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Commutable (P, Q) - Matrix Calculus

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Abstract: This paper delves into the (p, q)- derivative operator for matrix functions. The (p, q)-matrix exponential, (p, q)-binomial of two matrices and (p, q) -trigonometric and (p, q)–hyperbolic matrix functions are introduced. The (p, q)-derivatives of these newly defined functions are calculated in detail, and the fundamental properties of these structures are investigated. This study provides new insights into the interaction between (p, q)-calculus and matrix analysis.

Keywords: (p, q)-Calculus; Matrix Functions; (p, q)-Derivative Operator; Matrix Exponential; Quantum Calculus.

1 Introduction

In standard q-calculus, the derivative is inherently asymmetric, scaling the argument in only one direction. The (p, q)-calculus restores symmetry by scaling the argument by both p and q. When the functions involved are matrices whose elements are often non-commuting operators, this calculus becomes essential for describing the quantum groups.[5][7]
We quickly recall the basic terminology from the (p, q)-calculus. [8][13] The (p, q)-analogue of a real number n is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \quad n \in \mathbb{R}, \quad 0 < q < p \leq 1. \tag{1}$$

The (p, q)-binomial expansion is given as

$$(x+y)^n_{p,q} = (x+y)(px+qy)\ldots(p^{n-1}x+q^{n-1}y) \tag{2}$$

and the (p, q)-binomial coefficients are defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}. \tag{3}$$

The (p, q)-derivative of an arbitrary function $f(x)$ is defined as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0. \tag{4}$$

$0 < q < p \leq 1$.

Note that for $f(x) = x^n$,

$$D_{p,q}f(x) = nx^{n-1}. \tag{5}$$

Also note that for $p = 1$, the (p, q)-derivative reduces to the Jackson's derivative[10] given by

$$D_q f(x) = \frac{f(q)-f(x)}{(q-1)x}, \quad x \neq 0. \tag{6}$$

The two (p,q)-analogues of the classical exponential function[5][7] are defined as:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!}.p\frac{n(n-1)}{2}, \tag{7}$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.q\frac{n(n-1)}{2}. \tag{8}$$

The (p,q)-trigonometric functions are given as:

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$$Sin_{p,q}(x) = \frac{e_{p,q}^{(ix)} - e_{p,q}^{(-ix)}}{2i}, \quad (9)$$

$$Cos_{p,q}(x) = \frac{e_{p,q}^{(ix)} + e_{p,q}^{(-ix)}}{2}. \quad (10)$$

and

$$Sin_{p,q}(x) = \frac{E_{p,q}^{(ix)} - E_{p,q}^{(-ix)}}{2i}, \quad (11)$$

$$Cos_{p,q}(x) = \frac{E_{p,q}^{(ix)} + E_{p,q}^{(-ix)}}{2}. \quad (12)$$

The (p,q)-hyperbolic functions are given as

$$Sinh_{p,q}(x) = \frac{e_{p,q}^{(x)} - e_{p,q}^{(-x)}}{2}, \quad (13)$$

$$Cosh_{p,q}(x) = \frac{e_{p,q}^{(x)} + e_{p,q}^{(-x)}}{2}. \quad (14)$$

The (p, q)-Fundamental theory of Calculus[8][13] state that if F(x) is an anti-derivative of f(x), such that $D_{p,q}F(x) = f(x)$, then the (p,q)- definate integral is given by

$$\int_0^a f(x) d_{p,q}x = F(a) - F(0) \quad (15)$$

Matrix functions play an important role in various fields of mathematics and its applications, ranging from solving systems of differential equations to quantum mechanics. The extension of classical calculus concepts to matrices has enabled research into novel frameworks including matrix trigonometric, hyperbolic, and exponential functions.[1][4] A quick recall of these functions are defined using Taylor series expansion.

(i) The matrix exponential is defined as:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad (16)$$

(ii) The matrix sine and cosine functions are defined as:

$$\sin(A) = \frac{e^{(Ai)} - e^{-(Ai)}}{2i} \quad (17)$$

$$\cos(A) = \frac{e^{(Ai)} + e^{(-Ai)}}{2} \quad (18)$$

(iii) The matrix hyperbolic sinh(A) and cosH(A) functions are defined as:

$$\sinh(A) = \frac{e^{(A)} - e^{(-A)}}{2} \quad (19)$$

$$\cosh(A) = \frac{e^{(A)} + e^{(-A)}}{2} \quad (20)$$

Where A is a commutable square matrix.
Earlier works[2][3][6] have dealt with q-matrix calculus,in this paper, we introduce the (p,q)-derivative operator for matrix function, establishing a framework that combines the principles of p,q)-calculus theory. This approach allows the definition of new matrices and the (p,q)–matrix exponential, the (p,q)-trigonometric and the (p,q)-hyperbolic matrix functions. This approach also allows us to compute the derivatives of these newly defined functions and examine some of their properties.

2 Main Results

We begin with the definition of the (p, q) – derivative of a matrix function, which is very important in this paper.

2.1 Definition 2.1

Let $f(M)$ be an arbitrary matrix function for a non-singular matrix M. Then, the (p, q) – derivative of $f(M)$ with respect to M is defined by

$$D_{p,q}f(M) = \frac{f(pM) - f(qM)}{(p-q).M}. \quad (21)$$

2.2 Example 2.2

Let $f(M) = M^n$ for a positive integer n. Then, the difference quotient is defined as follows:

$$D^M_{(p,q)}f(M) = \frac{f(pM) - f(qM)}{(p-q)M} \quad (22)$$

Substituting $f(M) = M^n$:

$$= \frac{(pM)^n - (qM)^n}{(p-q)M} \quad (23)$$

This simplifies to:

$$= \frac{p^n M^n - q^n M^n}{(p-q)M} \quad (24)$$

Factoring out M^n:

$$= \frac{(p^n - q^n)}{(p-q)} M^{n-1} \quad (25)$$

By the finite difference formula, we have:

$$= [n]_{(p,q)} M^{n-1} \tag{26}$$

It is very obvious that the (p,q)-derivative operator is a linear operator. That is,

$$D^M_{(p,q)}(af(M)+bg(M)) = aD^M_{(p,q)}f(M)+bD^M_{(p,q)}g(M) \tag{27}$$

holds for any matrix functions $f(M)$ and $g(M)$ and constants a and b.

Proposition 2.3.

Let M be a non-singular matrix and $f(M)$ and $g(M)$ be matrix functions. The product rule of the (p,q)-derivative of the functions $f(M)$ and $g(M)$ is given by:

$$D^M_{(p,q)}[f(M)g(M)] = f(M_p)D^M_{(p,q)}g(M)+D^M_{(p,q)}f(M)g(M) \tag{28}$$

$$= g(M_p)D^M_{(p,q)}f(M)+f(M_q)D_{(p,q)}g(M) \tag{29}$$

Proof:
We start with the definition of the (p,q)-derivative:

$$D^M_{(p,q)}[f(M)g(M)] = \frac{f(pM)g(pM)-f(qM)g(qM)}{(p-q)M} \tag{30}$$

Rewriting this, we have:

$$= \frac{f(pM)g(pM)-f(pM)g(qM)+f(pM)g(qM)-f(qM)g(qM)}{(p-q)M} \tag{31}$$

This can be grouped as:

$$= f(pM)\frac{g(pM)-g(qM)}{(p-q)M}+g(qM)\frac{f(pM)-f(qM)}{(p-q)M} \tag{32}$$

Thus, we obtain:

$$= f(pM)D^M_{(p,q)}g(M)+D^M_{(p,q)}f(M)g(qM) \tag{33}$$

Hence proved.

Definition 2.4.

Let A and B be commutable matrices. Then the (p,q)-binomial is defined as:

$$(A-B)^n_{(p,q)} = (A-B)(pA-qB)(p^2A-q^2B)$$
$$(p^{n-1}A-q^{n-1}B) \quad \text{for } n \geq 1. \tag{34}$$

Note that for $n=0$:

$$(A-B)^0_{(p,q)} = I \tag{35}$$

where I is the identity matrix of the same order as A and B.

Proposition 2.5.

For $n \geq 1$,

$$D_{(p,q)}(A-B)^n_{(p,q)} = [n]_{(p,q)}(A-B)^{(n-1)}_{(p,q)} \tag{36}$$

Proof:
The (p,q)-derivative with respect to the matrix M is defined as:

$$D_{(p,q)}f(M) = \frac{f(pM)-f(qM)}{(p-q)M} \tag{37}$$

For $n=1$:

$$f(A) = (A-B)^1_{(p,q)} = (A-B) \tag{38}$$

Thus, we have:

$$D^M_{(p,q)}(A-B) = \frac{(pA-B)-(qA-B)}{(p-q)A} \tag{39}$$

This simplifies to:

$$= \frac{(pA-B-qA+B)}{(p-q)A} = \frac{(p-q)A}{(p-q)A} = 1 \tag{40}$$

Since $[1]_{(p,q)} = 1$ and $(A \oplus B)^0 = 1$, the formula holds.

Definition 2.6.

For a polynomial function $f(M)$ of order N, the (p,q)-Taylor formula is defined by:

$$f(M) = \sum_{n=0}^{N} \frac{(D^M_{(p,q)}f)(B)(A-B)^n_{(p,q)}}{[n]_{(p,q)}!} \tag{41}$$

where M is a non-singular matrix and B is any matrix of the same size.
Let A and B be commutable matrices and let n be a non-negative integer. We compute the (p,q)-Taylor series expansion of the function

$$f(A) = (A+B)^{n-1}_{(p,q)} \tag{42}$$

by evaluating its (p,q)-derivative at $A=0$.
For $j \leq n$:

$$D^A_{(p,q)} j\,(A+B)^n_{(p,q)} = [n]_{(p,q)}[n-1]_{(p,q)} \\ [n-j+1]_{(p,q)}(A+B)^{n-1}_{(p,q)} \tag{43}$$

is obtained.
The expansion is given by:

$$(A+B)^n_{(p,q)} = \sum_{j=0}^{n} \binom{n}{j}_{(p,q)} \\ P\frac{(n-j)(n-j-1)}{2} q^{i(j-1)/2} B^{(n-j)} \tag{44}$$

Finally, by replacing j with $n-j$, we get:

$$(A+B)^n_{(p,q)} = \sum_{j=0}^{n} \binom{n}{j}_{(p,q)} \\ p^{j(j-1)/2} q^{(n-j)(n-j-1)/2} B^j A^{(n-j)} \tag{45}$$

Proposition 2.8.

For $n \geq 1$:

$$D^A_{(p,q)}(A+B)^n_{(p,q)} = [n]_{(p,q)}(A+B)^{(n-1)}_{(p,q)} \tag{46}$$

$$D^B_{(p,q)}(A+B)^n_{(p,q)} = [n]_{(p,q)}(A+B)^{(n-1)}_{(p,q)} \tag{47}$$

Proof:
Starting with the derivative with respect to A:

$$D^A_{(p,q)}(A+B)^n_{(p,q)} = D^A_{(p,q)} \\ \left(\sum_{j=0}^{n} [\binom{n}{j}]_{(p,q)} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} B^j A^{(n-j)}\right) \tag{48}$$

This leads to:

$$D^A_{(p,q)}\left(\sum_{j=0}^{n} \frac{[n]_{(p,q)}!}{[n-j]_{(p,q)}![j]_{(p,q)}!} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} B^j A^{(n-j)}\right) \tag{49}$$

Now, we have:

$$\sum_{j=0}^{n-1} \frac{[n]_{(p,q)}!}{[n-j]_{(p,q)}![j]_{(p,q)}!} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} D^A_{(p,q)}(B^j A^{(n-j)}) \tag{50}$$

This simplifies further to:

$$\sum_{j=0}^{n-1} \frac{[n]_{(p,q)}!}{[n-j]_{(p,q)}![j]_{(p,q)}!} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} [n-j]_{(p,q)} A^{(n-j)} B^j \tag{51}$$

Thus, we conclude:

$$= [n]_{(p,q)} \sum_{j=0}^{n-1} \frac{[n-1]_{(p,q)}!}{[n-1-j]_{(p,q)}![j]_{(p,q)}!} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} A^{(n-j)} B^j \tag{52}$$

Therefore, we have:

$$= [n]_{(p,q)}(A+B)^{(n-1)}_{(p,q)} \tag{53}$$

Now, for the derivative with respect to B:

$$D^B_{(p,q)}(A+B)^n_{(p,q)} = D^B_{(p,q)} \\ \left(\sum_{j=0}^{n} [\binom{n}{j}]_{(p,q)} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} A^{(n-j)} B^j\right) \tag{54}$$

Following similar steps, we arrive at:

$$= \sum_{j=0}^{n-1} \frac{[n]_{(p,q)}!}{[n-j]_{(p,q)}![j]_{(p,q)}!} p^{\frac{j(j-1)}{2}} q^{\frac{(n-j)(n-j-1)}{2}} A^{(n-j)} D^B(B^j) \tag{55}$$

Continuing this leads us to:

$$= [n]_{(p,q)}(A+B)^{(n-1)}_{(p,q)} \tag{56}$$

Hence, we conclude the proof

Definition 2.9.

A (p,q)-analogue of the classical matrix exponential function e^M is defined for a square matrix M as:

$$e^M_{(p,q)} = \sum_{k=0}^{\infty} \frac{p^{(k/2)}}{[k]_{(p,q)}!} M^k \tag{57}$$

where $[k]_{(p,q)}!$ is the (p,q) factorial and $p^{(k/2)}$ is a weighting factor where $(k/2) = \frac{k(k-1)}{2}$.
Another (p,q)-analogue of the classical matrix exponential function e^M is defined as:

$$E^M_{(p,q)} = \sum_{k=0}^{\infty} \frac{q^{(k/2)}}{[k]_{(p,q)}!} M^k \tag{58}$$

Definition 2.10. Let M be a commutable square matrix. The (p,q)-analogues of some trigonometric and hyperbolic matrix functions are defined as follows:

$$\sin_{(p,q)}(M) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k+1)(2k)}{2}}}{[2k+1]_{(p,q)}!} M^{(2k+1)} \tag{59}$$

$$\cos_{(p,q)}(M) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k)(2k-1)}{2}}}{[2k]_{(p,q)}!} M^{(2k)} \tag{60}$$

$$\sinh_{(p,q)}(M) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k+1)(2k)}{2}}}{[2k+1]_{(p,q)}!} M^{(2k+1)} \tag{61}$$

$$\cosh_{(p,q)}(M) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k-1)}{2}}}{[2k]_{(p,q)}!} M^{(2k)} \tag{62}$$

According to Definition 2.9, it is clear that the (p,q)-trigonometric and (p,q)-hyperbolic matrix functions can be expressed as follows:

$$\sin_{(p,q)}(M) = \frac{e_{(p,q)}^{(iM)} - e_{(p,q)}^{(-iM)}}{2i} \tag{63}$$

$$\cos_{(p,q)}(M) = \frac{e_{(p,q)}^{(iM)} + e_{(p,q)}^{(-iM)}}{2} \tag{64}$$

$$\sinh_{(p,q)}(M) = \frac{e_{(p,q)}^{(M)} - e_{(p,q)}^{(-M)}}{2} \tag{65}$$

$$\cosh_{(p,q)}(M) = \frac{e_{(p,q)}^{(M)} + e_{(p,q)}^{(-M)}}{2} \tag{66}$$

Definition 2.11.

The (p,q)-trigonometric and (p,q)-hyperbolic matrix functions are defined for a square matrix M as follows:

$$\sin_{(p,q)}(M) = \frac{E_{(p,q)}^{(iM)} - E_{(p,q)}^{(-iM)}}{2i} \tag{67}$$

$$\cos_{(p,q)}(M) = \frac{E_{(p,q)}^{(iM)} + E_{(p,q)}^{(-iM)}}{2} \tag{68}$$

$$\sinh_{(p,q)}(M) = \frac{E_{(p,q)}^{(M)} - E_{(p,q)}^{(-M)}}{2} \tag{69}$$

$$\cosh_{(p,q)}(M) = \frac{E_{(p,q)}^{(M)} + E_{(p,q)}^{(-M)}}{2} \tag{70}$$

Proposition 2.12.

The following relations are true for the (p,q)-trigonometric and (p,q)-hyperbolic matrix functions:

$$\sinh_{(p,q)}(iM) = i\sin_{(p,q)}(M) \tag{71}$$

$$\cosh_{(p,q)}(iM) = \cos_{(p,q)}(M) \tag{72}$$

where M is a commutable square matrix.

Proof:

Considering Definition 2.10, we have:

$$\sinh_{(p,q)}(A) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k+1]_{(p,q)}!} A^{(2k+1)} \tag{73}$$

Substituting $A = iM$:

$$\sinh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k+1]_{(p,q)}!} (iM)^{(2k+1)} \tag{74}$$

This gives:

$$= \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k+1]_{(p,q)}!} i^{(2k+1)} M^{(2k+1)} \tag{75}$$

Since $i^{(2k+1)} = i(-1)^k$:

$$\sinh_{(p,q)}(iM) = i\sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k)(2k+1)}{2}}}{[2k+1]_{(p,q)}!} M^{(2k+1)} \tag{76}$$

Thus, we find:

$$\sinh_{(p,q)}(iM) = i\sin_{(p,q)}(M) \tag{77}$$

Hence proved.

Now for the cosine function:

$$\cosh_{(p,q)}(A) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} A^{(2k)} \tag{78}$$

Substituting $A = iM$:

$$\cosh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} (iM)^{2k} \tag{79}$$

This becomes:

$$= \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} i^{2k} M^{2k} \tag{80}$$

Since $i^{2k} = (-1)^k$:

$$\cosh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} M^{2k} \tag{81}$$

Thus, we conclude:

$$\cosh_{(p,q)}(iM) = \cos_{(p,q)}(M) \tag{82}$$

Hence proved.
ii. From the definition, we have:

$$\cosh_{(p,q)}(A) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} A^{(2k)} \tag{83}$$

Substituting $A = iM$:

$$\cosh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} (iM)^{(2k)} \tag{84}$$

This expands to:

$$= \sum_{k=0}^{\infty} \frac{p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} i^{2k} M^{2k} \tag{85}$$

Since $i^{2k} = (-1)^k$, we can write:

$$\cosh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} M^{2k} \tag{86}$$

Thus, we find:

$$\cosh_{(p,q)}(iM) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\frac{(2k)(2k+1)}{2}}}{[2k]_{(p,q)}!} M^{(2k)} \tag{87}$$

Hence, we conclude:

$$\cosh_{(p,q)}(iM) = \cos_{(p,q)}(M) \tag{88}$$

Hence proved.

Proposition 2.13. Let A and B be square matrices of the same size. Then, the following relation between (p,q)-exponential matrices holds:

$$e_{(p,q)}^{(A)} E_{(p,q)}^{(B)} = e_{(p,q)}^{(A+B)} \tag{89}$$

Proof. Considering Definition 2.9, we have:

$$e_{(p,q)}^{(A)} E_{(p,q)}^{(B)} = \left(\sum_{j=0}^{\infty} \frac{p^{\frac{(j)(j-1)}{2}}}{[j]_{(p,q)}!} A^j \right) \left(\sum_{k=0}^{\infty} \frac{q^{\frac{(k)(k-1)}{2}}}{[k]_{(p,q)}!} B^k \right) \tag{90}$$

This can be rewritten as:

$$= \sum_{n=0}^{\infty} \frac{1}{[n]_{(p,q)}!} \sum_{k=0}^{n} \binom{n}{k}_{(p,q)} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} A^{(n-k)} B^k \tag{91}$$

Noting that $n = j+k$, we can express it as:

$$= \sum_{n=0}^{\infty} \frac{1}{[n]_{(p,q)}!} (A+B)_{(p,q)}^n \tag{92}$$

This gives us:

$$e_{(p,q)}^{(A)} E_{(p,q)}^{(B)} = \sum_{n=0}^{\infty} \frac{1}{[n]_{(p,q)}!} (A+B)_{(p,q)}^n \tag{93}$$

By the definition of the (p,q)-exponential function, we also have:

$$e_{(p,q)}^{(A+B)} = \sum_{n=0}^{\infty} \frac{1}{[n]_{(p,q)}!} (A+B)_{(p,q)}^n \tag{94}$$

Thus, we conclude:

$$e_{(p,q)}^{(A)} E_{(p,q)}^{(B)} = e_{(p,q)}^{(A+B)} \tag{95}$$

Hence proved.

Proposition 2.14. The (p,q) derivatives of the (p,q)-analogue exponential matrices are given by:

$$D_{(p,q)} e_{(p,q)}^{(Mt)} = t e_{(p,q)}^{(pMt)} \tag{96}$$

$$D_{(p,q)} E_{(p,q)}^{(Mt)} = t E_{(p,q)}^{(qMt)} \tag{97}$$

for a commutable square matrix M.

Proof. Using Definition 2.9, we have:

$$e_{(p,q)}^{(Mt)} = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!} (Mt)^n \tag{98}$$

Taking the derivative:

$$D_{(p,q)} e_{(p,q)}^{(Mt)} = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!} D_{(p,q)}\left((Mt)^n\right) \tag{99}$$

Utilizing the power rule for derivatives:

$$D_{(p,q)} e_{(p,q)}^{(Mt)} = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!} t^n [n]_{(p,q)} M^{(n-1)} \tag{100}$$

Since $[n]_{(p,q)}! = [n]_{(p,q)}! [n-1]_{(p,q)}!$:

$$D_{(p,q)} e_{(p,q)}^{(Mt)} = \sum_{n=1}^{\infty} \frac{p^{\frac{(n-1)n}{2}} t^n}{[n-1]_{(p,q)}!} M^{(n-1)} \tag{101}$$

Reindexing the sum where $k = n-1$:

$$= \sum_{k=0}^{\infty} \frac{p^{\frac{(k+1)(k+2)}{2}} t^{k+1}}{[k]_{(p,q)}!} M^k \tag{102}$$

Using the triangular number property:

$$\frac{(k+1)(k+2)}{2} = \frac{k(k+1)}{2} + k \tag{103}$$

We find:

$$D_{(p,q)}e_{(p,q)}^{(Mt)} = t\sum_{k=0}^{\infty}\frac{p^{\frac{k(k+1)}{2}}}{[k]_{(p,q)}!}(pMt)^k \tag{104}$$

Thus, we conclude:

$$D_{(p,q)}e_{(p,q)}^{(Mt)} = te_{(p,q)}^{(pMt)} \tag{105}$$

Hence proved.
Proof: We need to show that:

$$D_{(p,q)}E_{(p,q)}^{(Mt)} = tE_{(p,q)}^{(qMt)} \tag{106}$$

Starting with:

$$E_{(p,q)}^{(Mt)} = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!}(Mt)^n \tag{107}$$

Taking the derivative:

$$D_{(p,q)}E_{(p,q)}^{(Mt)} = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!}t^nD_{(p,q)}\left((M)^n\right) \tag{108}$$

Utilizing the power rule for derivatives, we write:

$$= \sum_{n=1}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_{(p,q)}!}t^n[n]_{(p,q)}M^{(n-1)} \tag{109}$$

Reindexing with $k = n-1$:

$$= \sum_{k=0}^{\infty}\frac{q^{\frac{(k+1)(k+2)}{2}}}{[k+1]_{(p,q)}!}t^{k+1}M^k \tag{110}$$

Using the factorial relation:

$$= \sum_{k=0}^{\infty}\frac{q^{\frac{(k+1)(k+2)}{2}}}{[k]_{(p,q)}!}t^{k+1}M^k \tag{111}$$

Now expressing $q^{\frac{(k+1)(k+2)}{2}}$ in terms of q:

$$= \sum_{k=0}^{\infty}\frac{(q^{\frac{k(k+1)}{2}})q^k}{[k]_{(p,q)}!}tt^kM^k \tag{112}$$

Thus we find:

$$= t\sum_{k=0}^{\infty}\frac{q^{\frac{k(k+1)}{2}}}{[k]_{(p,q)}!}(qMt)^k \tag{113}$$

This leads to the conclusion:

$$D_{(p,q)}E_{(p,q)}^{(Mt)} = tE_{(p,q)}^{(qMt)} \tag{114}$$

Hence proved.

Proposition 2.15. Let M be a commutable square matrix. The (p,q)-derivatives of the (p,q)-trigonometric matrix functions are:

i. $D_{(p,q)}\sin_{(p,q)}(Mt) = t\cos_{(p,q)}(pMt)$ (115)

ii. $D_{(p,q)}\cos_{(p,q)}(Mt) = t\sin_{(p,q)}(pMt)$ (116)

Proof: Considering Proposition 2.14, we have:
i.

$$D_{(p,q)}\sin_{(p,q)}(Mt) = D_{(p,q)}\left(\frac{e_{(p,q)}^{(iMt)} - e_{(p,q)}^{(-iMt)}}{2i}\right) \tag{117}$$

This leads to:

$$= \frac{1}{2i}\left[D_{(p,q)}e_{(p,q)}^{(iMt)} - D_{(p,q)}^{M}e_{(p,q)}^{(-iMt)}\right] \tag{118}$$

Applying the derivative:

$$= \frac{1}{2i}\left[ite_{(p,q)}^{(ipMt)} - (-it)e_{(p,q)}^{(-ipMt)}\right] \tag{119}$$

Which simplifies to:

$$= \frac{it}{2i}\left[e_{(p,q)}^{(ipMt)} + e_{(p,q)}^{(-ipMt)}\right] \tag{120}$$

Hence:

$$= t\left(\frac{e_{(p,q)}^{(ipMt)} + e_{(p,q)}^{(-ipMt)}}{2}\right) \tag{121}$$

:

$$\cos_{(p,q)}(pMt) = \frac{e_{(p,q)}^{(ipMt)} + e_{(p,q)}^{(-ipMt)}}{2} \tag{122}$$

Therefore:

$$D_{(p,q)}\sin_{(p,q)}(Mt) = t\cos_{(p,q)}(pMt) \tag{123}$$

ii.

$$D_{(p,q)}\cos_{(p,q)}(Mt) = D_{(p,q)}\left(\frac{e_{(p,q)}^{(iMt)} + e_{(p,q)}^{(-iMt)}}{2}\right) \tag{124}$$

This gives us:

$$= \frac{1}{2}\left[D_{(p,q)}e_{(p,q)}^{(iMt)} + D_{(p,q)}e_{(p,q)}^{(-iMt)}\right] \tag{125}$$

Calculating the derivatives:

$$= \frac{1}{2}\left[ite_{(p,q)}^{(ipMt)} - ite_{(p,q)}^{(-ipMt)}\right] \tag{126}$$

Thus:

$$= \frac{it}{2}\left[e_{(p,q)}^{(ipMt)} - e_{(p,q)}^{(-ipMt)}\right] \tag{127}$$

Considering $i^2 = -1$:

$$= -t\left(\frac{e_{(p,q)}^{(ipMt)} - e_{(p,q)}^{(-ipMt)}}{2i}\right) \tag{128}$$

And we have:

$$\sin_{(p,q)}(pMt) = \frac{e_{(p,q)}^{(ipMt)} - e_{(p,q)}^{(-ipMt)}}{2i} \tag{129}$$

Thus:

$$D_{(p,q)}\cos_{(p,q)}(Mt) = t\sin_{(p,q)}(pMt) \tag{130}$$

Hence proved.

Proposition 2.16. The (p,q)-derivatives of the (p,q)-hyperbolic matrix functions are:

$$\text{i.}\quad D_{(p,q)}\sinh_{(p,q)}(Mt) = t\cosh_{(p,q)}(pMt) \tag{131}$$

$$\text{ii.}\quad D_{(p,q)}\cosh_{(p,q)}(Mt) = t\sinh_{(p,q)}(pMt) \tag{132}$$

for a commutable square matrix M.

Proof: Using Proposition 2.14, we obtain:

i.

$$D_{(p,q)}^{M}\sinh_{(p,q)}(Mt) = D_{(p,q)}\left(\frac{e_{(p,q)}^{(Mt)} - e_{(p,q)}^{(-Mt)}}{2}\right) \tag{133}$$

This gives:

$$= \frac{1}{2}\left[D_{(p,q)}e_{(p,q)}^{(Mt)} - D_{(p,q)}e_{(p,q)}^{(-Mt)}\right] \tag{134}$$

Applying the derivative:

$$= \frac{1}{2}\left[te_{(p,q)}^{(pMt)} - (-t)e_{(p,q)}^{(-pMt)}\right] \tag{135}$$

This results in:

$$= t\left(\frac{e_{(p,q)}^{(pMt)} + e_{(p,q)}^{(-pMt)}}{2}\right) \tag{136}$$

But:

$$\cosh_{(p,q)}(pMt) = \frac{e_{(p,q)}^{(pMt)} + e_{(p,q)}^{(-pMt)}}{2} \tag{137}$$

Therefore:

$$D_{(p,q)}\sinh_{(p,q)}(Mt) = t\cosh_{(p,q)}(pMt) \tag{138}$$

ii.

$$D_{(p,q)}\cosh_{(p,q)}(Mt) = D_{(p,q)}\left(\frac{e_{(p,q)}^{(Mt)} + e_{(p,q)}^{(-Mt)}}{2}\right) \tag{139}$$

This simplifies to:

$$= \frac{1}{2}\left[D_{(p,q)}e_{(p,q)}^{(Mt)} + D_{(p,q)}e_{(p,q)}^{(-Mt)}\right] \tag{140}$$

Calculating the derivatives yields:

$$= \frac{1}{2}\left[te_{(p,q)}^{(pMt)} + te_{(p,q)}^{(-pMt)}\right] \tag{141}$$

Thus:

$$= t\left(\frac{e_{(p,q)}^{(pMt)} - e_{(p,q)}^{(-pMt)}}{2}\right) \tag{142}$$

And we know:

$$\sinh_{(p,q)}(pMt) = \frac{e_{(p,q)}^{(pMt)} - e_{(p,q)}^{(-pMt)}}{2} \tag{143}$$

Therefore:

$$D_{(p,q)}\cosh_{(p,q)}(Mt) = t\sinh_{(p,q)}(pMt) \tag{144}$$

Hence proved.

Proposition 2.17. Let A and B be commutable square matrices of the same size. The (p,q)-derivatives of the function $e_{(p,q)}^{(A+B)}$ are as follows:

$$\text{i.}\quad D_{(p,q)}^{A}e_{(p,q)}^{(A+B)} = e_{(p,q)}^{(A+B)} \tag{145}$$

$$\text{ii.}\quad D_{(p,q)}^{B}e_{(p,q)}^{(A+B)} = e_{(p,q)}^{(A+B)} \tag{146}$$

Proof: Using Proposition 2.8, we have:

$$D_{(p,q)}^{A}e_{(p,q)}^{(A+B)} = D_{(p,q)}^{A}\left(\sum_{n=0}^{\infty}\frac{(A+B)_{(p,q)}^{n}}{[n]_{(p,q)}!}\right) \tag{147}$$

This leads to:

$$= \sum_{n=0}^{\infty}\frac{D_{(p,q)}^{A}\left((A+B)_{(p,q)}^{n}\right)}{[n]_{(p,q)}!} \tag{148}$$

Applying the derivative, we write:

$$= \sum_{n=1}^{\infty}\frac{[n]_{(p,q)}(A+B)_{(p,q)}^{(n-1)}}{[n]_{(p,q)}!} \tag{149}$$

This simplifies to:

$$= \sum_{n=1}^{\infty}\frac{[n]_{(p,q)}(A+B)_{(p,q)}^{(n-1)}}{[n]_{(p,q)}[n-1]_{(p,q)}!} \tag{150}$$

Thus we have:

$$= \sum_{n=1}^{\infty} \frac{1}{[n-1]_{(p,q)}!}(A+B)_{(p,q)}^{(n-1)} \tag{151}$$

Shifting the index by letting $k = n-1$:

$$= \sum_{k=0}^{\infty} \frac{(A+B)_{(p,q)}^{k}}{[k]_{(p,q)}!} \tag{152}$$

Therefore:

$$D_{(p,q)}^{A} e_{(p,q)}^{(A+B)} = e_{(p,q)}^{(A+B)} \tag{153}$$

Proof of (ii): Using the same approach as Proposition 2.8, we get:

$$D_{(p,q)}^{B} e_{(p,q)}^{(A+B)} = D_{(p,q)}^{B} \left(\sum_{n=0}^{\infty} \frac{(A+B)_{(p,q)}^{n}}{[n]_{(p,q)}!} \right) \tag{154}$$

This leads to:

$$= \sum_{n=0}^{\infty} \frac{D_{(p,q)}^{B}\left((A+B)_{(p,q)}^{n}\right)}{[n]_{(p,q)}!} \tag{155}$$

Applying the derivative gives us:

$$= \sum_{n=1}^{\infty} \frac{[n]_{(p,q)}(A+B)_{(p,q)}^{(n-1)}}{[n]_{(p,q)}!} \tag{156}$$

This simplifies to:

$$= \sum_{n=1}^{\infty} \frac{[n]_{(p,q)}(A+B)_{(p,q)}^{(n-1)}}{[n]_{(p,q)}[n-1]_{(p,q)}!} \tag{157}$$

Thus,

$$= \sum_{n=1}^{\infty} \frac{1}{[n-1]_{(p,q)}!}(A+B)_{(p,q)}^{(n-1)} \tag{158}$$

Shifting the index by letting $k = n-1$:

$$= \sum_{k=0}^{\infty} \frac{(A+B)_{(p,q)}^{k}}{[k]_{(p,q)}!} \tag{159}$$

Therefore:

$$D_{(p,q)}^{B} e_{(p,q)}^{(A+B)} = e_{(p,q)}^{(A+B)} \tag{160}$$

Hence proved.

3 Conclusion

The results presented in this paper extend the (p, q) – calculus to matrices, bridging the gap between (p, q) – calculus and linear algebra. The extension enables potential applications in quantum computation, signal processing, and design of discrete dynamical systems. The findings established in this paper will pave the way for further exploration of the (p, q) – matrix functions and their benefits in mathematical analysis. Though these functions provide a significant framework related to quantum mechanics, more research is needed to enhance their practical impact.

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