eta))$-expansion technique [19], Painlévé expansion [20], Cole-Hopf transformation approach [21], Bäcklund transformation [12], rational expansion method [22], $F-$expansion technique [23], tan-cot method [24], extended simplest equation method [25], Hirota technique [26], Lie symmetry analysis [27, 28], the $(G'/G)-$expansion method [29], Darboux transformation [30], sine-Gordon equation expansion technique [31], Kudryashov's method [32], exponential function technique [33], tanh-function technique [34] and so on.

In the field of mathematical physics, Kadomtsev-Petviashvili (KP) equation which was named after Boris B. Kadomtsev alongside his co-researcher Vladimir I. Petviashvili, is a partial differential equation that expounds nonlinear wave motion. Moreover, it is known that the study of nonlinear waves is very significant in the investigation of diverse nonlinear phenomena. Many researchers have investigated a variety

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of KP equations. For instance, the well-known (1+2)-dimensional KP equation expressed as [35, 36]

$$u_{tx} - 6u_x^2 - 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0, \quad (1)$$

which is an equation revealed to be a nonlinear partial differential equation with its existence in two spatial as well as one temporal coordinate has been studied. The KP equation (1) recounts the evolution of nonlinear alongside long waves that possess small-amplitude which is slowly dependent on the involved transverse coordinate. Kadomtsev and Petviashvili [35] relaxed the restriction that the waves be strictly one dimensional, to derive the completely integrable KP equation in the structure of (1). In addition, KP equation (1) delineates the evolution of shallow-water waves that are presented as quasi-one-dimensional, especially when effects of the viscosity, as well as the surface tension, are found to be negligible.

Kadomtsev-Petviashvili equation (1) has been engaged in modelling various natural happenstances. For instance, in the investigation of water waves, equation (1) is revealed in the recounting of a tsunami wave that is travelling in a zone which is non-homogeneous and appears at the bottom of the ocean [37]. Besides, it also emerges in the investigation of nonlinear ion-acoustic waves found in a magnetized dusty plasma [38]. Over the past few years, different types of research outcomes for 3D-KP equation (1) have been secured. Consequently, travelling wave solutions in [39, 40], rogue wave, as well as a pair of resonance stripe solitons in [41] have been achieved for the 3D-KP equation (1). The author in [42] investigated symmetry reductions along with conserved quantities of the equation. The Hirota bilinear structure of the 3D-KP equation (1) was utilized to gain mixed lump-kink solutions with the aid of Maple software in [43]. Furthermore, by using the exponential function as well as positive quadratic function, line soliton pairs together with rational lump solutions of the 3D-KP equation (1) were established by the authors in [44].

Our work investigates the (1+3)-dimensional extended Kadomtsev-Petviashvili-like equation ((1+3)-D extKPle) [45]

$$\left(u_t + \frac{3}{2}u_x^2 + \frac{3}{8}u^4 + \frac{3}{2}u^2u_x\right)_x + u_{yy} + u_{zz} = 0, \quad (2)$$

which was introduced as an extended Kadomtsev-Petviashvili-like equation via the engagement of a generalized bilinear differential equation of KP type of equation (1). The authors employed the bilinear representations in Hirota sense together with a transformation given, respectively, as

$$(D_xD_t + D_x^4 + D_y^2)f \cdot f = 0 \text{ and } u = 2\left[\ln f(t,x,y)\right]_{xx}, \quad (3)$$

for the derivation of the equation. Moreover, based on generalized bilinear equation as well as Bell polynomial theories [46], the authors constructed eighteen classes of rational solutions to (1+3)-D extKPle (2) with symbolic computation.

This study explicitly examines (1+3)-D extKPle (2) with the use of Lie symmetry technique in conjunction with some other standard methods to achieve new exact solutions of the equation and its conservation laws.

The paper is outlined as follows. Section 2 highlights the systematic way of carrying out Lie group analysis of (1+3)-D extKPle (2) from which Lie point symmetries of the equation are computed. In addition symmetry reductions of the underlying equation shall be performed. Moreover, Section 3 presents the general solution of (2) with the aid of Kudryashov's technique. We also secure the power series solution of (2). Furthermore, Section 4 presents the conservation laws of the underlying equation by utilizing Ibragimov's theorem for conserved vectors. Concluding remarks follow in Section 5.

2 Symmetry analysis

This section presents the computation of the Lie symmetries of (1+3)-D extKPle (2) which shall be used in the construction of exact solutions of the equation.

2.1 Lie point symmetries of (2)

The symmetry group of (1+3)-D extKPle (2) will be achieved by the use of vector field structured as

$$Q = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \xi^4\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u},$$

with the coefficient functions $(\xi^1, \xi^2, \xi^3, \xi^4, \eta)$, all depending on (t,x,y,z,u). Vector Q generates all the Lie point symmetry of (2) if the condition

$$Q^{[4]}\left[\left(u_t + \frac{3}{2}u_x^2 + \frac{3}{8}u^4 + \frac{3}{2}u^2u_x\right)_x + u_{yy} + u_{zz}\right] = 0 \quad (4)$$

holds on $\left(u_t + \frac{3}{2}u_x^2 + \frac{3}{8}u^4 + \frac{3}{2}u^2u_x\right)_x + u_{yy} + u_{zz} = 0$. Here $Q^{[4]}$ stands for the fourth prolongation of vector Q and is defined as

$$Q^{[4]} = Q + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} + \eta^{tx}\partial_{u_{tx}} + \eta^{xx}\partial_{u_{xx}} + \eta^{yy}\partial_{u_{yy}} + \eta^{zz}\partial_{u_{zz}}, \quad (5)$$

with η^t, η^x, η^{tx}, η^{xx}, η^{yy} and η^{zz} defined as

$$\begin{aligned}
\eta^t &= D_t(\varphi) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3) \\
&\quad - u_z D_t(\xi^4), \\
\eta^x &= D_x(\varphi) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) \\
&\quad - u_z D_x(\xi^4), \\
\eta^{tx} &= D_t(\eta^x) - u_{tx} D_t(\xi^1) - u_{xx} D_t(\xi^2) - u_{xy} D_t(\xi^3) \\
&\quad - u_{xz} D_t(\xi^4), \\
\eta^{xx} &= D_x(\eta^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) \\
&\quad - u_{xz} D_x(\xi^4), \\
\eta^{yy} &= D_y(\eta^y) - u_{ty} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{yy} D_y(\xi^3) \\
&\quad - u_{yz} D_y(\xi^4), \\
\eta^{zz} &= D_z(\eta^z) - u_{tz} D_z(\xi^1) - u_{xz} D_z(\xi^2) - u_{yz} D_z(\xi^3) \\
&\quad - u_{zz} D_z(\xi^4),
\end{aligned} \tag{6}$$

representing the coefficient functions in $Q^{[4]}$ and the total derivatives appearing in (6) are given as

$$\begin{aligned}
D_t &= \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{xt} \partial_{u_x} + \cdots, \\
D_x &= \partial_x + u_x \partial_u + u_{xt} \partial_{u_t} + u_{xx} \partial_{u_x} + \cdots, \\
D_y &= \partial_y + u_y \partial_u + u_{ty} \partial_{u_t} + u_{xy} \partial_{u_x} + \cdots, \\
D_z &= \partial_z + u_z \partial_u + u_{tz} \partial_{u_t} + u_{xz} \partial_{u_x} + \cdots.
\end{aligned} \tag{7}$$

Expansion of (4) using the values of the coefficients given in (6) and subsequently equating all the involved differential coefficients of u to zero, we get twenty-nine overdetermined system of linear partial differential equations (LPDE) which are

$$\begin{aligned}
&\eta_{xu} = 0,\ \xi^4_x = 0,\ \xi^3_x = 0,\ \xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_z = 0, \\
&\xi^3_u = 0,\ \xi^2_u = 0,\ \xi^1_u = 0,\ \xi^3_z + \xi^4_y = 0,\ 2\xi^2_z + \xi^4_t = 0, \\
&\xi^2_{xx} - u\xi^2_x - \eta = 0,\ \eta_u - 2\xi^2_x + \xi^1_t = 0,\ 2\eta_{zu} - \xi^4_{zz} \\
&- \xi^4_{yy} = 0,\ 2\xi^3_y - \xi^2_x - \xi^1_t = 0,\ 2\xi^4_z - \xi^2_x - \xi^1_t = 0, \\
&2\xi^2_y + \xi^3_t = 0,\ 2\eta_{yu} + \xi^4_{yz} - \xi^3_{yy} = 0, \\
&3\xi^2_x u^2 - 3\xi^1_t u^2 - 6\eta u - 6\eta_x + 2\xi^2_t = 0,\ \eta_{uu} = 0, \\
&3\eta_x u^3 + 3\eta_{xx} u^2 + 2\eta_{zz} + 2\eta_{yy} + 2\eta_{tx} = 0,\ \xi^4_u = 0, \\
&3\xi^1_t u^3 + 9\eta u^2 - 3\xi^1_{tx} u^2 + 6\eta_x u + 2\eta_{tu} + \xi^4_{tz} + \xi^3_{ty} = 0,
\end{aligned}$$

whose solution gives the values of infinitesimals as

$$\begin{aligned}
&\xi^1 = \mathbf{c}_1 + \frac{3}{2}\mathbf{c}_6 t,\ \xi^2 = \frac{1}{2}\left(2\mathbf{c}_2 + \mathbf{c}_6 x - \mathbf{c}_4 y - \mathbf{c}_8 z\right), \\
&\xi^3 = \mathbf{c}_3 + \mathbf{c}_4 t + \mathbf{c}_6 y - \mathbf{c}_7 z,\ \xi^4 = \mathbf{c}_5 + \mathbf{c}_8 t + \mathbf{c}_7 y + \mathbf{c}_6 z, \\
&\eta = -\frac{1}{2}\mathbf{c}_6 u.
\end{aligned}$$

Thus, we have eight Lie point symmetries of (2) listed as:

$$\begin{aligned}
&Q_1 = \frac{\partial}{\partial t},\ Q_2 = \frac{\partial}{\partial x},\ Q_3 = \frac{\partial}{\partial y}, \\
&Q_4 = \frac{\partial}{\partial z},\ Q_5 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \\
&Q_6 = 2t\frac{\partial}{\partial z} - z\frac{\partial}{\partial x},\ Q_7 = 2t\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \\
&Q_8 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}.
\end{aligned}$$

Hence, (1+3)-D extKPle (2) admits an eight-dimensional Lie algebra spanned by vectors $Q_1, \ldots, Q_8$.

2.2 Symmetry reduction and exact solution

Consider a linear combination of symmetries Q_1, Q_2, Q_3 and Q_4 secured earlier as $Q = Q_1 + \gamma Q_2 + Q_3 + Q_4$ with constant $\gamma \neq 0$. We use Q to reduce (1+3)-D extKPle (2) to a PDE in three independent variables. In continuation, by solving the corresponding Lagrangian system to Q, we gain invariants

$$w = x - \gamma t,\ g = x - \gamma y,\ h = y - z,\ H = u. \tag{8}$$

Letting θ serve as the current dependent variables as well as w, g and h as latest independent variables, (1+3)-D extKPle (2) then alters to

$$\begin{aligned}
&3H_g H^3 + 3H_w H^3 + 3H_{gg} H^2 + 6H_{wg} H^2 + 3H_{ww} H^2 \\
&+ 6H_g^2 H + 6H_w^2 H + 12H_g H_w H + 2\gamma^2 H_{gg} - 4\gamma H_{gh} \\
&- 2\gamma H_{wg} - 2\gamma H_{ww} + 4H_{hh} + 6H_g H_{gg} + 6H_{gg} H_w \\
&+ 12H_g H_{wg} + 12H_w H_{wg} + 6H_g H_{ww} + 6H_w H_{ww} = 0.
\end{aligned} \tag{9}$$

Clearly, (9) is a nonlinear partial differential equation existing in terms of three independent variables. We utilize the generators of (9) to further make a reduction of the equation to a PDE in two independent variables. Equation (9) possesses generators given as

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial w},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = \frac{\partial}{\partial h}, \\
&\Gamma_4 = 2\gamma\left(w - h\gamma - \frac{h}{2} - g\right)\frac{\partial}{\partial g} - \gamma h\frac{\partial}{\partial w} \\
&\qquad + 2\left(h\gamma - 2w + 2g\right)\frac{\partial}{\partial h}.
\end{aligned}$$

If we combine generators Γ_1, Γ_2 and Γ_3 as $\Gamma = \Gamma_1 + \Gamma_2 + \vartheta\Gamma_3$, $(\vartheta \neq 0)$, we reduce equation (9). The solution of the related characteristic equations to Γ yields the invariants

$$r = h - \vartheta w,\ s = h - \vartheta g,\ H = G. \tag{10}$$

Now suppose G is taken as dependent variable alongside r and s as independent variables, then (1+3)-D extKPle (2) becomes

$$
\begin{aligned}
&6\vartheta^2 G_s^2 G + 6\vartheta^2 G_r^2 G + 3\vartheta^2 G_{ss} G^2 + 12\vartheta^2 G_s G_r G \\
&+ 6\vartheta^2 G_{rs} G^2 + 3\vartheta^2 G_{rr} G^2 - 3\vartheta G_s G^3 - 3\vartheta G_r G^3 \\
&+ 2\gamma^2\vartheta^2 G_{ss} - 2\gamma\vartheta^2 G_{rs} - 2\gamma\vartheta^2 G_{rr} + 4\gamma\vartheta G_{ss} \\
&+ 4\gamma\vartheta G_{rs} - 6\vartheta^3 G_s G_{ss} - 6\vartheta^3 G_{ss} G_r - 12\vartheta^3 G_s G_{rs} \\
&- 12\vartheta^3 G_r G_{rs} - 6\vartheta^3 G_s G_{rr} - 6\vartheta^3 G_r G_{rr} + 4G_{ss} \\
&+ 8G_{rs} + 4G_{rr} = 0. \qquad (11)
\end{aligned}
$$

We invoke the Lie point symmetries of (11) to transform it to an ordinary differential equation (ODE). The symmetries of (11) include the translational symmetries

$$\Sigma_1 = \frac{\partial}{\partial r} \text{ and } \Sigma_2 = \frac{\partial}{\partial s}. \qquad (12)$$

Contemplating the combined form of the two translation symmetries Σ_1 and Σ_2, as $\Sigma = \Sigma_1 + \nu\Sigma_2$, with ν regarded as a constant, we gain invariants

$$p = s - \nu r,\ G = F \qquad (13)$$

and then we achieve a group invariant solution $G = F(p)$. Application of the secured invariants to equation (11), the equation is transformed into

$$
\begin{aligned}
&2\gamma^2\vartheta^2 F'' - 2\gamma\nu^2\vartheta^2 F'' + 2\gamma\nu\vartheta^2 F'' - 4\gamma\nu\vartheta F'' + 4\gamma\vartheta F'' \\
&+ 6\nu^3\vartheta^3 F'F'' - 18\nu^2\vartheta^3 F'F'' + 4\nu^2 F'' + 18\nu\vartheta^3 F'F'' \\
&- 8\nu F'' + 6\nu^2\vartheta^2 F'^2 F + 4F'' + 3\nu^2\vartheta^2 F'' F^2 + 6\vartheta^2 F'^2 F \\
&- 12\nu\vartheta^2 F'^2 F - 6\nu\vartheta^2 F'' F^2 + 3\vartheta^2 F'' F^2 + 3\nu\vartheta F' F^3 \\
&- 3\vartheta F' F^3 - 6\vartheta^3 F' F'' = 0 \qquad (14)
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\alpha_0 F''(p) + \alpha_1 F' F''(p) + \alpha_2 F(p) F'^2(p) + \alpha_3 F^3(p) F'(p) \\
&+ \alpha_4 F^2(p) F''(p) = 0, \qquad (15)
\end{aligned}
$$

which is a fourth-order nonlinear ordinary differential equation (NODE), where
$\alpha_0 = 2(\gamma^2\vartheta^2 + \nu^2(2 - \gamma\vartheta^2) + \nu(\gamma(\vartheta - 2)\vartheta - 4) + 2\gamma\vartheta + 2)$, $\alpha_1 = 6(\nu - 1)^3\vartheta^3$, $\alpha_2 = 6(\nu - 1)^2\vartheta^2$, $\alpha_3 = 3(\nu - 1)\vartheta$, $\alpha_4 = 3(\nu - 1)^2\vartheta^2$ and $p = \vartheta(\nu - 1)x - \vartheta\gamma\nu t + (\gamma\vartheta - \nu + 1)y + (\nu - 1)z$.

3 Solution of (2) via the Kudryashov's technique

This section engages the Kudryashov's approach [47] to construct the exact solution of the (1+3)-D extKPle (2).

Hyperbolic solution

We now assume that the solution to the fourth-order ODE (14) using the Kudryashov's technique can be written in a formal structure given as

$$F(p) = \sum_{k=0}^{M} A_k H^k(p), \qquad (16)$$

with $H(p)$ satisfying the first-order ODE

$$H'(p) = H^2(p) - H(p), \qquad (17)$$

whose solution is

$$H(p) = \frac{1}{\cosh p + \sinh p}. \qquad (18)$$

We note that constant parameters $A_k, k = 0, 1, \cdots, M$ must be decided but M should first be determined in (16) by invoking the balancing procedure, see [48]. In our case we get $M = 1$, and consequently solution (16) can be presented as

$$F(p) = A_0 + A_1 H(p). \qquad (19)$$

Inserting the value of $F(p)$ in (19) into NODE (14) in consonance with (17) we secure an algebraic equation with regards to $H(p)$ as

$$
\begin{aligned}
&3\vartheta\nu A_1^4 H(p)^5 + 12\vartheta^2 A_1^3 H(p)^5 - 3\vartheta A_1^4 H(p)^5 \\
&+ 12\vartheta^2\nu^2 A_1^3 H(p)^5 - 24\vartheta^2\nu A_1^3 H(p)^5 - 12\vartheta^3 A_1^2 H(p)^5 \\
&+ 12\vartheta^3\nu^3 A_1^2 H(p)^5 - 36\vartheta^3\nu^2 A_1^2 H(p)^5 + 36\vartheta^3\nu A_1^2 H(p)^5 \\
&- 3\vartheta\nu A_1^4 H(p)^4 - 21\vartheta^2 A_1^3 H(p)^4 - 21\vartheta^2\nu^2 A_1^3 H(p)^4 \\
&+ 42\vartheta^2\nu A_1^3 H(p)^4 - 9\vartheta A_0 A_1^3 H(p)^4 + 3\vartheta A_1^4 H(p)^4 \\
&+ 9\vartheta\nu A_0 A_1^3 H(p)^4 + 30\vartheta^3 A_1^2 H(p)^4 - 30\vartheta^3\nu^3 A_1^2 H(p)^4 \\
&+ 90\vartheta^3\nu^2 A_1^2 H(p)^4 + 18\vartheta^2 A_0 A_1^2 H(p)^4 + 18\vartheta^2\nu^2 A_0 A_1^2 H(p)^4 \\
&- 36\vartheta^2\nu A_0 A_1^2 H(p)^4 + 9\vartheta^2 A_1^3 H(p)^3 - 18\vartheta^2\nu A_1^3 H(p)^3 \\
&+ 9\vartheta A_0 A_1^3 H(p)^3 - 9\vartheta\nu A_0 A_1^3 H(p)^3 - 24\vartheta^3 A_1^2 H(p)^3 \\
&+ 24\vartheta^3\nu^3 A_1^2 H(p)^3 - 72\vartheta^3\nu^2 A_1^2 H(p)^3 - 9\vartheta A_0^2 A_1^2 H(p)^3 \\
&+ 9\vartheta\nu A_0^2 A_1^2 H(p)^3 + 72\vartheta^3\nu A_1^2 H(p)^3 - 30\vartheta^2\nu^2 A_0 A_1^2 H(p)^3 \\
&+ 60\vartheta^2\nu A_0 A_1^2 H(p)^3 + 4\gamma^2\vartheta^2 A_1 H(p)^3 - 4\gamma\vartheta^2\nu^2 A_1 H(p)^3 \\
&+ 8\nu^2 A_1 H(p)^3 + 6\vartheta^2 A_0^2 A_1 H(p)^3 + 6\vartheta^2\nu^2 A_0^2 A_1 H(p)^3 \\
&- 12\vartheta^2\nu A_0^2 A_1 H(p)^3 + 8\gamma\vartheta A_1 H(p)^3 + 4\gamma\vartheta^2\nu A_1 H(p)^3 \\
&- 8\gamma\vartheta\nu A_1 H(p)^3 - 16\nu A_1 H(p)^3 + 8A_1 H(p)^3 + 6\vartheta^3 A_1^2 H(p)^2 \\
&- 6\vartheta^3\nu^3 A_1^2 H(p)^2 + 18\vartheta^3\nu^2 A_1^2 H(p)^2 + 9\vartheta A_0^2 A_1^2 H(p)^2 \\
&- 9\vartheta\nu A_0^2 A_1^2 H(p)^2 - 18\vartheta^3\nu A_1^2 H(p)^2 + 12\vartheta^2 A_0 A_1^2 H(p)^2 \\
&+ 12\vartheta^2\nu^2 A_0 A_1^2 H(p)^2 - 24\vartheta^2\nu A_0 A_1^2 H(p)^2 - 3\vartheta A_0^3 A_1 H(p)^2 \\
&+ 3\vartheta\nu A_0^3 A_1 H(p)^2 - 6\gamma^2\vartheta^2 A_1 H(p)^2 + 6\gamma\vartheta^2\nu^2 A_1 H(p)^2 \\
&- 12\nu^2 A_1 H(p)^2 - 9\vartheta^2 A_0^2 A_1 H(p)^2 - 9\vartheta^2\nu^2 A_0^2 A_1 H(p)^2 \\
&+ 18\vartheta^2\nu A_0^2 A_1 H(p)^2 - 12\gamma\vartheta A_1 H(p)^2 - 6\gamma\vartheta^2\nu A_1 H(p)^2 \\
&+ 24\nu A_1 H(p)^2 + 9\vartheta^2\nu^2 A_1^3 H(p)^3 + 12\gamma\vartheta\nu A_1 H(p)^2 \\
&- 12A_1 H(p)^2 + 3\vartheta A_0^3 A_1 H(p) - 3\vartheta\nu A_0^3 A_1 H(p) \\
&+ 2\gamma^2\vartheta^2 A_1 H(p) - 2\gamma\vartheta^2\nu^2 A_1 H(p) + 3\vartheta^2 A_0^2 A_1 H(p) \\
&+ 4\nu^2 A_1 H(p) + 3\vartheta^2\nu^2 A_0^2 A_1 H(p) - 30\vartheta^2 A_0 A_1^2 H(p)^3 \\
&- 90\vartheta^3\nu A_1^2 H(p)^4 - 6\vartheta^2\nu A_0^2 A_1 H(p) + 4\gamma\vartheta A_1 H(p) \\
&+ 2\gamma\vartheta^2\nu A_1 H(p) - 4\gamma\vartheta\nu A_1 H(p) - 8\nu A_1 H(p) \\
&+ 4A_1 H(p) = 0.
\end{aligned}
$$

Further splitting the above algebraic equation into diverse powers of $H(p)$ furnishes

$$
\begin{aligned}
H(p):\ & 2A_1\gamma^2\vartheta^2 - 2A_1\gamma\nu^2\vartheta^2 + 2A_1\gamma\nu\vartheta^2 - 4A_1\gamma\nu\vartheta \\
& + 4A_1\gamma\vartheta + 4A_1\nu^2 - 8A_1\nu + 3A_1A_0^2\nu^2\vartheta^2 \\
& - 6A_1A_0^2\nu\vartheta^2 + 3A_1A_0^2\vartheta^2 - 3A_1A_0^3\nu\vartheta + 4A_1 \\
& + 3A_1A_0^3\vartheta = 0, \\
H(p)^2:\ & 6A_1\gamma\nu^2\vartheta^2 - 6A_1\gamma^2\vartheta^2 - 6A_1\gamma\nu\vartheta^2 + 12A_1\gamma\nu\vartheta \\
& - 12A_1\gamma\vartheta + 24A_1\nu - 6A_1^2\nu^3\vartheta^3 + 18A_1^2\nu^2\vartheta^3 \\
& - 18A_1^2\nu\vartheta^3 + 6A_1^2\vartheta^3 + 12A_0A_1^2\nu^2\vartheta^2 - 12A_1\nu^2 \\
& - 24A_0A_1^2\nu\vartheta^2 + 18A_0^2A_1\nu\vartheta^2 + 12A_0A_1^2\vartheta^2 \\
& - 9A_0^2A_1\nu^2\vartheta^2 - 9A_0^2A_1^2\nu\vartheta + 3A_0^3A_1\nu\vartheta \\
& - 9A_0^2A_1\vartheta^2 + 9A_0^2A_1^2\vartheta - 3A_0^3A_1\vartheta - 12A_1 = 0, \\
H(p)^3:\ & 4A_1\gamma^2\vartheta^2 - 4A_1\gamma\nu^2\vartheta^2 + 4A_1\gamma\nu\vartheta^2 - 8A_1\gamma\nu\vartheta \\
& + 8A_1\gamma\vartheta + 8A_1\nu^2 + 24A_1^2\nu^3\vartheta^3 - 72A_1^2\nu^2\vartheta^3 \\
& + 72A_1^2\nu\vartheta^3 + 9A_1^3\nu^2\vartheta^2 - 30A_0A_1^2\nu^2\vartheta^2 \\
& + 6A_0^2A_1\nu^2\vartheta^2 - 18A_1^3\nu\vartheta^2 + 60A_0A_1^2\nu\vartheta^2 \\
& - 12A_0^2A_1\nu\vartheta^2 + 9A_1^3\vartheta^2 - 30A_0A_1^2\vartheta^2 + 6A_0^2A_1\vartheta^2 \\
& - 9A_0A_1^3\nu\vartheta + 9A_0^2A_1^2\nu\vartheta + 9A_0A_1^3\vartheta - 9A_0^2A_1^2\vartheta \\
& - 16A_1\nu - 24A_1^2\vartheta^3 + 8A_1 = 0, \\
H(p)^4:\ & -30A_1^2\nu^3\vartheta^3 + 90A_1^2\nu^2\vartheta^3 - 90A_1^2\nu\vartheta^3 + 30A_1^2\vartheta^3 \\
& - 21A_1^3\nu^2\vartheta^2 + 18A_0A_1^2\nu^2\vartheta^2 + 18A_0A_1^2\vartheta^2 \\
& + 42A_1^3\nu\vartheta^2 - 36A_0A_1^2\nu\vartheta^2 - 3A_1^4\nu\vartheta + 9A_0A_1^3\nu\vartheta \\
& - 21A_1^3\vartheta^2 + 3A_1^4\vartheta - 9A_0A_1^3\vartheta = 0, \\
H(p)^5:\ & 12A_1^2\nu^3\vartheta^3 - 36A_1^2\nu^2\vartheta^3 + 36A_1^2\nu\vartheta^3 - 12A_1^2\vartheta^3 \\
& + 12A_1^3\nu^2\vartheta^2 - 24A_1^3\nu\vartheta^2 + 12A_1^3\vartheta^2 + 3A_1^4\nu\vartheta \\
& - 3A_1^4\vartheta = 0.
\end{aligned}
$$

The above system yields

$$
\begin{aligned}
& A_0 = \vartheta(1-\nu),\ A_1 = 2\vartheta(1-\nu), \\
& \gamma = \frac{\nu-1}{2\vartheta}\left\{\vartheta(\nu+1) + 2 - \sqrt{\nu^2\vartheta^2 + 4\nu\vartheta - 4}\right\}.
\end{aligned}
$$

In consequence, the associated general solution to the stated values of A_0, A_1 and γ is expressed with regards to hyperbolic functions as

$$
u(t,x,y,z) = \vartheta(1-\nu) + \frac{2\vartheta(1-\nu)}{\cosh p + \sinh p}, \tag{20}
$$

with $p = \vartheta(\nu-1)x - \vartheta\gamma\nu t + (\gamma\vartheta - \nu + 1)y + (\nu-1)z$. The dynamics of the solution can be viewed with dissimilar values of the involved parameters in Figures 1, 2 and 3.

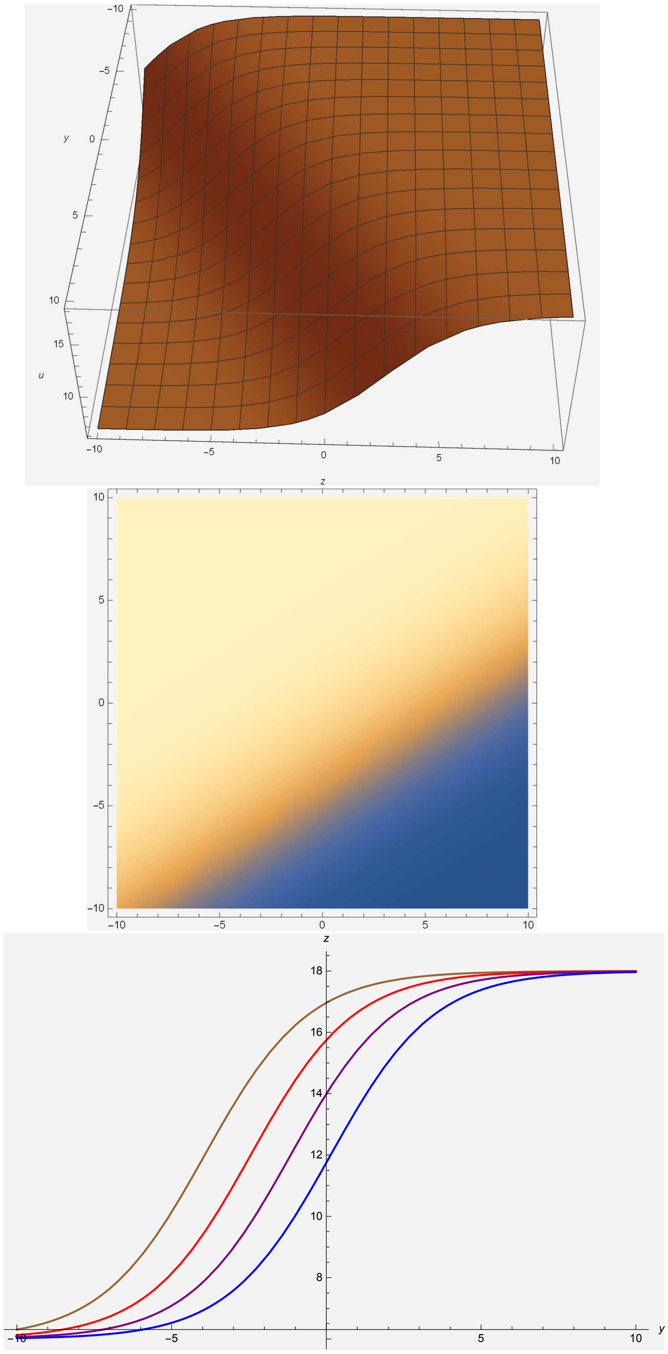

Fig. 1: Kink-type wave profile of hyperbolic solution (20) at $t = 0.5$ and $x = 0.4$.

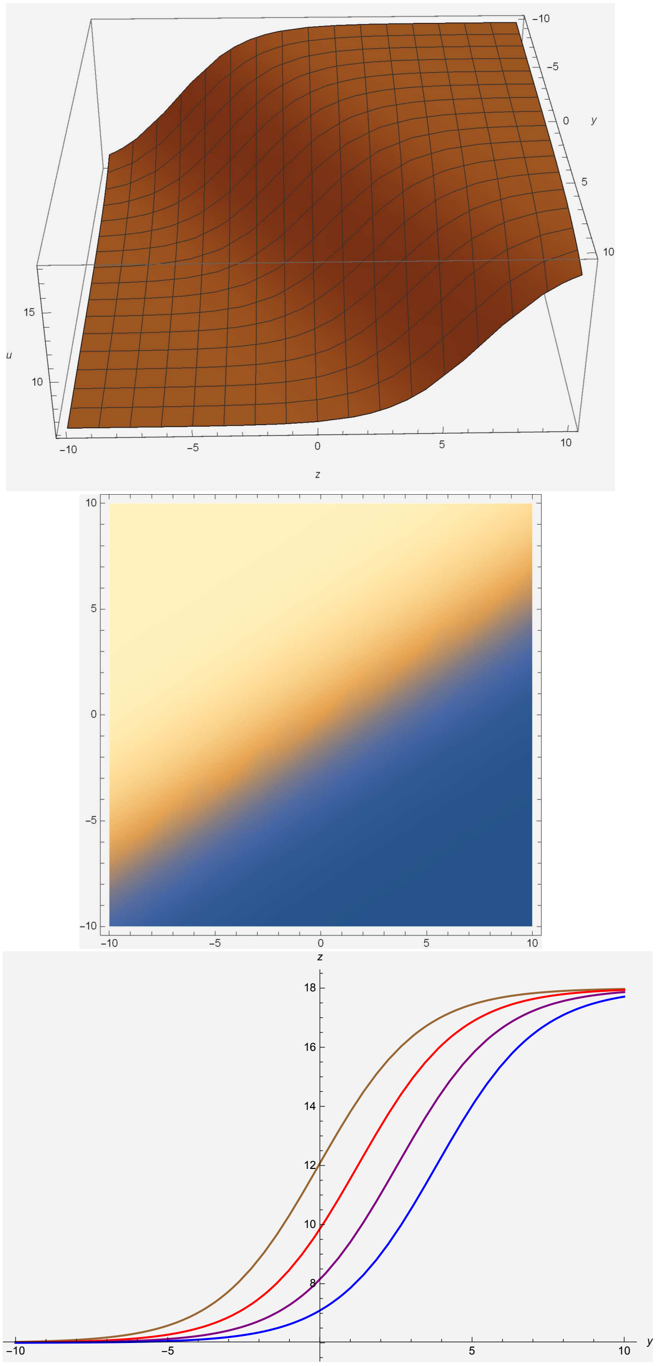

Fig. 2: Kink-type wave profile of hyperbolic solution (20) at $t = -0.3$ and $x = 0$.

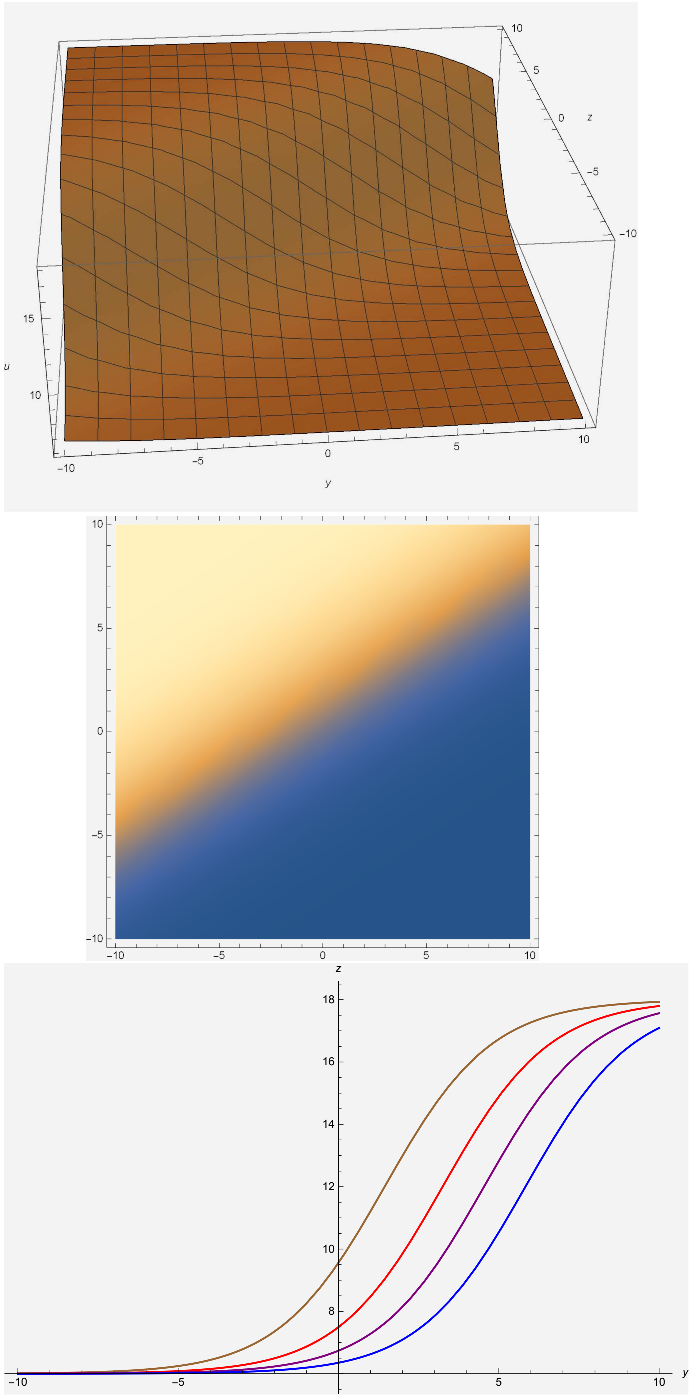

Fig. 3: Kink-type wave profile of hyperbolic solution (20) at $t = 0.6$ and $x = -0.2$.

3.1 Power series solution of NODE (15)

This subsection furnishes a solution to NODE (15) by employing power series technique [49]. In this case we seek a solution of equation (15) in a power series which assumes the form

$$F(p) = \sum_{m=0}^{\infty} a_m p^m, \tag{21}$$

with first and second derivatives given, respectively, as

$$F'(p) = \sum_{m=0}^{\infty} m a_m p^{m-1}, \text{ and } F''(p) = \sum_{m=0}^{\infty} m(m-1) a_m p^{m-2}. \tag{22}$$

Inserting the value of $F(p)$ from (21) and (22) into (15), one realizes

$$\alpha_0 \sum_{m=2}^{\infty} m(m-1) a_m p^{m-2} + \alpha_1 \left[\sum_{m=1}^{\infty} m a_m p^{m-1}\right]$$

$$\times \left[\sum_{m=2}^{\infty} m(m-1) a_m p^{m-2}\right] + \alpha_2 \left[\sum_{m=0}^{\infty} a_m p^m\right] \tag{23}$$

$$\times \left[\sum_{m=1}^{\infty} \sum_{m=1}^{\infty} m a_m p^{m-1}\right] + \alpha_3 \left[\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_m p^m\right]$$

$$\times \left[\sum_{m=1}^{\infty} m a_m p^{m-1}\right] + \alpha_4 \left[\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_m p^m\right] \tag{24}$$

$$\times \left[\sum_{m=2}^{\infty} m(m-1) a_m p^{m-2}\right] = 0, \tag{25}$$

which leads to a more simplified structure given as

$$\begin{aligned}
&2\alpha_0 a_2 + 6\alpha_0 a_3 + \alpha_0 \sum_{m=2}^{\infty} (m+1)(m+2) a_{m+2} p^m \\
&+ 2\alpha_1 a_1 a_2 + 12\alpha_1 a_2 a_3 + \alpha_1 \sum_{m=2}^{\infty} \left[\sum_{k=0}^{m} (k+1)\right. \\
&\left.\times\, (m-k+1)(m-k+2) a_{k+1} a_{m-k+2}\right] p^m \\
&+ \alpha_2 a_0 a_1^2 + 4\alpha_2 a_1 a_2^2 + \alpha_2 \sum_{m=2}^{\infty} \left[\sum_{k=0}^{m} \sum_{i=1}^{k+1} i(k-i\right. \\
&\left.+2) a_i a_{k-i+2} a_{m-k}\right] p^m + \alpha_3 a_0^3 a_1 + 2\alpha_3 a_1^3 a_2 \\
&+ \alpha_3 \sum_{m=2}^{\infty} \left[\sum_{k=0}^{m} \sum_{j=0}^{k} \sum_{i=0}^{j} (m-k+1) a_i a_{j-i} a_{k-j-i}\right. \\
&\left.\times a_{m-k+1}\right] p^m + 2\alpha_4 a_0^2 a_2 + 6\alpha_4 a_1^2 a_3 + \alpha_4 \sum_{m=2}^{\infty} \left[\sum_{k=1}^{m}\right. \\
&\left.\times \sum_{i=0}^{k} (m-k+1)(m-k+2) a_i a_{k-i} a_{m-k+2}\right] p^m = 0.
\end{aligned} \tag{26}$$

Next, comparing various coefficients of p in equation (26), one achieves for $m = 0, 1$;

$$2\alpha_0 a_2 + 2\alpha_1 a_1 a_2 + \alpha_2 a_0 a_1^2 + \alpha_3 a_0^3 a_1 + 2\alpha_4 a_0^2 a_2 = 0, \tag{27}$$

$$6\alpha_0 a_3 + 12\alpha_1 a_2 a_3 + 4\alpha_2 a_1 a_2^2 + 2\alpha_3 a_1^3 a_2 + 6\alpha_4 a_1^2 a_3 = 0 \tag{28}$$

and by generally considering $m \geq 2$, we gain

$$\begin{aligned}
a_{m+2} = &-\frac{1}{\alpha_0 (m+1)(m+2)} \left[\alpha_1 \sum_{k=0}^{m} (k+1)\right. \\
&\times (m-k+1)(m-k+2) a_{k+1} a_{m-k+2} \\
&+ \alpha_2 \sum_{k=0}^{m} \sum_{i=1}^{k+1} i(k-i+2) a_i a_{k-i+2} a_{m-k} \\
&+ \alpha_3 \sum_{k=0}^{m} \sum_{j=0}^{k} \sum_{i=0}^{j} (m-k+1) a_i a_{j-i} a_{k-j-i} a_{m-k+1} \\
&+ \alpha_4 \sum_{k=1}^{m} \sum_{i=0}^{k} (m-k+1)(m-k+2) a_i a_{k-i} \\
&\left.\times a_{m-k+2}\right].
\end{aligned} \tag{29}$$

Using equations (27) and (28), for arbitrary constants a_0 and a_1, one generates

$$a_2 = -\frac{\alpha_2 a_0 a_1^2 + \alpha_3 a_0^3 a_1}{2\alpha_0 + 2\alpha_1 a_1 + 2\alpha_4 a_0^2}, \tag{30}$$

$$a_3 = -\frac{4\alpha_2 a_1 a_2^2 + 2\alpha_3 a_1^3 a_2}{6\alpha_0 + 12\alpha_1 a_2 + 6\alpha_4 a_1^2}. \tag{31}$$

Invoking recursion formula (29), one can successfully secure other terms $a_m, m = 4, 5, \ldots, \infty$. Therefore, the power series solution of (15) can be presented as

$$\begin{aligned}
F(p) &= a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \sum_{m=2}^{\infty} a_{m+2} p^{m+2} \\
&= a_0 + a_1 p - \frac{\alpha_2 a_0 a_1^2 + \alpha_3 a_0^3 a_1}{2\alpha_0 + 2\alpha_1 a_1 + 2\alpha_4 a_0^2} p^2 \\
&\quad - \frac{4\alpha_2 a_1 a_2^2 + 2\alpha_3 a_1^3 a_2}{6\alpha_0 + 12\alpha_1 a_2 + 6\alpha_4 a_1^2} p^3 \\
&\quad + \sum_{m=2}^{\infty} \left[-\frac{1}{\alpha_0 (m+1)(m+2)} \left(\alpha_1 \sum_{k=0}^{m} (k+1)\right.\right. \\
&\quad \times (m-k+1)(m-k+2) a_{k+1} a_{m-k+2} + \alpha_2 \sum_{k=0}^{m} \\
&\quad \times \sum_{i=1}^{k+1} i(k-i+2) a_i a_{k-i+2} a_{m-k} + \alpha_3 \sum_{k=0}^{m} \sum_{j=0}^{k} \\
&\quad \times \sum_{i=0}^{j} (m-k+1) a_i a_{j-i} a_{k-j-i} a_{m-k+1} + \alpha_4 \sum_{k=1}^{m} \\
&\quad \left.\left.\times \sum_{i=0}^{k} (m-k+1)(m-k+2) a_i a_{k-i} a_{m-k+2}\right)\right] p^{m+2},
\end{aligned}$$

$m = 2, 3, 4, 5, \ldots$

and so the power series solution of equation (2) is given as

$$
\begin{aligned}
u(t,x,y,z) &= a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \sum_{m=2}^{\infty} a_{m+2} p^{m+2},\\
&= a_0 + a_1 p - \frac{\alpha_2 a_0 a_1^2 + \alpha_3 a_0^3 a_1}{2\alpha_0 + 2\alpha_1 a_1 + 2\alpha_4 a_0^2} p^2\\
&\quad - \frac{4\alpha_2 a_1 a_2^2 + 2\alpha_3 a_1^3 a_2}{6\alpha_0 + 12\alpha_1 a_2 + 6\alpha_4 a_1^2} p^3\\
&\quad + \sum_{m=2}^{\infty} \Bigg[-\frac{1}{\alpha_0 (m+1)(m+2)} \Bigg(\alpha_1 \sum_{k=0}^{m} (k+1)\\
&\quad \times (m-k+1)(m-k+2) a_{k+1} a_{m-k+2}\\
&\quad + \alpha_2 \sum_{k=0}^{m} \sum_{i=1}^{k+1} i(k-i+2) a_i a_{k-i+2} a_{m-k}\\
&\quad + \alpha_3 \sum_{k=0}^{m} \sum_{j=0}^{k} \sum_{i=0}^{j} (m-k+1) a_i a_{j-i}\\
&\quad \times a_{k-j-i} a_{m-k+1} + \alpha_4 \sum_{k=1}^{m} \sum_{i=0}^{k} (m-k+1)\\
&\quad \times (m-k+2) a_i a_{k-i} a_{m-k+2} \Bigg) \Bigg] p^{m+2},\\
&\quad m = 2,3,4,5,\cdots
\end{aligned}
\tag{32}
$$

with $a_i, (i = 0,1,2,\dots)$ regarded as arbitrary constants. It is noteworthy to declare here that approximate structure of (32) can be expressed as

$$
\begin{aligned}
u(t,x,y,z) &= a_0 + a_1(\vartheta(\nu-1)x - \vartheta\gamma\nu t + (\gamma\vartheta - \nu + 1)y\\
&\quad + (\nu-1)z) - \frac{\alpha_2 a_0 a_1^2 + \alpha_3 a_0^3 a_1}{2\alpha_0 + 2\alpha_1 a_1 + 2\alpha_4 a_0^2}\\
&\quad \times (\vartheta(\nu-1)x - \vartheta\gamma\nu t + (\gamma\vartheta - \nu + 1)y\\
&\quad + (\nu-1)z)^2 - \frac{4\alpha_2 a_1 a_2^2 + 2\alpha_3 a_1^3 a_2}{6\alpha_0 + 12\alpha_1 a_2 + 6\alpha_4 a_1^2}\\
&\quad \times (\vartheta(\nu-1)x - \vartheta\gamma\nu t + (\gamma\vartheta - \nu + 1)y\\
&\quad + (\nu-1)z)^3 + \cdots .
\end{aligned}
\tag{33}
$$

4 Conservation laws of (1+3)-D extKPle (2)

In this section, we do the computation of conserved vectors for (2) by exploiting the Ibragimov's theorem [50, 51] for conserved quantities. In doing that, we first supply some salient characteristics of the technique.

Preliminaries

Contemplate a system expressed in the structure

$$
H_\alpha(x,u,u_{(1)},\dots,u_{(q)}) = 0, \quad \alpha = 1,\dots,m, \tag{34}
$$

which is a qth-order system of partial differential equations existing with n independent variables $x = (x^1,x^2,\dots,x^n)$ as well as m dependent variables $u = (u^1,u^2,\dots,u^m)$. In this case, we have $u_{(1)},u_{(2)},\dots,u_{(q)}$ connoting the collections of all first, second up until the qth-order partial derivatives, which implies, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u_i^\alpha)$,..., accordingly where we have total derivative operators expressed with regards to x^i as

$$
D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \; i = 1,\dots,n \tag{35}
$$

The Euler-Lagrange operator, for each α included in (35), is given as

$$
\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \tag{36}
$$

$$
\alpha = 1,\dots,m, \tag{37}
$$

with the Lie-Bäcklund operator expressed as

$$
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \; \xi^i, \eta^\alpha \in \mathscr{A}, \tag{38}
$$

where the space $\mathscr{A}$ is taken as the space of differential functions and operator (38) can be presented as

$$
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \tag{39}
$$

with

$$
\begin{aligned}
\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha,\\
\zeta^\alpha_{i_1 \dots i_s} &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_s}, \; s > 1.
\end{aligned}
\tag{40}
$$

In this case, we express the Lie characteristic function W^α as

$$
W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{41}
$$

Consequently, the Lie-Bäcklund operator (39) in a characteristic structure can be presented as

$$
X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}. \tag{42}
$$

Assuming we associate the system of adjoint equations to (34) as

$$
H^*_\alpha(x,u,v,\dots,u_{(q)},v_{(q)}) = 0, \quad \alpha = 1,\dots,m, \tag{43}
$$

with H^*_α given as

$$
\begin{aligned}
H^*_\alpha(x,u,v,\dots,u_{(q)},v_{(q)}) &= \frac{\delta(v^\beta H_\beta)}{\delta u^\alpha}, \quad \alpha, = 1,\dots,m,\\
v &= v(x),
\end{aligned}
\tag{44}
$$

where $v = (v^1,v^2,\dots,v^m)$ are newly-introduced dependent variables. Suppose that (34) admits the symmetry generator (38), it follows that system (44) also admits the operator

$$
Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha}, \quad \eta^\alpha_* = -[\lambda^\alpha_\beta + v^\alpha D_i(\xi^i)], \tag{45}
$$

with Y in (45) serving as an extension of (38) to the new variable v^α. Besides, λ_β^α can be achieved from the relation

$$X(H_\alpha) = \lambda_\alpha^\beta H_\beta. \tag{46}$$

Thus, we give Ibragimov's theorem for conserved vectors.

Theorem 41 *Every Lie Bäcklund, Lie point as well as non-local symmetry (38) that is admitted by the system of equation presented in (34) produces a conserved quantity for the system comprising equation (34) alongside adjoint (43) with the components of C^i of the conserved vector $C = (C^1, \cdots, C^n)$ decided by the relation*

$$C^i = \xi^i \mathscr{L} + W^\alpha \frac{\delta \mathscr{L}}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta \mathscr{L}}{\delta u_{ii_1 i_2 \dots i_s}^\alpha},$$
$$i = 1, \dots, n, \tag{47}$$

with a formal Lagrangian expressed as

$$\mathscr{L} = v^\alpha H_\alpha(x, u, \dots, u_{(q)}). \tag{48}$$

Remark 41 *We remark that system (34) is self-adjoint if the replacement $v = u$ in the system of adjoint equation (43) produces the same system (34). For a more detailed understanding of the proof and more information on the results presented here, the reader is directed to [50, 51].*

4.1 Derivation of conservation laws via Ibragimov's theorem

This subsection presents the conservation laws of (1+3)-D extKPle (2) by invoking Ibragimov's theorem [51]. Using the outlined information given earlier we have the theorem:

Theorem 42 *The adjoint equation of (1+3)-D extKPle (2) is expressed as*

$$H^* \equiv v_{tx} - \frac{3}{2} v_x u^3 + \frac{3}{2} v_{xx} u^2 + 3u_{xx} v_x + 3u_x v_{xx} + v_{yy} + v_{zz} = 0 \tag{49}$$

and the Lagrangian given as

$$\mathscr{L} = vH = v\Big\{ u_{tx} + \frac{3}{2} u_x u^3 + \frac{3}{2} u_{xx} u^2 + 3u_x^2 u + 3u_x u_{xx} + u_{yy} + u_{zz} \Big\}, \tag{50}$$

where

$$H = u_{tx} + \frac{3}{2} u_x u^3 + \frac{3}{2} u_{xx} u^2 + 3u_x^2 u + 3u_x u_{xx} + u_{yy} + u_{zz}. \tag{51}$$

It is clear from the adjoint equation (49) and remark (41) that (1+3)-D extKPle (2) is not self-adjoint. Therefore, using the earlier outlined information, we have the conserved vectors associated with Lie symmetries previously obtained as subsequently presented. Therefore, time translation symmetry $Q_1 = \partial/\partial t$ possesses conservation law

$$\begin{aligned}
C_1^t &= \frac{3}{2} v u_x u^3 + \frac{3}{2} v u_{xx} u^2 + 3v u u_x^2 + v u_{zz} + v u_{yy} + 3v u_x u_{xx} \\
&\quad + \frac{1}{2} v u_{tx} + \frac{1}{2} u_t v_x, \\
C_1^x &= \frac{3}{2} u_t v_x u^2 - \frac{3}{2} v u_t u^3 - \frac{3}{2} v u_{tx} u^2 - 3u_t u_x u v - 3u_x u_{tx} v v \\
&\quad - \frac{1}{2} u_{tt} + 3u_t u_x v_x + \frac{1}{2} u_t v_t, \\
C_1^y &= u_t v_y - u_{ty} v, \\
C_1^z &= u_t v_z - u_{tz} v.
\end{aligned}$$

The corresponding conserved vectors to generator $Q_2 = \partial/\partial x$ which is a Lie point symmetry in x- translation, are

$$\begin{aligned}
C_2^t &= \frac{1}{2} u_x v_x - \frac{1}{2} u_{xx} v, \\
C_2^x &= \frac{3}{2} u_x v_x u^2 + u_{zz} v + u_{yy} v + \frac{1}{2} u_{tx} v + \frac{1}{2} v_t u_x + 3u_x^2 v_x, \\
C_2^y &= u_x v_y - u_{xy} v, \\
C_2^z &= u_x v_z - u_{xz} v.
\end{aligned}$$

Associated conservation law to y-translation Lie point symmetry $Q_3 = \partial/\partial y$ is

$$\begin{aligned}
C_3^t &= \frac{1}{2} u_y v_x - \frac{1}{2} u_{xy} v, \\
C_3^x &= -\frac{3}{2} u_y u^3 v + \frac{3}{2} u_y v_x u^2 - \frac{3}{2} u_{xy} u^2 v - 3u_x u_y u v \\
&\quad - 3u_x u_{xy} v - \frac{1}{2} u_{ty} v + \frac{1}{2} v_t u_y + 3u_x u_y v_x, \\
C_3^y &= \frac{3}{2} u_x u^3 v + \frac{3}{2} u_{xx} u^2 v + 3u_x^2 u v + u_{zz} v + 3u_x u_{xx} v \\
&\quad + u_{tx} v + u_y v_y, \\
C_3^z &= u_y v_z - u_{yz} v.
\end{aligned}$$

We present the related conservation law to z-translation symmetry $Q_4 = \partial/\partial z$ as

$$\begin{aligned}
C_4^t &= \frac{1}{2} u_z v_x - \frac{1}{2} u_{xz} v, \\
C_4^x &= -\frac{3}{2} u_z u^3 v + \frac{3}{2} u_z v_x u^2 - \frac{3}{2} u_{xz} u^2 v - 3u_x u_z u v \\
&\quad - 3u_x u_{xz} v v - \frac{1}{2} u_{tz} + \frac{1}{2} v_t u_z + 3u_x u_z v_x, \\
C_4^y &= u_z v_y - u_{yz} v, \\
C_4^z &= \frac{3}{2} u_x u^3 v + \frac{3}{2} u_{xx} u^2 v + 3u_x^2 u v + u_{yy} v + 3u_x u_{xx} v \\
&\quad + u_{tx} v + u_z v_z.
\end{aligned}$$

Rotation Lie point symmetry $Q_5 = y\partial/\partial z - z\partial/\partial y$ has conserved quantity

$$\begin{aligned}
C_5^t =& -\frac{1}{2}yu_{xz}v+\frac{1}{2}zu_{xy}v+\frac{1}{2}yu_zv_x-\frac{1}{2}zu_yv_x,\\
C_5^x =& -\frac{3}{2}yu_zu^3v+\frac{3}{2}zu_yu^3v+\frac{3}{2}yu_zv_xu^2-\frac{3}{2}zu_yv_xu^2\\
&+\frac{3}{2}zu_{xy}u^2v-3yu_xu_zuv+3zu_xu_yuv-3yu_xu_{xz}v\\
&-\frac{1}{2}yu_{tz}v+\frac{1}{2}zu_{ty}v+\frac{1}{2}yv_tu_z-\frac{1}{2}zv_tu_y+3yu_xu_zv_x\\
&-\frac{3}{2}yu_{xz}u^2v+3zu_xu_{xy}v-3zu_xu_yv_x,\\
C_5^y =& -\frac{3}{2}zu_xu^3v-\frac{3}{2}zu_{xx}u^2v-3zu_x^2uv-u_zv-zu_{zz}v\\
&-yu_{yz}v-3zu_xu_{xx}v-zu_{tx}v+yu_zv_y-zu_yv_y,\\
C_5^z =& \frac{3}{2}yu_xu^3v+\frac{3}{2}yu_{xx}u^2v+3yu_x^2uv+u_yv+zu_{yz}v\\
&+yu_{yy}v+3yu_xu_{xx}v+yu_{tx}v+yu_zv_z-zu_yv_z.
\end{aligned}$$

We computed the conservation law for generator $Q_6 = 2t\partial/\partial z - z\partial/\partial x$ as

$$\begin{aligned}
C_6^t =& -tu_{xz}v+\frac{1}{2}zu_{xx}v+tu_zv_x-\frac{1}{2}zu_xv_x,\\
C_6^x =& -3tu_zu^3v+3tu_zv_xu^2-\frac{3}{2}zu_xv_xu^2-3tu_{xz}u^2vv\\
&-zu_{yy}v-6tu_xu_{xz}v-tu_{tz}v-\frac{1}{2}zu_{tx}v+6tu_xu_zv_x\\
&-\frac{1}{2}zv_tu_x+tv_tu_z-3zu_x^2v_x-u_zv-zu_{zz}\\
&-6tu_xu_zuv,\\
C_6^y =& -2tu_{yz}v+zu_{xy}v+2tu_zv_y-zu_xv_y,\\
C_6^z =& 3tu_xu^3v+3tu_{xx}u^2v+6tu_x^2uv+2tu_{yy}v+zu_{xz}v\\
&+u_xv+6tu_xu_{xx}v+2tu_{tx}v+2tu_zv_z-zu_xv_z.
\end{aligned}$$

Operator $Q_7 = 2t\partial/\partial y - y\partial/\partial x$ possesses the conserved quantity

$$\begin{aligned}
C_7^t =& -tu_{xy}v+\frac{1}{2}yu_{xx}v+tu_yv_x-\frac{1}{2}yu_xv_x,\\
C_7^x =& -3tu_yu^3v+3tu_yv_xu^2-\frac{3}{2}yu_xv_xu^2-3tu_{xy}u^2v\\
&-u_yv-yu_{yy}v-6tu_xu_{xy}v-tu_{ty}v-\frac{1}{2}yu_{tx}v\\
&+6tu_xu_yv_x-\frac{1}{2}yv_tu_x-6tu_xu_yuv-yu_{zz}v\\
&+tv_tu_y-3yu_x^2v_x,\\
C_7^y =& 3tu_xu^3v+3tu_{xx}u^2v+6tu_x^2uv+2tu_{zz}v+u_xv\\
&+yu_{xy}v+6tu_xu_{xx}v+2tu_{tx}v+2tu_yv_y-yu_xv_y,\\
C_7^z =& -2tu_{yz}v+yu_{xz}v+2tu_yv_z-yu_xv_z.
\end{aligned}$$

Finally, for $Q_8 = 3t\partial/\partial t + x\partial/\partial x + 2y\partial/\partial y + 2z\partial/\partial z - u\partial/\partial u$, we calculated conserved vectors in the structure

$$\begin{aligned}
C_8^t =& \frac{9}{2}tu_xu^3v+\frac{9}{2}tu_{xx}u^2v+9tu_x^2uv+\frac{1}{2}v_xu+3tu_{zz}v\\
&-zu_{xz}v-yu_{xy}v-\frac{1}{2}xu_{xx}v+9tu_xu_{xx}v+\frac{3}{2}tu_{tx}v\\
&+\frac{3}{2}tu_tv_x+yu_yv_x+3tu_{yy}v-u_xv+zu_zv_x+\frac{1}{2}xu_xv_x,\\
C_8^x =& -\frac{3}{2}vu^4-3zvu_zu^3-3yvu_yu^3+\frac{3}{2}v_xu^3-\frac{9}{2}tvu_tu^3\\
&+3zu_zv_xu^2+3yu_yv_xu^2+\frac{3}{2}xu_xv_xu^2-3zvu_{xz}u^2\\
&+\frac{9}{2}tv_xu_tu^2-\frac{9}{2}tvu_{tx}u^2-6zvu_zu_xu-6yvu_yu_xuu\\
&-9tvu_xu_tu+\frac{1}{2}v_tu-6vu_x^2+xvu_{zz}+xvu_{yy}+3xu_x^2v_xv_x\\
&+6yu_yu_xv_x-6zvu_xu_{xz}-6yvu_xu_{xy}-2vu_t+9tu_xv_xu_t\\
&+yu_yv_t+\frac{1}{2}xu_xv_t+\frac{3}{2}tu_tv_t-zvu_{tz}-yvu_{ty}+\frac{1}{2}xvu_{tx}\\
&-9tvu_xu_{tx}+zu_zv_t-\frac{3}{2}tvu_{tt}+6zu_zu_x+3u_xv_x\\
&-3yvu_{xy}u^2-6vu_xu^2,\\
C_8^y =& 3yu_xu^3v+3yu_{xx}u^2v+6yu_x^2uv+v_yu+2yu_{zz}v\\
&-xu_{xy}v+6yu_xu_{xx}v-3tu_{ty}v+2yu_{tx}v+3tu_tv_y\\
&+2yu_yv_y-3u_yv-2zu_{yz}v+xu_xv_y+2zu_zv_y,\\
C_8^z =& 3zu_xu^3v+3zu_{xx}u^2v+6zu_x^2uv+v_zu-3u_zv\\
&-xu_{xz}v+6zu_xu_{xx}v-3tu_{tz}v+2zu_{tx}v+3tu_tv_z\\
&+xu_xv_z+2yu_yv_z-2yu_{yz}v+2zu_{yy}v+2zu_zv_z.
\end{aligned}$$

5 Concluding remarks

In this paper, we investigated an extended (1+3)-dimensional Kadomtsev-Petviashvili-like equation (2). We gained exact solutions of the equation with the use of Lie symmetry reductions as well as direct integration. We obtained a general solution of the equation by making use of the Kudryashov's technique. In consequence, we gain a hyperbolic function solution of the equation. Besides, power series solution of the underlying equation was also found. Furthermore, we derived conserved quantities of (2) by invoking Ibragimov's theorem. These conserved quantities include the conservation of energy and momentum. We note that the conserved vectors contain variable v which is an indication that the extended (1+3)-dimensional Kadomtsev-Petviashvili-like equation (2) possesses infinite conserved vectors.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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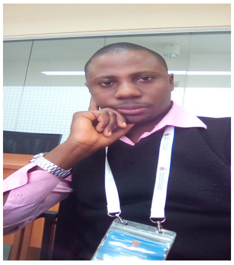

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