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Quantum Riemann Surfaces and the Measure of Noncommutativity

Simon Davis

Research Foundation of Southern California 8861 Villa La Jolla Drive #13595 La Jolla, CA 92039, USA

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Abstract: It is known that there exists a formulation of the quantum Riemann surface with a value of the deformation parameter related to the genus. A specific value of $\hbar$ may be derived from breaking of the quantum symmetry. The partition function for string theory is generalized to include a sum over quantum surfaces. A Fock space is constructed at the ideal boundary consisting of a countable set of points.

Keywords: Quantum surface, deformation parameter, ideal boundary

1 Introduction

Quantum commutation relations for operators representing position and metric variables would introduce noncommutative coordinates into the geometry. The characterization of the C^* algebras for various manifolds has been developed for Riemann surfaces that arise in the sum over string histories in the expansion of the scattering matrix. There it is found that the deformation parameter is quantized in inverse units of $2g-2$ for hyperbolic surfaces with genus $g \geq 2$ [1]. This result also has been derived through a consideration of the maximal uncertainty in the area resulting from the generalization of the commutation relations of the position and momentum operators to the Riemann surface [2].

The deformation parameter first occurs in the quantization of Lie algebras. Consider a group G and $A = Fun(G) = \{smooth\ functions\ on\ G|G\ is\ a\ Lie\ group\}$ with $f : G \times G \to G$ introducing comultiplication in the Hopf algebra $\Delta : A \to A \otimes A$ and $Spec\ A$ being the quantum space corresponding to A.

A simple Lie algebra can be quantized by considering the universal enveloping algebra $U_g \subset (C^\infty(G))^*$, $U_g = \{\phi \in C_0^{-\infty}|Supp(\phi) \subset \{e\}\}$. Then $U_h(g)$ is generated by h_i, X_i^+ and X_i^-, such that

$$[a_1, a_2] = 0 \quad [a, X_i^+] = \alpha_i(a) X_i^+ \quad [a, X_i^-] = -\alpha_i(a) X_i^- \quad [X_i^+, X_i^-] = 2\delta_{ij} h^{-1} \tag{1}$$

with $a_1, a_2 \in h$ and $i, j = 1, 2., 3, ..., ord\{X_i\}$ and

$$q = exp\left[\frac{h}{2}\langle H_i, H_i \rangle\right] \tag{2}$$

$$\sum_{k=1}^{m} (-1)^m \binom{n}{k} q - \frac{k(n-k)}{2} (X_i^-)^k X_j^+ (X_+ i^+)^{n-k} \quad = 0. \tag{3}$$

$$\sum_{k}^{n} (-1)^k \binom{n}{k} q^{-\frac{k(n-k)}{2}} (X_i^-)^k X_j^- (X_i^-)^{n-k} \quad = 0 \tag{4}$$

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)...(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)...(q - 1)}.$$

The comultiplication [3] is defined by

$$\begin{aligned} \Delta(a) &= a \otimes 1 \oplus 1 \otimes a & (5) \\ \Delta(X_i^-) &= X_i^+ \otimes exp\left(h\frac{H_i}{4}\right) + exp\left(-h\frac{H_i}{4}\right) \otimes X_i^- \\ \Delta(X_i^-) &= X_i^- \otimes exp\left(h\frac{H_i}{4}\right) + exp\left(-h\frac{H_i}{4}\right) \otimes X_i^- \end{aligned}$$

The algebra $(U_h(g), \Delta)$ is a quantization of g for which there exists a cocommutative Hopf subalgebra $C \subset A$ and a mapping $\theta : A \to A$ such that the mapping $C/hC \to U_g$,

* Corresponding author e-mail: sbdavis@resfdnsca.org

$C = U_h[[h]]$, is injective and its image is equal to U_h, while $\theta(X_i^+) = -X_i$, $\theta(X_i^-) = -X_i^+$, $\theta|_h = -id$, $\theta^2 = id$, $\theta(C) = 0$.

The commutation relations between the roots in the Chevalley basis of a Lie algebra

$$[h_i, h_j] = 0 \quad [h_i, X_j^{\pm}] = \alpha_{ij}^{\pm} X_j^{\pm} \quad [X_i^+, X_i^-] = \delta_{ij} h_j \qquad (6)$$

are replaced by

$$[X_i^+, X_i^-] = \delta_{ij} \frac{q^{h_j} - q^{-h_j}}{q - q^{-1}} \qquad (7)$$

in the quantum algebra $U_q(g)$ [4]. If $q = 1 + q_1\hbar + q_2\hbar^2 + \ldots$, the commutators would tend to $\delta_{ij}h_j\left(1 - \left(\frac{1}{2}q_1 - \frac{q_2}{q_1}\right)\hbar + \mathscr{O}(\hbar^2)\right)$ and $\hbar$ would represent a deformation parameter of the root system.

Therefore, the deformation parameter may be computed from the quantum symmetries of the theory. The string path integral can be generalized by including a summation over quantum surfaces. Then the correction will be determined by the value of the deformation parameter. The expansion of the scattering matrix is given in Theorem 1 of §3. The coefficients at genus $g \geq 2$ are changed according to the replacement of $\hbar$ by a multiple of $\frac{1}{2g-2}$. The terms at genus 0 and 1 are left invariant to corroborate the phenomenologically realistic calculations of string amplitudes. A theoretical explanation for these values through symmetries and Hilbert space methods is introduced.

The noncommutative C^* algebras of the ends of Riemann surfaces is developed to establish the form of the Hilbert space at the ideal boundary with a countable number of components. It is found in Theorem 2 of §4 that the Fock space is second countable and isomorphic to a multiplier algebra defined with respect to the C^* algebra. Therefore, the the physical states can be prepared in the neighbourhood of the boundary through conventional methods of quantum field theory.

2 Dimensions of Hilbert Spaces on Riemann Surfaces

Let

$$U = space\ of\ meromorphic\ functions\ on\ S\ which\ are\ holomorphic\ S\ \{P_0, P_\infty\} \qquad (8)$$

$$= U_0 \oplus U_1 \oplus U_1 \oplus U_+$$

$$U_- = \{f \mid ord_{P_0} f \geq 1\}$$

$$U_+ = \{f \mid ord_{P_\infty} f \geq 1\}$$

$$U_1 = \{f \mid ord_{P_0} f \geq 0\ and\ ord_{P_\infty} f \leq 0\}$$

with $dim\ U_1 = g + 1$ by the Riemann-Roch theorem, and the inner product be

$$(f, g) = Res_{P_0}(df \cdot g) = -Res_{\ P_\infty}(df \cdot g) = \frac{1}{2\pi i}\int_C df \cdot g \qquad (9)$$

where C is any cycle separating from P_0 to P_∞, such that (f, g) vanishes on $U_+ \times U_+$ and $U_- \times U_-$ [5]. The off-diagonal form of the inner product algebra is isomorphic with that of the Heisenberg algebra. An involution on the space U can be defined by the antiholomorphic involution on a Schottky double, $(\Theta f)(z) = \overline{f(\vartheta(z))}$, where ϑ is an antiholomorphic involution such that $\vartheta T = \bar{T}$. The Krichever-Novikov basis of U is

$$\begin{aligned} U_- &= \langle f_n^{(0)},\ n \geq 1\rangle \\ U_+ &= \langle f_n^{(0)},\ n \leq -(g+1)\rangle \\ U_1 &= \langle f_n^{(0)},\ -g \leq n \leq 0\rangle. \end{aligned} \qquad (10)$$

The difference in the dimensions of the spaces defined by the creation and annihilation operators is indicative of an inherent particle creation connected with the curvature of the genus g-surface.

The action of the classical group $SU(1,1)$ can be rotated in the two-dimensional complex plane to that of $SU(2)$. The quantum group $SU_q(2)$ may be represented as

$$\begin{aligned} &A(R)/(\tau_c \tau^b{}_d \varepsilon^{cd} = \varepsilon^{ab}) \\ &\varepsilon_{ab} = \begin{pmatrix} 0 & \frac{1}{\sqrt{q}} \\ -\frac{1}{\sqrt{q}} & 0 \end{pmatrix}. \end{aligned} \qquad (11)$$

The set of generators $\{\tau\}$ of $SU_q(2)$ would be separated into the generators $\{\Theta^A{}_B = S\tau^{b_0}{}_{a_0}\tau^{a_1}{}_{b_1}\}$ of the subalgebra of even elements, $SO_3(q)$, where

$$\begin{aligned} &(\Theta^A{}_B)^* = \Theta^{\bar{A}}{}_{\bar{B}} = S\Theta^A{}_B \\ &(\Theta)^A{}_B satisfies\ relations\ of\ A(R_1)\ where \\ &(R_1)^{AB}{}_{CD} = (R^{-1})^{\gamma b_1}{}_{a_0\beta} R^{c_0\alpha}{}_{\gamma b_0} R^{a_1\beta}{}_{\delta d_1} \tilde{R}^{\delta d_0}{}_{c_1\alpha} \\ &(\tilde{R}) = ((R^t)^{-1})^t \end{aligned} \qquad (12)$$

and the generator of odd elements [6]. It can be embedded in L_q, a quasitriangular $*$-Hopf algebra with an R-matrix that satisfies the quantum Yang-Baxter equations, since there is an isomorphism

$$\rho:\ L_q \to\ SL_q(2:C) = SU_q(2) \times SU_q(2). \qquad (13)$$

The rotation of $SU_q(1,1)$ into $SU_q(2)$ and the embedding into the quantum Lorentz group provides a method for translating the inherent quantum fluctuations in string dynamics to the uncertainty in the description of scattering in Minkowski space-times.

Quantizations of graded Hope algebras and affine Kac-Moody algebras involve commutation relations of quantum generators with classical limits that are generators of the classical algebras. There exists a representations $\rho : U_h(sl(n)) \to Mat(n, \mathbb{C}[[h]])$ [7] that

$$\rho(H_i) = e_{ii} - e_{i+1,i+1} \tag{14}$$
$$\rho(X_i^+) = e_{i,i+1}$$
$$\rho(X_i^-) = 2h^{-1}\, sinh\left(\frac{h}{2}\right) e_{i+1,i}$$
$$\rho_{ij}\rho_{i\ell} = e^{-\frac{h}{2}}\rho_{i\ell}\rho_{ij} \qquad j < \ell$$
$$\rho_{ij}\rho_{kj} = e^{-\frac{h}{2}}\rho_{kj}\rho_{ij} \qquad i < K$$
$$\rho_{i\ell}\rho_{kj} = \rho_{kj}\rho_{i\ell} \qquad i < k,\ \ell > j$$

and the commutation relations

$$[\rho_{i\ell}, \rho_{kj}] = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})\rho_{ij}\rho_{k\ell} \qquad i > k,\ \ell > j \tag{15}$$

and the defining relation

$$\sum_{i_1,\ldots i_n} \rho_{1i_1}\ldots\rho_{ni_n}\cdot(-e^{-\frac{h}{2}})^{\ell(i_1,\ldots i_n} = 1 \tag{16}$$

where $\ell(i_1,\ldots i_n) = \#\ of\ inversions$. Then $(U_h(sl(n)))^*$ is the quotient of the ring of noncommutative power seris in $\bar{\rho}_{ij} = \rho_{ij} - \delta_{ij}$ by the ideal generated by these relations.

3 The Planck Constant as a Deformation Parameter

It has been demonstrated that there is a quantum Riemann surface, defined by the quantization of the isometry group of the hyperbolic disk by a group, and the deformation parameter, which may be selected to be the Planck constant, can be identified with a function of the genus.

The variation in the area of a Riemann surface [2] based on a curved-space generalization of the uncertainty principle is

$$2\pi\cdot 2. \tag{17}$$

Furthermore, the relative uncertainty is

$$\frac{2\pi\cdot 2}{2\pi\cdot 2g-2} = \frac{1}{g-1} \tag{18}$$

It follows that $\hbar$ can be identified with $\frac{1}{g-1}$ and $\frac{\hbar}{2}$ would be equivalent to $\frac{1}{2g-2}$.

The C^* algebra of the surface Δ/Γ would be defined by the set of automorphic functions on the disk with respect to the uniformizing group [1]. The quantum group parameter for Riemann surfaces is found to coincide with this value following the conditions on the coefficients of automorphy of weight r, $\gamma_1'(\gamma_2(\zeta))^{-\frac{r}{2}}\gamma'(\zeta)\nu(\gamma_1)\nu(\gamma_2) = (\gamma_1\gamma_2)'(\zeta)^{-\frac{r}{2}}\nu(\gamma_1\gamma_2)$, which have solutions when $r = \frac{n}{2g-2}$ [8]. A similar result has been derived in quantum mechanics on a quotient of two-dimensional anti-de Sitter space by a discrete subgroup of $SO(2,1)$, where the image of the relation is

$$V\left(\prod_{i=1}^{g} g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}\right) = e^{4\pi i(g-1)j} \tag{19}$$

in the discrete representation $\mathscr{D}_j^+$, which requires $\pm s - j = \frac{n}{2g-2}$ with $U(g_\alpha) \in D_s^\pm$, the discrete representation in s [9]. Therefore, althought it is conventional to fix the value of $\hbar$, there is a basis for identifying in a relative sense with this function of the genus. A fixed value of the deformation parameter to be identified with $\hbar$ in this formalism can be achieved only if the quantization is developed for the disk as a universal cover with isometry group $SU(1,1)$. The deformation of a root system in a quantum algebra therefore would be given by the genus-dependent value $\frac{1}{g-1}$. It follows that the classical Lie algebra is recovered in the infinite-genus limit.

Since there is no change in the dimensions of a boundary identified with a point, the physical state would represent a particle that does not have a finite width in a semiclassical theory. The neutrino, for example, is a particle with no apparent dimension. The wavelength of light is given by $\frac{\hbar c}{E} = \frac{\hbar}{p}$, and yet it cannot be a characteristic of th dimensions of the photon at any instant in time because the lower bound in the uncertainty in the momentum Δp, increases indefinitely by $\Delta p\Delta x \geq \frac{\hbar}{2}$ as the position is restricted to a point. The width of the resonance is equal to the inverse of the lifetime. Given that the lifetimes of the photon and the neutrino are indefinite, the width in the energy spectrum would be reduced to be infinitesimal.

The external state particles in a string scattering diagram are given by semi-infinite cylinders, which may be conformally mapped to punctured disks on the surface. By contrast, a conformal transformation of a surface of infinite genus causes the handles to accumulate to a finite endpoint. Any neighbourhood of this accumulation point resembles a cusp. When the fundamental domain of the uniformizing Fuchsian group has a cusplike end, and the Hausdorff dimension of the ideal boundary is null, the escape rate of a random walk to infinity vanishes, while the spectrum of the Laplacian of a quantum field on the surface yields a lower bound for the imaginary part of the eigenvalue of $\frac{1}{2} - \delta$, where δ is the Hausdorff dimension of the limit set of the Fuchsian group which would equal $\frac{1}{2}$ for this domain, and represent a non-zero decay [10]. Suppose that the cusplike region is conformally expanded after a scattering process. The decay of a point like particle produced at the boundary point would be consistent with vanishing width and an indefinite lifetime, provided that complex eigenvalues are not introduced into the spectrum of the Hamiltonian. The relative uncertainty tends to zero as $g \to \infty$ corroborating the triviality in the transformation from the vacuum state to the infinite-genus state in a free fermion theory allowing the preparation of states of definite momentum [11]. There would be no corrections from a quantum deformation of this surface.

The calculation of a energy and width of a resonance are affected by radiative corrections. Rigorously, the value of the mass in the quantum propagators in the perturbative diagrams at each order must be left arbitrary until it is determined through the renormalization procedure. The mass of a resonance may be deduced classically from momentum space transform of the Green function or the inverse of the differential operator representing the wave equation for the field. It receives quantum corrections in string theory through the higher-genus correlation functions $\langle 0|V_1V_2|g\rangle$. The sum of the terms at each genus yields

$$\langle V_1V_2\rangle_{class.\ surf.} = \langle 0|V_1V_2|0\rangle + \sum_g c_g \int_{\bar{\mathscr{M}}_g} \sum_{g\geq 1} \langle 0|V_1V_2|g\rangle. \tag{20}$$

The coefficient in superstring theory would decrease as a $\frac{B^g}{f(g)}$, where $f(g)$ is a function that interpolates between an exponential and a factorial function of the genus [12].

Quantum deformations of the Riemann surfaces would alter the formula for the two-point correlation fucnction.

Theorem 1. The shift in the mass resulting from the quantum string theory perturbation expansion is given by the residue of the Fourier transform of

$$\langle 0|V_1V_2|0\rangle + c_1 \int_{\bar{\mathscr{M}}_1} \langle 0|V_1V_2|1\rangle + \sum_{g=2}^{\infty} c_g \int_{\bar{\mathscr{M}}_g} \left(1+\frac{1}{g-1}\right) \langle 0|V_1V_2|g\rangle$$

.

Proof. The perturbative expansion of the partition function of a field theory is

$$Z[\phi] = \int D[\phi] e^{iI[\phi]}. \tag{21}$$

The power series in $\hbar$ defines the order of the diagrammatic formula for the scattering matrix which is defined by functional derivatives of the partition function

$$\langle \phi(k_1)...\phi(k_n)\rangle = \frac{\delta}{\delta J(k_1)} \cdots \frac{\delta}{\delta J(k_n)} Z[\phi, J] \tag{22}$$

where $Z[\phi, J]$ is the partition function with a source term.

The string perturbation series in the Euclidean formalism would be given by a sum over Riemann surfaces with a weighting factor defined by e^{-I}, where I is the sigma model action for the theory,

$$\langle V_1V_2\rangle_{class\ surf.} = \langle 0|V_1V_2|0\rangle + \sum_g c_g \hbar^g \int_{\bar{\mathscr{M}}_g} d\mu_g \langle 0|V_1V_2|g\rangle. \tag{23}$$

where $\hbar$ is set equal to 1 in the formula (15). If quantum deformations of the surface are included, there is another contribution to the amplitudes arising from these surfaces $\hbar$ replaced by $\frac{1}{g-1}$. Adding this fraction to $\frac{1}{g-1}$ gives the multiplicative factor $1+\frac{1}{g-1}$ and the expansion

$$\langle V_1(z_1)V_2(z_2)\rangle_{quantum\ surf.} = \langle 0|V_1V_2|0\rangle + c_1 \langle 0|V_1V_2|1\rangle \tag{24}$$

$$+ \sum_g c_g \left(1+\frac{1}{g-1}\right) \int_{\bar{\mathscr{M}}_g} d\mu_g \langle 0|V_1V_2|g\rangle.$$

The Fourier transform

$$\langle V_1V_2\rangle(p) = \int d^2p e^{ip\cdot(z_1-z_2)} \langle V_1(z_1)V_2(z_2)\rangle_{quantum\ surf.}. \tag{25}$$

The classical limt of the two-point correlation function is the Green function for $\Delta_\Sigma - \frac{1}{4} - m^2$, which is $G(z_1,z_2) = \sum_{\gamma\in\Gamma} G_0(\gamma z_1, z_2)$, where $G_0(z_1,z_2) = \frac{2^{-2-2im}\Gamma\left(\frac{1}{2}+im\right)}{\sqrt{\pi}\Gamma(1+im)}$

$$cosh^{-1}\left(\frac{d(z_1,z_2)}{2}\right) F\left(\frac{1}{2}+im, \frac{1}{2}+im, 1+2im;\right.$$

$$\left. cosh^2\left(\frac{d(z_1,z_2)}{2}\right)\right)$$

[10] such that the integral transform yields $\frac{1}{p^2-m^2}$ in the limit of coincidence of z_1 and z_2, where m is the mass of the resonance. Therefore, the nonperturbative mass will equal

$$\frac{1}{2\pi i}\langle V_1V_2\rangle(p) = \frac{1}{2\pi i} Res \int d^2p\ e^{ip\cdot(z_1-z_2)} \langle V_1(z_1)V_2(z_2)\rangle_{quantum\ surf.}. \tag{26}$$

The coefficients of the moduli space integrals are left invariant at genus 0 and 1 in the series expansion of the scattering amplitudes. A non-zero value of the deformation parameter affects the symmetry algebra and the conformal symmetries on the sphere and the torus. The measure can be defined through a factorization by the volume of the conformal group and it is not possible to preserve the form of the moduli space without setting the parameter equal to zero. Furthermore, the dimensions of the subspaces of the Hilbert space defined by the Krichever-Novikov basis of meromorphic functions is given by $U_- = \langle f_n^{(0)} |\ n \geq 1\rangle$, $U_+(g = 0) = \langle F_n^{(0)} |\ n \geq -1\rangle$, $U_+(g=1) = \langle f_n^{(0)} |\ n \leq -2\rangle$, $U_1(g=0) = \langle f_n^{(0)} |\ n = 0\rangle$ and $U_1(g=1) = \langle f_n^{(0)} |\ n = -1,\ 0\rangle$, where the functions

$f_n^{(0)}$ have poles of order n. The Hilbert space U_1 for $g=0$ consists of only constant functions and cannot be deformed, while the space of meromorphic functions with simple poles wil yield fixed expectation values of opertators upon normalization.

Unlike the perturbative diagrams of the point-particle field theories, where the momentum-space propagator with non-zero mass is required in the evaluation of the amplitudes, it is not necessary in the moduli space integral. Instead, the fundamental masses are given by vibrational modes of the string and the energies of the excitations are derived by the action of the Virasoro algebra on the sphere and the Krichever-Novikov algebras on higher-genus surfaces.

4 Conformal Field Theory on Quantum Surfaces

The physical states are prepared by performing a path integral for the conformal field theory on a Riemann surface with an ideal boundary. The boundary may have zero harmonic measure and be identified with a point or the capacity can be non-zero and the state is identified with a configuration of non-zero linear extent or a string [13].

A physical state identified with a boundary of non-zero linear extent could have a finite width resulting from the effect of quantum fluctuations at the ideal boundary. A superselection principle would be required for the creation of a set of particle states, which may follow from methods [14] that have been developed for Hilbert spaces constructed over ideal boundaries with fractional dimension.

The effect of quantum corrections to a surface with ends is the blurring of the boundary. States are prepared at points on the ideal boundary are likely to be observed to be magnified. The C^* algebra of a Raum space, that has a Hausdorff topology which is connected, locally connected, locally compact and σ-compact, has been defined [15]. The Raum spaces include the ends of Riemann surfaces, and the C^* algebra is a *Raum*$^+$ algebra, allowing an algorithmic enumeration of the ends. It has been established that the quotient of the multiplier algebra $M(\mathscr{U})$ of a *Raum*$^+$ algebra $\mathscr{U}$, $M(\mathscr{U})/\mathscr{U}$ has cardinality $\aleph$ [15].

Theorem 2. The Fock space of states that can be constructed at the ends of an O_G surface is isomorphic to $M(C(\cup_n E_n))\backslash C(E_n)$ where $C(E_n)$ is a *Raum*$^+$ algebra.

Proof. The class of O_G surfaces with zero capacity has been demonstrated to be conformally equivalent to the set of spheres with an infinite number of handles. The number of sequences of handles that can be placed on a sphere without introducing a non-zero harmonic measure is countable [13], and the space continues to have a Hausdorff topology.

Suppose that the ends of an O_G surface are labelled by $E_n,\ n\in\mathbb{N}$. The C^* algebra of the product of two noncompace Raum spaces X and Y is isomorphic to $C(X)$ [15] . Consequently,

$$C\left(\prod_n E_n\right)\simeq C(E) \tag{27}$$

where E is a representative of $\{E_n,\ n\in\mathbb{N}\}$. Since $E\subseteq \cup_n E_n\subseteq\otimes_n E_n$,

$$C(\subset_n E_n)\simeq C\left(\prod_n E_n\right)C(E). \tag{28}$$

The C^* algebra $C(E)$ contains a Hilbert space $\mathfrak{H}(E)$ because the space of L^2 functions with prescribed boundary values is a subspace of $C(E)$.

Suppose that a field theory on the Riemann surface is quantized with operators $a(p)$ and $a^\dagger(p)$ such that $a(p)|0\rangle=0$ and $a^\dagger|0\rangle=|p\rangle$, where p is a label that could be given, for example, by the momentum. Then the Fock space at the boundary of the Riemann surface is

$$\mathscr{F}(\partial\Sigma)=\cup_n\left\{|p_{i_1},...,p_{i_n}\rangle|\ |p\rangle=a^\dagger|0\rangle,\ p\in\mathbb{R}^2,\ p_{+-}^2\geq 0\right\}. \tag{29}$$

where $p_{+-}^2=p_\tau^2-p_\sigma^2$. Consequently, the space of states in the Hilber space $\mathfrak{H}(E_n)$ all can be collected into a single Hilbert space $\mathfrak{H}(E)$. Furthermore, the union of states in $E_{i_1},...,E_{i_k}$ may be defined to be multiparticle state $|p_{i_1},...,p_{i_n}\rangle$ in the Fock space $\mathscr{F}(\cup_n E_n)$.

The cardinality of the states in the Fock space, given that the enumeration of the states in the Hilbert space $\mathfrak{H}(E_n)$ yields a countable set, would be $2^{\aleph_0}$. Therefore, it has the same cardinality as $M(C(\cup_n E_n))/C(\cup_n E_n))$. The factorization by $C(U_n E_n)$ is equivalent to the identification of all of the vacuum states $|0\rangle,|0,\ 0\rangle,\ ...,\ |0,\ ,...,0,...\rangle$.

5 Perspective

The allowed values of the deformation parameter in a quantum Riemann surface are $\left\{\frac{1}{2g-2}n\right\}$. The relative uncertainty in the area on the curved space generalization of the uncertainty relation would be $\frac{2}{2g-2}=\frac{1}{g-1}$. Given commutation relations $[x,p]=i\hbar$, it is customary to use $\Delta x\Delta p\geq\frac{h}{2}$ which is relevant for the additional factor of 2.

It follows from this result that the parameter in the Hopf algebra can be related to that derived in a normal coordinate expansion on the manifold on which the group act. For a Riemann surface, the group is $SU(1,1)$ acting on the hyperbolic disk factored by a Fuchsian group, which yields the inverse of the linear function of the genus after quantization.

The effect of quantum surfaces on the masses and the widths of resonances may be given by a multiplicative factor in the perturbative expansion of the correlation function that includes the reciprocal of $g-1$. Consequently, ti decreases with increasing genus and the contribution reduces to that of the classical Riemann surfaces in the string perturbation series. The residue of the Fourier transform fo the two-point correlation would be sufficient to determine the nonperturbative mass.

The noncommutative theory of ends includes a result on the C^* algebra of the tensor product of Raum spaces that is developed to prove a theorem on the Hilbert spaces that can be constructed on the ideal boundaries of effective closed surfaces in the sum over string histories. It is concluded that a Fock space may be defined in terms of multiparticle states with a countable basis for each Hilbert space.

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