

Generalized System of Variational Inequalities in Banach Spaces

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Abstract: In this paper, we introduce and consider a new system of extended general variational inequalities in Banach spaces. We establish the equivalence between the extended general variational inequalities and the fixed point problems. We use this equivalent formulation to suggest some iterative methods for solving this new system. We prove the convergence analysis of the proposed iterative methods under some suitable conditions. Several special cases, which can be obtained from main results are discussed. The idea and technique of this paper may stimulate further research activities in these fields.

Keywords: Sunny nonexpansive retraction, generalized variational inequalities system, q - uniformly smooth Banach spaces, iterative algorithm, relaxed cocoercive mappings.

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1 Introduction

Variational inequality and complementarity problems are of fundamental importance in a wide range of mathematical and applied sciences problems, such as mathematical programming, traffic engineering, economics and equilibrium problems, see [1–18]. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. The fixed-point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [6] to study the existence of a solution of the variational inequalities. Projection methods and its variant forms represent important tools for finding the approximate solution of variational inequalities. We now have a variety of techniques to suggest and analyze

various iterative algorithms for solving variational inequalities and the related optimization problems. In recent years, some new and interesting problems, which are called the system of variational inclusions were introduced and studied. Chang et al. [2], Huang and Noor [5], Noor and Noor [10], Noor [9], Verma [18, 19, 20] and Yang et al. [22] introduced and studied a system of variational inclusions involving four, three, two different nonlinear operators.

Inspired and motivated by research going on in this area, we introduce and consider a new system of extended general variational inequalities involving six different nonlinear operators in Banach spaces. We establish the equivalence between this system of extended general variational inequalities and the fixed point problems using the projection operator technique. This equivalent formulation is used to suggest and analyze some new iterative methods for solving the extended general variational inequalities. We also prove the convergence analysis of the proposed algorithm under some suitable mild conditions. Since this class of systems includes the system of variational inequalities involving four, three,

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two operators and the classical variational inequalities as special cases, results obtained in this paper continue to hold for these problems. It is expected that these results may inspire and motivate others to find novel and innovative applications in various branches of pure and applied sciences.

The readers are encouraged to explore the novel and innovative applications of the system of extended general variational inequalities and its variant forms in different areas of pure and applied sciences.

2 Preliminaries

Throughout this article, let X be a real Banach space with its dual space X^* . We usually use $\langle \cdot, \cdot \rangle$ to denote the pairing between X and X^* , and 2^X denote the family of all the nonempty subsets of X . The generalized duality mapping $J_q(x) : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $q > 1$ is a constant. In particular, $J_2 = J$ is the usual normalized duality mapping. It is known that, in general, $J_q = \|x\|^{q-2}J_2$, for all $x \neq 0$, and $J_q(x)$ is single-valued if X^* is strictly convex. If $X = H$ is a Hilbert space then J_2 becomes the identity mapping of H . Note that X is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of X .

Let K be a nonempty, closed and convex subset of X . A mapping $Q : X \rightarrow K$ is said to be sunny if

$$Q(Q(x) + t(x - Q(x))) = Q(x), \forall x \in X, \forall t \geq 0.$$

A mapping $Q : X \rightarrow K$ is said to be a retraction or a projection if $Q(x) = x; \forall x \in K$. If X is smooth then the sunny nonexpansive retraction of X onto K is uniquely decided (see [4]). Using $J_q = \|x\|^{q-2}J_2$, replace J by J_q , it is easy to obtain the following results by Bruck [1], Goebel and Reich [3].

Proposition 2.1 Let X be a q -uniformly smooth Banach spaces and let K be a nonempty subset of X . Let $Q_K : X \rightarrow K$ be a retraction and let J_q be the generalized duality mapping on X . Then the following are equivalent:

- Q_K is sunny and nonexpansive.
- $\|Q_K(x) - Q_K(y)\|^2 \leq \langle x - y, J_q(Q_K(x) - Q_K(y)) \rangle, \forall x, y \in X$.
- $\langle x - Q_K(x), J_q(Q_K(x) - y) \rangle \geq 0, \forall y \in K$.

Lemma 2.1 (See [21]) Let X be a real q -uniformly smooth Banach space ($q > 1$), then there exists a constant $c_q > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q x \rangle + c_q \|y\|^q, \forall x, y \in X.$$

In particular, if X be a real 2-uniformly smooth Banach space, then there exists a constant $c_2 > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J_2 x \rangle + c_2 \|y\|^2, \forall x, y \in X.$$

Let K be a nonempty, closed and convex subset of X . For given nonlinear operators $T_i(\cdot, \cdot) : X \times X \rightarrow X$ and $g_i, h_i : X \rightarrow X (i = 1, 2)$, we consider problem of finding $(x^*, y^*) \in X \times X : (h_1(x^*), h_2(y^*)) \in K \times K$ such that

$$\left. \begin{aligned} &\langle \rho T_1(y^*, x^*) + h_1(x^*) - g_1(y^*), \\ &J_q(g_1(x) - h_1(x^*)) \rangle \geq 0; \forall x \in X, g_1(x) \in K \text{ and } \rho > 0, \\ &\langle \eta T_2(x^*, y^*) + h_2(y^*) - g_2(x^*), \\ &J_q(g_2(x) - h_2(y^*)) \rangle \geq 0; \forall x \in X, g_2(x) \in K \text{ and } \eta > 0. \end{aligned} \right\} \quad (1)$$

which is called the system of extended general variational inequalities.

We now discuss some applications of the system of extended general variational inequalities (1).

If $X = H$ is a Hilbert space, $J_q = I, h_i = I$, the identity operator, then problem (1) reduces to finding $(x^*, y^*) \in K \times K$ such that

$$\left. \begin{aligned} &\langle \rho T_1(y^*, x^*) + x^* - g_1(y^*), \\ &g_1(x) - x^* \rangle \geq 0; \\ &\forall x \in H, g_1(x) \in K \text{ and } \rho > 0, \\ &\langle \eta T_2(x^*, y^*) + y^* - g_2(x^*), g_2(x) - y^* \rangle \geq 0; \\ &\forall x \in H, g_2(x) \in K \text{ and } \eta > 0, \end{aligned} \right\} \quad (2)$$

which were introduced and studied by Noor and Noor [10].

If $g_i = g(i = 1, 2)$, then the system (2) is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\left. \begin{aligned} &\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^* \rangle \geq 0; \\ &\forall x \in H, g(x) \in K \text{ and } \rho > 0, \\ &\langle \eta T_2(x^*, y^*) + y^* - g(x^*), g(x) - y^* \rangle \geq 0; \\ &\forall x \in H, g(x) \in K \text{ and } \eta > 0, \end{aligned} \right\} \quad (3)$$

which was considered and investigated by Noor [9].

If $X = H$ is a Hilbert space, $J_q = I, T_i(x, y) = T_i(x)$, $h_i = g_i = g(i = 1, 2)$, then problem (1) reduces to finding $(x^*, y^*) \in H \times H : (g(x^*), g(y^*)) \in K \times K$ such that

$$\left. \begin{aligned} &\langle \rho T_1(y^*) + g(x^*) - g(y^*), g(x) - g(x^*) \rangle \geq 0; \\ &\forall x \in H, g(x) \in K \text{ and } \rho > 0, \\ &\langle \eta T_2(x^*) + g(y^*) - g(x^*), g(x) - g(y^*) \rangle \geq 0; \\ &\forall x \in H, g_2(x) \in K \text{ and } \eta > 0, \end{aligned} \right\} \quad (4)$$

which has been introduced and studied by Yang et al. [22].

If $X = H$ is a Hilbert space, $x^* = y^*, J_q = I, T_i(x, y) = T_1(x) = T_2(x) (i = 1, 2), h_1 = h_2, g_1 = g_2$, then problem (1) reduces to finding $x^* \in H : h_1(x^*) \in K \times K$ such that

$$\langle \rho T_1(x^*) + h_1(x^*) - g_1(x^*), g_1(x) - h_1(x^*) \rangle \geq 0; \quad (5)$$

$\forall x \in H : g_1(x) \in K$ and $\rho > 0$, which is called the extended general variational inequality, introduced and studied by Noor [11] in 2009. For the applications, numerical results, generalizations and other aspects of extended general variational inequalities, see [11, 12, 13, 14] and the references therein.

If $g = I$, then problem (3) reduces of finding $(x^*, y^*) \in K \times K$ such that

$$\left. \begin{aligned} \langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \rho &> 0, \\ \langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \eta &> 0, \end{aligned} \right\} \quad (6)$$

which has been considered and studied by Huang and Noor [5].

If $T_1 = T_2 = T$, then problem (6) reduces of finding $(x^*, y^*) \in K \times K$ such that

$$\left. \begin{aligned} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \rho &> 0, \\ \langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \eta &> 0, \end{aligned} \right\} \quad (7)$$

The system (7) has been studied and investigated by Chang et al. [2] and Verma [19].

If $T_1 = T_2 = T$, and $g = I$, then problem (4) reduces of finding $(x^*, y^*) \in K \times K$ such that

$$\left. \begin{aligned} \langle \rho T(y^*) + x^* - y^*, x - x^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \rho &> 0, \\ \langle \eta T(x^*) + y^* - x^*, x - y^* \rangle &\geq 0; \\ \forall x \in K \quad \text{and} \quad \eta &> 0, \end{aligned} \right\} \quad (8)$$

which has been introduced and studied by Verma [18,20].

If $x^* = y^*$, then problem (8) collapses to finding $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \quad (9)$$

Inequality of type (9) is called variational inequality, which was introduced and studied by Stampacchia [17] in 1964. The system of extended general variational inequalities (1) includes several classes of variational inequalities and related optimization problems as special cases. For the recent research, see [1-18] and the references therein.

We now recall some well-known results and concepts, which are needed.

Definition 2.1 A mapping $h : X \rightarrow X$ is called

- (a) r -strongly monotone if, there exists a constant $r > 0$ such that

$$\langle h(x) - h(y), J_q(x - y) \rangle \geq r \|x - y\|^2, \forall x, y \in X;$$

- (b) (ξ, ς) -relaxed cocoercive if, there exist constants $\xi, \varsigma > 0$ such that

$$\langle h(x) - h(y), J_q(x - y) \rangle \geq -\xi \|h(x) - h(y)\|^2 + \varsigma \|x - y\|^2, \forall x, y \in X;$$

- (c) γ -Lipschitz continuous if, there exists a constant $\gamma > 0$ such that

$$\|h(x) - h(y)\| \leq \gamma \|x - y\|, \forall x, y \in X.$$

Definition 2.2 Let $T : X \times X \rightarrow X$ and $g : X \rightarrow X$. Then T is called

- (a) r -strongly monotone with respect to g if, there exists a constant $r > 0$ such that

$$\langle T(x_1, y) - T(x_2, y), J_q(g(x_1) - g(x_2)) \rangle \geq r \|x_1 - x_2\|^2, \forall x_1, x_2, y \in X;$$

- (b) (ξ, ς) -relaxed cocoercive with respect to g if, there exist constants $\xi, \varsigma > 0$ such that

$$\langle T(x_1, y) - T(x_2, y), J_q(g(x_1) - g(x_2)) \rangle \geq -\xi \|T(x_1, y) - T(x_2, y)\|^2 + \varsigma \|x_1 - x_2\|^2, \forall x_1, x_2, y \in X;$$

- (c) γ -Lipschitz continuous in the first variable if, there exists a constant $\gamma > 0$ such that

$$\|T(x_1, y) - T(x_2, y)\| \leq \gamma \|x_1 - x_2\|, \forall x_1, x_2, y \in X.$$

- (d) γ -Lipschitz continuous in the second variable if, there exists a constant $\gamma > 0$ such that

$$\|T(x, y_1) - T(x, y_2)\| \leq \gamma \|y_1 - y_2\|, \forall x, y_1, y_2 \in X.$$

3 Main Results

In this section, we first establish the equivalence between the system of the extended general variational inequality (1) and the fixed point problems. Then by using the obtained fixed point formulation, we construct a new iterative algorithm for solving the systems (1).

Lemma 3.1 Let X be a smooth Banach space, for given nonlinear operators $T_i : X \times X \rightarrow X, g_i, h_i : X \rightarrow X (i = 1, 2)$, and $\rho, \eta > 0$. Then the point $(x^*, y^*) \in K \times K$ is a solution of (1) if and only if

$$\left. \begin{aligned} h_1(x^*) &= Q_K[g_1(y^*) - \rho T_1(y^*, x^*)] \\ h_2(x^*) &= Q_K[g_2(x^*) - \eta T_2(x^*, y^*)]. \end{aligned} \right\} \quad (10)$$

Proof. The first variational inequality of (1) can be written as follows:

$$\langle [g_1(y^*) - \rho T_1(y^*, x^*)] - h_1(x^*), J_q(h_1(x^*) - g_1(x)) \rangle \geq 0; \quad \forall x \in X, g_1(x) \in K \quad \text{and} \quad \rho > 0.$$

We can deduce from Proposition 2.1 (c) that the above inequality is equivalent to

$$h_1(x^*) = Q_K[g_1(y^*) - \rho T_1(y^*, x^*)].$$

Similar, the second variational inequality of (1) is equivalent to

$$h_2(x^*) = Q_K[g_2(x^*) - \eta T_2(x^*, y^*)]. \quad \square$$

Lemma 3.1 implies that the system of extended general variational inequalities (1) and the fixed point problems

(10) are equivalent. This equivalent formulation plays a crucial role in developing the iterative methods for solving (1). We rewrite (10) in the following form:

$$\begin{cases} h_1(x) = Q_K[z] \\ h_2(y) = Q_K[t] \\ z = (1 - \alpha)z + \alpha[g_1(y) - \rho T_1(y, x)] \\ t = (1 - \alpha)t + \alpha[g_2(x) - \eta T_2(x, y)], \end{cases}$$

which enables to suggest the following iterative method,

Algorithm 3.1 Let $T_i, g_i, h_i (i = 1, 2)$, be nonlinear operators, $\rho, \eta > 0$. For arbitrary chosen initial points $z^0, t^0 \in K$, compute the iterative sequence $\{(x^k, y^k)\}$ by using

$$\begin{cases} h_1(x^k) = Q_K[z^k] \\ h_2(y^k) = Q_K[t^k] \\ z^{k+1} = (1 - \alpha)z^k + \alpha[g_1(y^k) - \rho T_1(y^k, x^k)] \\ t^{k+1} = (1 - \alpha)t^k + \alpha[g_2(x^k) - \eta T_2(x^k, y^k)], \end{cases} \quad (11)$$

where Q_K is the sunny nonexpansive retraction of X onto K and $0 < \alpha \leq 1$.

We now establish the strongly convergence of the sequences generated by Algorithm 3.1.

Theorem 3.1 Let $T_i, g_i, h_i (i = 1, 2)$, be nonlinear operators, such that for each $i = 1, 2$,

- (a) T_i is (κ_i, θ_i) -relaxed cocoercive with respect to g_i and γ_i -Lipschitz continuous in the first variable, and μ_i -Lipschitz continuous in the second variable;
- (b) h_i is (ξ_i, v_i) -relaxed cocoercive and δ_i -Lipschitz continuous;
- (c) g_i is σ_i -Lipschitz continuous.

If there exist constants $\rho, \eta > 0$ such that

$$\left. \begin{aligned} & \sqrt[q]{(\sigma_2^q - \eta q \theta_2) + (\eta q \kappa_2 + C_q \eta^q) \gamma_2^q} \\ & < 1 - k_1 - \rho \mu_1, k_1 + \rho \mu_1 < 1 \\ & \sqrt[q]{((\sigma_1^q - \rho q \theta_1) + (\rho q \kappa_1 + C_q \rho^q) \gamma_1^q)} \\ & < 1 - k_2 - \eta \mu_2, k_2 + \eta \mu_2 < 1, \\ & \text{and} \\ & q v_i < 1 + (C_q + q \xi_i) \delta_i^q, (i = 1, 2), \\ & \text{where} \\ & k_i = \sqrt[q]{(1 - q v_i) + (C_q + q \xi_i) \delta_i^q}, (i = 1, 2), \end{aligned} \right\} \quad (12)$$

then for arbitrarily chosen initial points $x_0, y_0 \in K$, x^k and y^k obtained from Algorithm 3.1 converge strongly to x^* and y^* respectively.

Proof. It follows from (11) that

$$\begin{aligned} & \|z^{k+1} - z^k\| \\ &= \|(1 - \alpha)z^k + \alpha[g_1(y^k) - \rho T_1(y^k, x^k)] - (1 - \alpha)z^{k-1} \\ & \quad - \alpha[g_1(y^{k-1}) - \rho T_1(y^{k-1}, x^{k-1})]\| \\ &\leq (1 - \alpha)\|z^k - z^{k-1}\| + \alpha\|g_1(y^k) - g_1(y^{k-1}) \\ & \quad - \rho(T_1(y^k, x^k) - T_1(y^{k-1}, x^{k-1}))\| \end{aligned} \quad (13)$$

From T_1 μ_1 -Lipschitz continuous in the second variable, we have

$$\|g_1(y^k) - g_1(y^{k-1}) - \rho(T_1(y^k, x^k) - T_1(y^{k-1}, x^{k-1}))\|$$

$$\begin{aligned} &\leq \|g_1(y^k) - g_1(y^{k-1}) - \rho(T_1(y^k, x^k) - T_1(y^{k-1}, x^k))\| \\ & \quad + \rho\|T_1(y^{k-1}, x^k) - T_1(y^{k-1}, x^{k-1})\| \\ &\leq \|g_1(y^k) - g_1(y^{k-1}) - \rho(T_1(y^k, x^k) - T_1(y^{k-1}, x^k))\| \\ & \quad + \rho\mu_1\|x^k - x^{k-1}\| \end{aligned} \quad (14)$$

Since T_1 is (κ_1, θ_1) -relaxed cocoercive with respect to g_1 and γ_1 -Lipschitz continuous in the first variable, and g_1 is σ_1 -Lipschitz continuous, we can conclude that

$$\begin{aligned} & \|g_1(y^k) - g_1(y^{k-1}) - \rho(T_1(y^k, x^k) - T_1(y^{k-1}, x^k))\|^q \\ &= \|g_1(y^k) - g_1(y^{k-1})\|^q - \rho q \langle T_1(y^k, x^k) - T_1(y^{k-1}, x^k), \\ & \quad J_q(g_1(y^k) - g_1(y^{k-1})) \rangle \\ & \quad + C_q \rho^q \|T_1(y^k, x^k) - T_1(y^{k-1}, x^k)\|^q \\ &\leq \|g_1(y^k) - g_1(y^{k-1})\|^q \\ & \quad + \rho q \kappa_1 \|T_1(y^k, x^k) - T_1(y^{k-1}, x^k)\|^q - \rho q \theta_1 \|y^k - y^{k-1}\|^q \\ & \quad + C_q \rho^q \|T_1(y^k, x^k) - T_1(y^{k-1}, x^k)\|^q \\ &\leq ((\sigma_1^q - \rho q \theta_1) + (\rho q \kappa_1 + C_q \rho^q) \gamma_1^q) \|y^k - y^{k-1}\|^q \end{aligned} \quad (15)$$

Substituting (14) and (15) in (13), we get

$$\begin{aligned} & \|z^{k+1} - z^k\| \\ &\leq (1 - \alpha)\|z^k - z^{k-1}\| + \alpha \sqrt[q]{((\sigma_1^q - \rho q \theta_1) \\ & \quad + (\rho q \kappa_1 + C_q \rho^q) \gamma_1^q)} \|y^k - y^{k-1}\| \\ & \quad + \alpha \rho \mu_1 \|x^k - x^{k-1}\|. \end{aligned} \quad (16)$$

In a similar way, we can prove that

$$\begin{aligned} & \|t^{k+1} - t^k\| \\ &\leq (1 - \alpha)\|t^k - t^{k-1}\| \\ & \quad + \alpha \sqrt[q]{((\sigma_2^q - \eta q \theta_2) + (\eta q \kappa_2 + C_q \eta^q) \gamma_2^q)} \|x^k - x^{k-1}\| \\ & \quad + \alpha \eta \mu_2 \|y^k - y^{k-1}\|. \end{aligned} \quad (17)$$

On the other hand, by using (11), we find that

$$\begin{aligned} & \|x^k - x^{k-1}\| \\ &\leq \|x^k - x^{k-1} - (h_1(x^k) - h_1(x^{k-1}))\| \\ & \quad + \|a_k(z^k) - Q_k(x^{k-1})\| \\ &\leq \|x^k - x^{k-1} - (h_1(x^k) - h_1(x^{k-1}))\| + \|z^k - z^{k-1}\|. \end{aligned} \quad (18)$$

From (ξ_1, v_1) -relaxed cocoercive and δ_1 -Lipschitz continuous of h_1 , we have

$$\begin{aligned} & \|x^k - x^{k-1} - (h_1(x^k) - h_1(x^{k-1}))\|^q \\ &\leq \|x^k - x^{k-1}\|^q - q \langle h_1(x^k) - h_1(x^{k-1}), J_q(x^k - x^{k-1}) \rangle \\ & \quad + C_q \|h_1(x^k) - h_1(x^{k-1})\|^q \\ &\leq \|x^k - x^{k-1}\|^q + q \xi_1 \|h_1(x^k) - h_1(x^{k-1})\|^q \\ & \quad - q v_1 \|x^k - x^{k-1}\|^q \\ & \quad + C_q \|h_1(x^k) - h_1(x^{k-1})\|^q \\ &\leq ((1 - q v_1) + (C_q + q \xi_1) \delta_1^q) \|x^k - x^{k-1}\|^q \end{aligned} \quad (19)$$

Substituting (19) in (18), we get

$$\|x^k - x^{k-1}\| \leq \frac{1}{1 - \sqrt[q]{(1 - qv_1) + (C_q + q\xi_1)\delta_1^q}} \|z^k - z^{k-1}\| \quad (20)$$

Similarly, we can prove that

$$\|y^k - y^{k-1}\| \leq \frac{1}{1 - \sqrt[q]{(1 - qv_2) + (C_q + q\xi_2)\delta_2^q}} \|t^k - t^{k-1}\|. \quad (21)$$

From (16), (17), (20) and (21), it follows that

$$\begin{aligned} & \|z^{k+1} - z^k\| \\ & \leq (1 - \alpha(1 - \frac{\rho\mu_1}{1 - \sqrt[q]{(1 - qv_1) + (C_q + q\xi_1)\delta_1^q}})) \\ & \quad \times \|z^k - z^{k-1}\| \\ & \quad + \frac{\alpha\sqrt[q]{((\sigma_1^q - \rho q\theta_1) + (\rho q\kappa_1 + C_q\rho^q)\gamma_1^q)}}{1 - \sqrt[q]{(1 - qv_2) + (C_q + q\xi_2)\delta_2^q}} \\ & \quad \times \|t^k - t^{k-1}\| \end{aligned} \quad (22)$$

and

$$\begin{aligned} & \|t^{k+1} - t^k\| \\ & \leq (1 - \alpha(1 - \frac{\eta\mu_2}{1 - \sqrt[q]{(1 - qv_2) + (C_q + q\xi_2)\delta_2^q}})) \\ & \quad \times \|t^k - t^{k-1}\| \\ & \quad + \frac{\alpha\sqrt[q]{((\sigma_2^q - \eta q\theta_2) + (\eta q\kappa_2 + C_q\eta^q)\gamma_2^q)}}{1 - \sqrt[q]{(1 - qv_1) + (C_q + q\xi_1)\delta_1^q}} \\ & \quad \times \|z^k - z^{k-1}\| \end{aligned} \quad (23)$$

Now, we define $\|\cdot\|_*$ on $X \times X$ by $\|(x, y)\|_* = \|x\| + \|y\|$, for all $(x, y) \in X \times X$. From (22) and (23), it follows that

$$\begin{aligned} & \|(z^{k+1}, t^{k+1}) - (z^k, t^k)\|_* \\ & \leq (1 - \alpha)\|(z^k, t^k) - (z^{k-1}, t^{k-1})\|_* \\ & \quad + \alpha\Lambda\|(z^k, t^k) - (z^{k-1}, t^{k-1})\|_* \end{aligned} \quad (24)$$

where

$$\Lambda = \max \left(\frac{\rho\mu_1 + \sqrt[q]{((\sigma_2^q - \eta q\theta_2) + (\eta q\kappa_2 + C_q\eta^q)\gamma_2^q)}}{1 - \sqrt[q]{(1 - qv_1) + (C_q + q\xi_1)\delta_1^q}}, \frac{\eta\mu_2 + \sqrt[q]{((\sigma_1^q - \rho q\theta_1) + (\rho q\kappa_1 + C_q\rho^q)\gamma_1^q)}}{1 - \sqrt[q]{(1 - qv_2) + (C_q + q\xi_2)\delta_2^q}} \right)$$

In view of the condition (12), we know that $0 \leq \Lambda < 1$. Thus it follows from (24) that, for each $k \geq k_0$,

$$\begin{aligned} & \|(z^{k+1}, t^{k+1}) - (z^k, t^k)\|_* \\ & \leq (1 - \alpha(1 - \Lambda))\|(z^k, t^k) - (z^{k-1}, t^{k-1})\|_* \\ & \leq (1 - \alpha(1 - \Lambda))^2\|(z^{k-1}, t^{k-1}) - (z^{k-2}, t^{k-2})\|_* \\ & \vdots \\ & \leq (1 - \alpha(1 - \Lambda))^{k-k_0}\|(z^{k_0+1}, t^{k_0+1}) - (z^{k_0}, t^{k_0})\|_*. \end{aligned}$$

Hence, for any $m \geq n > k_0$, we have

$$\begin{aligned} & \|(z^m, t^m) - (z^n, t^n)\|_* \\ & \leq \sum_{j=n}^{m-1} \|(z^{j+1}, t^{j+1}) - (z^j, t^j)\|_* \\ & \leq \sum_{j=n}^{m-1} (1 - \alpha(1 - \Lambda))^{j-k_0} \|(z^{k_0+1}, t^{k_0+1}) - (z^{k_0}, t^{k_0})\|_*. \end{aligned} \quad (25)$$

Since $1 - \alpha(1 - \Lambda) < 1$, it follows from (25) that $\|(z^m, t^m) - (z^n, t^n)\|_* = \|z^m - z^n\| + \|t^m - t^n\| \rightarrow 0$, as $n \rightarrow \infty$. Hence $\{z^n\}, \{t^n\}$ are both Cauchy sequences in K and so there exist $z^*, t^* \in K$ such that $z^n \rightarrow z^*$ and $t^n \rightarrow t^*$ as $n \rightarrow \infty$. From (20) and (21) it follows that the sequences $\{x^n\}$ and $\{y^n\}$ are also both Cauchy sequences. Thus there exist $x^*, y^* \in K$ such that $x^n \rightarrow x^*$ and $y^n \rightarrow y^*$ as $n \rightarrow \infty$.

Since the mappings g_1, g_2, h_1, h_2 and Q_K are continuous, it follows from (11) that

$$h_1(x^*) = Q_K[g_1(y^*) - \rho T_1(y^*, x^*)]$$

and

$$h_2(x^*) = Q_K[g_2(x^*) - \eta T_2(x^*, y^*)].$$

Now, Lemma 3.1 guarantees that (x^*, y^*) is a solution of the problem (1). This completes the proof. \square

Theorem 3.2 Let $T_i, g_i, h_i (i = 1, 2)$ be nonlinear operators, such that for each $i = 1, 2$,

- (a) T_i is θ_i -strongly monotone with respect to g_i and γ_i -Lipschitz continuous in the first variable, and μ_i -Lipschitz continuous in the second variable;
- (b) h_i is v_i -strongly monotone and δ_i -Lipschitz continuous;
- (c) g_i is σ_i -Lipschitz continuous.

If there exist constants $\rho, \eta > 0$ such that

$$\begin{cases} \sqrt[q]{(\sigma_2^q - \eta q\theta_2) + C_q\eta^q\gamma_2^q} < 1 - k_1 - \rho\mu_1, k_1 + \rho\mu_1 < 1 \\ \sqrt[q]{((\sigma_1^q - \rho q\theta_1) + C_q\rho^q\gamma_1^q)} < 1 - k_2 - \eta\mu_2, k_2 + \eta\mu_2 < 1, \\ \text{and} \\ qv_i < 1 + C_q\delta_i^q, (i = 1, 2), \\ \text{where} \\ k_i = \sqrt[q]{1 - (qv_i - C_q\delta_i^q)}, (i = 1, 2), \end{cases}$$

then for arbitrarily chosen initial points $x_0, y_0 \in K$, x^k and y^k obtained from Algorithm 3.1 converge strongly to x^* and y^* respectively.

4 Conclusion

A new system of extended general variational inequalities in Banach spaces is introduced and studied. It is shown that the new system is equivalent to the system of fixed point problems. This alternative equivalent formulation is used to suggest an iterative method for solving system along with convergence criteria. Some special cases are studied. Results proved in this paper continue for these known and new systems. The implementation and comparison of these methods with other methods is a subject of the future research.

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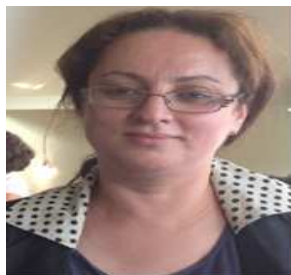
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