

Stress Strength Reliability Modeling with the Power Garhy Distribution under Progressive Censoring: A Comparative Bayesian and Classical Study

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Abstract: This study introduces a new two-parameter lifetime model, the power Garhy distribution (PGD), obtained by applying the power transformation to the Garhy distribution (GD). Closed-form expressions are derived for the quantile function (through the generalized Lambert W function), Bowley's skewness, Moors' kurtosis, the raw moments, the Shannon entropy and the stress–strength reliability $R = P(Y_2 < Y_1)$. The parameters and R are estimated by maximum likelihood and Bayesian methods under progressive Type-II censoring. A Monte Carlo study shows that, in most settings, the maximum likelihood estimators have smaller bias and mean squared error and coverage closer to the nominal level, whereas the Bayesian estimators are more stable in very small samples, where the likelihood surface is flat. The effects of the censoring scheme and of the prior are also examined. Two real data sets (vinyl chloride concentrations and bank waiting times) show that the PGD provides a competitive fit compared with the GD and several power-transformed models.

Keywords: Garhy distribution; power transformation; stress–strength reliability; progressive Type-II censoring; Bayesian estimation; simulation

1 Introduction

Probability distributions are important tools for explaining and forecasting real-world phenomena in a variety of disciplines. Choosing suitable distributions for modeling real-world data is essential for making sensible decisions in disciplines such as environmental policy and others with substantial effects. Inversely, selecting inappropriate distributions can result in incorrect inferences and undesirable actions.

Environmental and biomedical science datasets frequently contain complicated properties such as skewness, large tails, and high variability. Environmental data, for example, can include measurements of groundwater pollutant levels and soil contaminant concentrations, which can have high values or asymmetric distributions. Similarly, biomedical data, such as disease severity scores, also show high variation and nonstandard patterns. This complexity also exists in financial and engineering databases. Such complicated properties present considerable problems for classical probability models, which may lack the flexibility required to effectively represent the underlying patterns in these datasets.

To solve these difficulties, researchers created more adaptable distributions by modifying existing ones with a variety of strategies. One well-known strategy is the power transformation technique, which adds a shape parameter to improve the

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adaptability of a baseline distribution. This technique greatly increases the flexibility of the baseline distribution, allowing for more detailed modeling of complicated data behaviors.

Scientists have recently created a variety of power-transformed distributions to better depict the complexity of real-world data. As an example, the power Lindley distribution was studied by [1], power Lomax distribution was discussed by [2], power half-logistic distribution was investigated by [3], power Shanker distribution was investigated by [4], power Gompertz distribution was studied by [5], power Pranav distribution was discussed by [6], power Prakaamy distribution was introduced by [7], power Topp-Leone distribution was proposed by [8], power Hamza distribution was discussed by [9], power-modified kies-exponential distribution was studied by [10], power Zeghdoudi distribution was proposed by [12], power Darna distribution was discussed by [13], power Komal distribution was studied by [14], power modified XLindley distribution was proposed by [15], power new XLindley distribution was investigated by [16], and power length-biased new xLindley distribution studied by [17].

Most of these models are power transformations of one-parameter Lindley-type mixtures. Their common limitation is that the baseline mixture has fixed mixing weights, or weights that depend on the scale parameter only through simple forms, so the resulting power models mainly differ in how much mass they place near the origin. For several of them, reliability measures such as the stress–strength reliability are available only in special cases or are evaluated by numerical integration, and the quantile function is often left implicit. The GD is a mixture of $\text{Gamma}(2, \alpha)$ and $\text{Gamma}(3, \alpha)$ components with weight $\alpha^3/(6 + \alpha^3)$, so it has no exponential component, its density vanishes at the origin and it is always unimodal. These features make it a natural baseline for data with a clearly positive mode, such as waiting times and concentrations above a detection limit. Applying the power transformation keeps the gamma-mixture structure of Y^δ , which is what yields the closed-form moments, the entropy and a closed-form stress–strength reliability in the equal-shape case, while the shape parameter δ controls tail weight and skewness.

Ref. [18] introduced a new one parameter distribution called the Garhy distribution (GD) as a mixture of $\text{Gamma}(2, \alpha)$, $\text{Gamma}(3, \alpha)$ and $w = \frac{\alpha^3}{6 + \alpha^3}$. Then, the probability density function (PDF) and cumulative distribution function (CDF) of the GD are given by

$$g(x; \alpha) = \frac{\alpha^3 x}{6 + \alpha^3} (\alpha^2 + 3x) e^{-\alpha x} \quad x > 0, \alpha > 0, \quad (1)$$

and

$$G(x; \alpha) = 1 - \left[\frac{3\alpha^2 x^2}{6 + \alpha^3} + (\alpha x + 1) \right] e^{-\alpha x} \quad x > 0, \alpha > 0. \quad (2)$$

The GD has recently attracted attention in reliability analysis: its stress–strength reliability was studied under different record schemes [19], and Markov chain Monte Carlo and Tierney–Kadane methods were developed for progressively censored Garhy reliability models [20]. More generally, stress–strength reliability has been estimated for several lifetime models under progressive, hybrid and record-based sampling [21–23], step-stress models have been analysed under progressive Type-II censoring [24], and new flexible lifetime models continue to be proposed for engineering, medical and actuarial data [25–27].

This research proposes an extension of the GD using the power transformation approach. The resulting model, called the power Garhy distribution (PGD), is intended to increase the flexibility and applicability of the original distribution. To our knowledge, this is the first study of the PGD. We are particularly interested in it for the following reasons:

1. It is quite simple, with only two parameters: a scale-type parameter α and a shape parameter δ .
2. The PDF of the PGD can be decreasing (reverse-J) or unimodal, and it is right-skewed. The hazard rate function (HRF) can be increasing with different curvatures.
3. Some important statistical properties for the PGD such as; quantile function, Bowley's skewness and Moors' kurtosis, the raw moments are subsequently derived, producing fundamental descriptive statistics including mean, variance, Shannon entropy, and stress–strength reliability measure are calculated.
4. Bayesian and non-Bayesian estimation approaches can be applied to the PGD. In addition, classical and Bayesian approaches are used to estimate the stress–strength model for the PGD under progressive censored samples.

5. The PGD provides a competitive fit to positive real data. This is illustrated with two real data sets, where the PGD is compared with the GD, its one-parameter Lindley-type relatives and several power-transformed models.

The rest of the article is organized as follows. The construction of the PGD is addressed in Section 2. Its main statistical properties are derived in Section 3. Maximum likelihood and Bayesian estimation of the parameters and of the stress-strength reliability under progressive Type-II censoring are presented in Section 4. Section 5 reports the Monte Carlo simulation results. Section 6 illustrates the applicability of the PGD using two real data sets, and Section 7 concludes the paper.

2 Construction of Power Garhy Distribution

In this section, we introduce the new two-parameter lifetime distribution called the power Garhy distribution (PGD) with an additional shape parameter to improve the characteristics of the Garhy distribution. Using the transformation $Y = X^{1/\delta}$ where X has the CDF (2) and δ is the shape parameter. Then, the CDF and PDF of the PGD are given by

$$F(y; \alpha, \delta) = 1 - \left[\frac{3\alpha^2 y^{2\delta}}{6 + \alpha^3} + (\alpha y^\delta + 1) \right] e^{-\alpha y^\delta} \quad y > 0, \alpha, \delta > 0, \quad (3)$$

and

$$f(y; \alpha, \delta) = \frac{\delta \alpha^3 y^{2\delta-1}}{6 + \alpha^3} (\alpha^2 + 3y^\delta) e^{-\alpha y^\delta} \quad y > 0, \alpha, \delta > 0. \quad (4)$$

The survival function (SF), hazard rate function (HRF), reversed HRF and cumulative HRF of PGD are provided via

$$S(y; \alpha, \delta) = \left[\frac{3\alpha^2 y^{2\delta}}{6 + \alpha^3} + (\alpha y^\delta + 1) \right] e^{-\alpha y^\delta} \quad y > 0, \alpha, \delta > 0,$$

$$h(y; \alpha, \delta) = \frac{\frac{\delta \alpha^3 y^{2\delta-1}}{6 + \alpha^3} (\alpha^2 + 3y^\delta)}{\left[\frac{3\alpha^2 y^{2\delta}}{6 + \alpha^3} + (\alpha y^\delta + 1) \right]} \quad y > 0, \alpha, \delta > 0,$$

$$\tau(y; \alpha, \delta) = \frac{\frac{\delta \alpha^3 y^{2\delta-1}}{6 + \alpha^3} (\alpha^2 + 3y^\delta) e^{-\alpha y^\delta}}{1 - \left[\frac{3\alpha^2 y^{2\delta}}{6 + \alpha^3} + (\alpha y^\delta + 1) \right] e^{-\alpha y^\delta}} \quad y > 0, \alpha, \delta > 0,$$

and

$$H(y; \alpha, \delta) = \log[6 + \alpha^3] - \log \left[3\alpha^2 y^{2\delta} + (\alpha y^\delta + 1)(6 + \alpha^3) \right] - \alpha y^\delta \quad y > 0, \alpha, \delta > 0.$$

The survival function in (3) and $S(y; \alpha, \delta)$ above are written in the same form, $S = 1 - F$, with the common term $3\alpha^2 y^{2\delta} / (6 + \alpha^3)$.

A useful representation follows directly from (4). If $Y \sim \text{PGD}(\alpha, \delta)$, then $T = Y^\delta$ has the density $kg_2(t) + (1 - k)g_3(t)$, where g_j is the Gamma(j, α) density and $k = \alpha^3 / (6 + \alpha^3)$. Equivalently, αY^δ is a mixture of Gamma(2, 1) and Gamma(3, 1) with weights k and $1 - k$. This representation gives a simple way to generate random variates and is used repeatedly in Section 3.

Figure 1 shows the plots of the two most crucial functions of modeling importance: PDF and HRF. From Figure 1, the PDF is unimodal and right-skewed for the values shown, with a mode that moves towards the origin and becomes sharper as α and δ increase. The HRF is increasing, with a rate of increase that grows with α and δ . Moreover, since $f(y) \sim \delta \alpha^5 y^{2\delta-1} / (6 + \alpha^3)$ as $y \rightarrow 0$, the density is unbounded at the origin and decreasing near it (reverse-J shape) when $\delta < 1/2$. Shapes close to this one occur in applications, as in the fit to Data set I in Section 6.

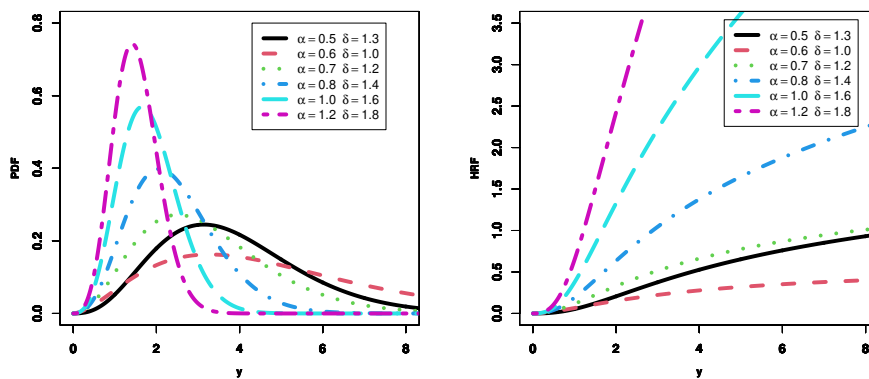


Fig. 1: PDF (left) and HRF (right) of the PGD for selected values of α and δ .

3 Some Important statistical properties

This section investigates the statistical properties of the proposed PGD. These properties offer valuable insight into both the mathematical structure and the practical behavior of the distribution. An analytical expression for the quantile function is obtained using the generalized Lambert W function, allowing for efficient quantile calculation and straightforward simulation. From this, quantile-based measures of skewness and kurtosis are examined using Bowley and Moors coefficients to characterize the shape of the distribution. The general formula for the raw moments is subsequently derived, yielding fundamental descriptive statistics including the mean, variance, and higher-order moments. In addition, the Shannon entropy of the PGD is established as a measure of the uncertainty inherent in the distribution. Finally, the stress–strength reliability measure is developed for two independent PGD random variables.

3.1 Quantile Function

Theorem 1. For any $0 < u < 1$, the quantile function $Q(u)$ of $Y \sim \text{PGD}(\alpha, \delta)$ can be expressed in closed form using the generalized Lambert W function as

$$Q(u) = \left[-\frac{1}{\alpha} W \left(-r_1, -r_2; \frac{(6 + \alpha^3)(1 - u)}{3} \right) \right]^{\frac{1}{\delta}}, \quad 0 < u < 1. \quad (5)$$

where r_1, r_2 are the roots of $1 + t + bt^2 = 0$, with $t = \alpha y^\delta$, $b = \frac{3}{6 + \alpha^3}$, and $r_{1,2} = \frac{-1 \pm \sqrt{1 - 4b}}{2b}$. Here $y = Q(u)$ denotes the quantile of order u , and $W(a_1, a_2; \cdot)$ is the generalized Lambert function of [28].

Proof. We start with the CDF in Equation (3) of the PGD. For a given probability $u \in (0, 1)$, the quantile $Q(u)$ satisfies $F(Q(u)) = u$. Hence

$$1 - u = \left[1 + \alpha y^\delta + \frac{3\alpha^2 y^{2\delta}}{6 + \alpha^3} \right] e^{-\alpha y^\delta}, \quad \text{with } y = Q(u). \quad (6)$$

Set $t = \alpha y^\delta > 0$. Then $y^\delta = \frac{t}{\alpha}$, $y^{2\delta} = \frac{t^2}{\alpha^2}$. Substituting these into (6) gives

$$1 - u = \left[1 + t + \frac{3t^2}{6 + \alpha^3} \right] e^{-t}.$$

Let the constant $b = \frac{3}{6 + \alpha^3}$. Then the equation becomes

$$1 - u = (1 + t + bt^2)e^{-t}. \quad (7)$$

The quadratic $1 + t + bt^2$ (equivalently, $bt^2 + t + 1$) has roots $r_{1,2} = \frac{-1 \pm \sqrt{1-4b}}{2b}$. Thus, we can write $bt^2 + t + 1 = b(t - r_1)(t - r_2)$. Substituting this factorization into (7) gives

$$1 - u = b(t - r_1)(t - r_2)e^{-t}. \quad (8)$$

Dividing both sides by b yields

$$\frac{1-u}{b} = (t - r_1)(t - r_2)e^{-t}. \quad (9)$$

To bring the equation into a suitable form for applying the generalized Lambert function W , introduce the transformation $x = -t$. Substituting $t = -x$ into (9) gives

$$\frac{1-u}{b} = (-x - r_1)(-x - r_2)e^x$$

Hence

$$\frac{1-u}{b} = (x + r_1)(x + r_2)e^x.$$

Equivalently, this can be written as

$$\frac{1-u}{b} = (x - (-r_1))(x - (-r_2))e^x.$$

This equation is now in the canonical form associated with the generalized Lambert function $W(a_1, a_2, \dots, a_n; \cdot)$, defined as the multivalued inverse of mappings of the form $x \mapsto e^x \prod_{j=1}^n (x - a_j)$, as introduced by [28]. Hence, the solution for x can be expressed as

$$x = W\left(-r_1, -r_2; \frac{1-u}{b}\right).$$

Recalling that $b = \frac{3}{6+\alpha^3}$, we then obtain

$$x = W\left(-r_1, -r_2; \frac{(6+\alpha^3)(1-u)}{3}\right).$$

Since $x = -t$, it follows that

$$t = -W\left(-r_1, -r_2; \frac{(6+\alpha^3)(1-u)}{3}\right).$$

Finally, using the substitution $t = \alpha y^\delta$, then $y^\delta = \frac{t}{\alpha}$, $y = \left(\frac{t}{\alpha}\right)^{\frac{1}{\delta}}$. Since $y = Q(u)$, the quantile function can be written as

$$Q(u) = \left[-\frac{1}{\alpha} W\left(-r_1, -r_2; \frac{(6+\alpha^3)(1-u)}{3}\right) \right]^{\frac{1}{\delta}}, \quad 0 < u < 1.$$

Remark(Numerical evaluation). Since $1 - 4b = (\alpha^3 - 6)/(\alpha^3 + 6)$, the roots r_1, r_2 are real when $\alpha^3 \geq 6$ and complex conjugates when $\alpha^3 < 6$. General-purpose implementations of the generalized Lambert function $W(a_1, a_2; \cdot)$ are not yet available in standard statistical software. In practice, $Q(u)$ is therefore computed by solving (7) numerically for t . The function $\psi(t) = (1 + t + bt^2)e^{-t}$ satisfies $\psi(0) = 1$, $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\psi'(t) = -t(1 - 2b + bt)e^{-t} < 0$ for $t > 0$ because $b \leq 1/2$. Hence (7) has exactly one positive root for every $u \in (0, 1)$, which can be found reliably by bisection or by Newton's method applied to $\log \psi(t) - \log(1 - u)$, for example with `uniroot` in R or `scipy.optimize.brentq` in Python. Then $Q(u) = (t/\alpha)^{1/\delta}$. This is the procedure used to generate PGD samples in Section 5.

3.1.1 Quantile Based Measures of Skewness and Kurtosis

Let $Q(u)$ be the quantile function of the Power Garhy distribution given in Theorem 1. Quantile-based measures are attractive alternatives to moment-based measures because they always exist (they do not require the existence of moments) and are robust to the presence of outliers. In this context, we present the Bowley coefficient of skewness and the Moors coefficient of kurtosis.

Bowley's Coefficient of Skewness: Bowley's measure of skewness is based on the quartiles of the distribution; see [29]. It is defined as

$$S_B = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$$

Here, Q_1 , Q_2 , and Q_3 denote the first quartile, median, and third quartile, respectively. The numerator measures the asymmetry between the distances from the median to the two quartiles. A positive value indicates right skewness, a negative value indicates left skewness, and a value equal to zero corresponds to a symmetric distribution.

To examine the influence of the distribution parameters on the skewness behavior of the proposed distribution, the Bowley coefficient is evaluated for different values of the parameters α and δ . The resulting values are illustrated in the surface plot shown in Figure 2, which describes how the skewness varies with respect to these parameters.

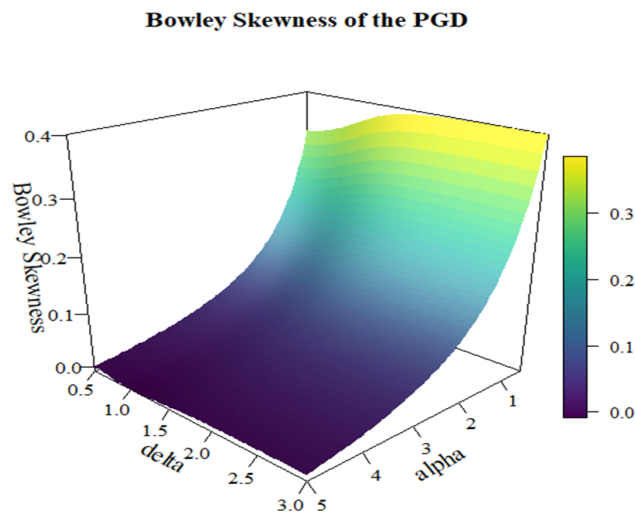


Fig. 2: Surface plot of Bowley’s skewness coefficient for the PGD as a function of the parameters α and δ .

As shown in Figure 2, Bowley’s skewness is positive over almost the whole parameter range considered, so the PGD is mainly right-skewed. The shape parameter δ has the dominant effect. Skewness is largest for small δ (about 0.34–0.40 at $\delta = 0.5$) and decreases quickly as δ increases, becoming close to zero (nearly symmetric quartiles) for $\delta \approx 3$. The parameter α has a weaker effect, and skewness increases slightly with α . These results highlight the flexibility of the proposed distribution for modeling positively skewed data.

Moors’ kurtosis coefficient: Moors’ kurtosis measure is based on octiles, see [30] and provides information about the tail weight of the distribution. It is defined as:

$$K_M = \frac{(Q_{0.875} - Q_{0.625}) + (Q_{0.375} - Q_{0.125})}{Q_{0.75} - Q_{0.25}}$$

To study the effect of the distribution parameters on the tail behavior of the proposed model, the Moors kurtosis values are calculated for different combinations of parameters α and δ . The resulting values are illustrated by a 3-D surface plot presented in Figure 3, which shows how kurtosis varies in parameter space.

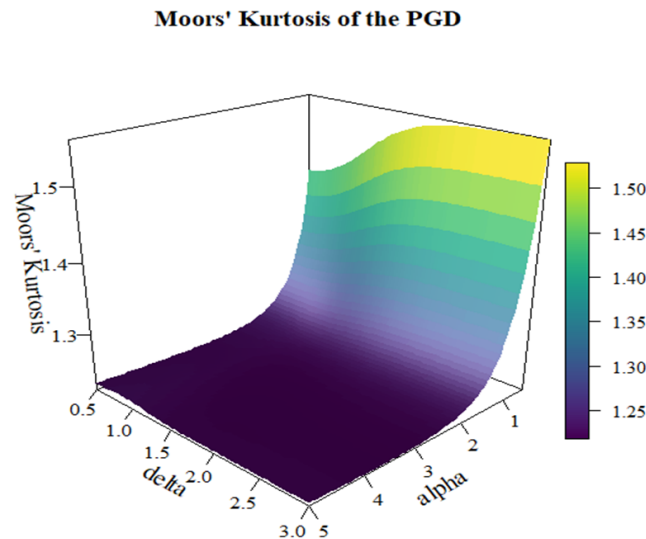


Fig. 3: Surface plot of Moors' kurtosis for the PGD as a function of the parameters α and δ .

Figure 3 shows that Moors' kurtosis is also driven mainly by δ . It is highest for small δ (about 1.45–1.55 at $\delta = 0.5$), which indicates heavier tails, and decreases towards about 1.23 as δ increases, which is the value for the normal distribution. Increasing α raises the kurtosis slightly. Thus small values of δ produce heavy-tailed, strongly skewed shapes, while large values of δ produce light-tailed, nearly symmetric shapes.

3.2 Raw Moments of the PGD

Let $Y \sim \text{PGD}(\alpha, \delta)$. The raw moments of the distribution play an important role in describing its main characteristics, such as the mean, variance, and higher-order measures.

Theorem 2. For any real $r > -2\delta$, the r -th raw moment of Y is given by

$$\mu'_r = \frac{\alpha^{-\frac{r}{\delta}}}{6 + \alpha^3} \left[\alpha^3 \Gamma\left(\frac{r}{\delta} + 2\right) + 3\Gamma\left(\frac{r}{\delta} + 3\right) \right] \quad (10)$$

Equivalently, using the recurrence relation of the gamma function, the above expression can be written as

$$\mu'_r = \frac{\alpha^{-\frac{r}{\delta}}}{(6 + \alpha^3)} \left(\Gamma\left(\frac{r}{\delta} + 2\right) \right) \left(\alpha^3 + \frac{3r}{\delta} + 6 \right).$$

Proof. By definition, the r -th raw moment of Y is

$$\mu'_r = E(Y^r) = \int_0^\infty y^r f(y) dy.$$

By substituting the integral with PDF (4) of the PGD, we obtain

$$\mu'_r = \frac{\delta \alpha^3}{(6 + \alpha^3)} \int_0^\infty y^{r+2\delta-1} (\alpha^2 + 3y^\delta) e^{-\alpha y^\delta} dy.$$

Expanding the integrand gives

$$\mu'_r = \frac{\delta \alpha^3}{(6 + \alpha^3)} \left[\alpha^2 \int_0^\infty y^{r+2\delta-1} e^{-\alpha y^\delta} dy + 3 \int_0^\infty y^{r+3\delta-1} e^{-\alpha y^\delta} dy \right].$$

Now, consider the transformation $t = \alpha y^\delta$, then

$$\mu'_r = \frac{\delta \alpha^3}{(6 + \alpha^3)} \left[\alpha^2 \frac{1}{\delta} \alpha^{-\frac{r}{\delta} - 2} \Gamma\left(\frac{r}{\delta} + 2\right) + 3 \frac{1}{\delta} \alpha^{-\frac{r}{\delta} - 3} \Gamma\left(\frac{r}{\delta} + 3\right) \right].$$

After simplification, this yields

$$\mu'_r = \frac{\alpha^{-\frac{r}{\delta}}}{(6 + \alpha^3)} \left[\alpha^3 \Gamma\left(\frac{r}{\delta} + 2\right) + 3 \Gamma\left(\frac{r}{\delta} + 3\right) \right] = \frac{\alpha^{-\frac{r}{\delta}} \Gamma\left(\frac{r}{\delta} + 2\right)}{(6 + \alpha^3)} \left(\alpha^3 + \frac{3r}{\delta} + 6 \right).$$

Additionally, the equivalent skewness is supplied as $E\{[Y - E(Y)]^3/[var(Y)]^{1.5}\}$, the kurtosis is stated as $E\{[Y - E(Y)]^4/[var(Y)]^2\}$, the coefficient of variation (CV) can be determined as $[var(Y)]^{0.5}/E(Y)$, and the index of dispersion (ID) is specified as $var(Y)/E(Y)$. Figure 4 shows 3D plots of these measures for $\alpha \in [0.5, 2.5]$ and $\delta \in [0.8, 2.5]$, computed from Theorem 2. The mean, the variance and the ID decrease as either parameter increases, and they become very large when both α and δ are small. The moment skewness (about 0.09–1.88) and kurtosis (about 2.9–8.6) are governed mainly by δ : small values of δ give strongly skewed, heavy-tailed shapes, while large values give nearly symmetric, light-tailed shapes. The CV ranges from about 0.24 to 0.87 over this range (below the exponential value of 1), and the ID ranges from about 0.08 to 5.2. Hence the PGD can describe both under- and over-dispersed data in the sense of the ID.

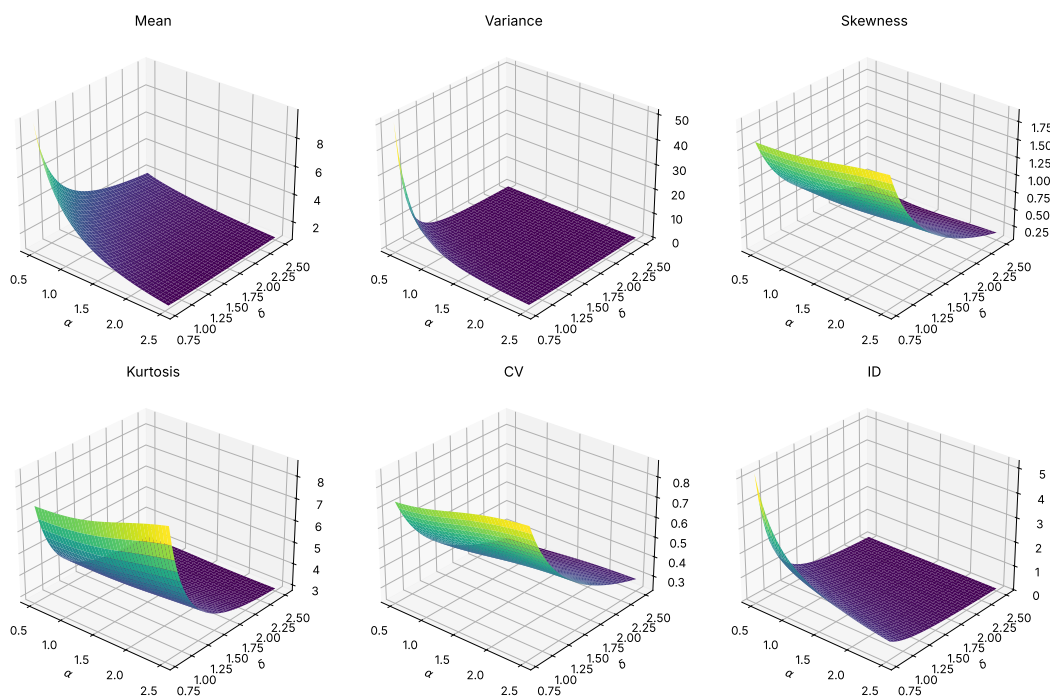


Fig. 4: 3D plots of the mean, variance, skewness, kurtosis, coefficient of variation (CV) and index of dispersion (ID) of the PGD as functions of α and δ .

3.3 Shannon Entropy of the PGD

Entropy is a fundamental concept in information theory and statistics, measuring the uncertainty or randomness associated with a random variable. Among the most widely used entropy measures is the Shannon entropy, introduced by [31]. For a continuous random variable with PDF $f(y)$, the Shannon entropy is defined as

$$H(Y) = -E[\log f(Y)].$$

This measure provides useful information about the dispersion and uncertainty of distribution. In this subsection, we derive the Shannon entropy for the PGD.

Theorem 3. Let $Y \sim PGD(\alpha, \delta)$ with PDF given in equation (4). Then the Shannon entropy has the form

$$H(Y) = -\log\left(\frac{\delta\alpha^3}{6+\alpha^3}\right) - \left(\frac{2\delta-1}{\delta}\right) [-\log(\alpha) + k\psi(2) + (1-k)\psi(3)] - [kA_2(\alpha) + (1-k)A_3(\alpha)] + (3-k), \quad (11)$$

where $\psi(\cdot) = \frac{d}{dx} \log(\Gamma(x))$ denotes the digamma function, $k = \frac{\alpha^3}{6+\alpha^3}$,

$$A_2(\alpha) = 1 + 2\log(\alpha) + (1-\beta)\Omega(\beta), \quad A_3(\alpha) = \frac{3}{2} + 2\log(\alpha) - \frac{\beta}{2} + \left(\frac{\beta^2 - 2\beta + 2}{2}\right)\Omega(\beta),$$

$\beta = \frac{\alpha^3}{3}$, $\Omega(\beta) = e^\beta E_1(\beta)$, and $E_1(\beta) = \int_\beta^\infty \frac{e^{-u}}{u} du$ is the exponential integral function.

Proof. Using the logarithmic form of the probability density function given in Equation (4), we obtain

$$\log f(y) = \log\left(\frac{\delta\alpha^3}{6+\alpha^3}\right) + (2\delta-1)\log(y) + \log(\alpha^2 + 3y^\delta) - \alpha y^\delta.$$

Taking expectations on both sides yields

$$H(Y) = -\log\left(\frac{\delta\alpha^3}{6+\alpha^3}\right) - (2\delta-1)E(\log(Y)) - E(\log(\alpha^2 + 3Y^\delta)) + \alpha E(Y^\delta).$$

To evaluate these expectations, consider the transformation $T = Y^\delta$. The PDF of T can be expressed as a mixture of gamma distributions see [32].

$$f_T(t) = kg_2(t) + (1-k)g_3(t), t > 0$$

where $g_r(t) = \frac{\alpha^r}{\Gamma(r)} t^{r-1} e^{-\alpha t}$, $r = 2, 3$ and $k = \frac{\alpha^3}{6+\alpha^3}$. Since $\log(Y) = \frac{1}{\delta} \log(T)$, we obtain $E(\log(Y)) = \frac{1}{\delta} E(\log(T))$. Because T follows a mixture of gamma distributions,

$$E(h(T)) = kE_{g_2}(h(T)) + (1-k)E_{g_3}(h(T)).$$

Taking, $h(T) = \log(T)$, then

$$E(\log(T)) = kE_{g_2}(\log(T)) + (1-k)E_{g_3}(\log(T)).$$

Using the known property of the gamma distribution (see [33])

$$E_{g_r}(\log(T)) = \psi(r) - \log(\alpha),$$

where $\psi(\cdot)$ denotes the digamma function, we obtain

$$E(\log(Y)) = \frac{1}{\delta} [-\log(\alpha) + k\psi(2) + (1-k)\psi(3)].$$

Now we will evaluate $E(\log(\alpha^2 + 3Y^\delta))$ as follows

$$E(\log(\alpha^2 + 3Y^\delta)) = E(\log(\alpha^2 + 3T)).$$

Since T admits a gamma mixture representation, this expectation can be written as $E(\log(\alpha^2 + 3T)) = kE_{g_2}(\log(\alpha^2 + 3T)) + (1-k)E_{g_3}(\log(\alpha^2 + 3T))$, then

$$E(\log(\alpha^2 + 3T)) = [kA_2(\alpha) + (1-k)A_3(\alpha)],$$

where $A_2(\alpha) = E_{g_2}(\log(\alpha^2 + 3T))$, $A_3(\alpha) = E_{g_3}(\log(\alpha^2 + 3T))$. These expectations correspond to the following integrals:

$$A_2(\alpha) = \int_0^\infty \log(\alpha^2 + 3t)g_2(t)dt, \quad A_3(\alpha) = \int_0^\infty \log(\alpha^2 + 3t)g_3(t)dt.$$

Moreover,

$$A_2(\alpha) = 1 + 2\log(\alpha) + (1 - \beta)\Omega(\beta), \quad A_3(\alpha) = \frac{3}{2} + 2\log(\alpha) - \frac{\beta}{2} + \left(\frac{\beta^2 - 2\beta + 2}{2}\right)\Omega(\beta),$$

where, as in the statement of the theorem, $\beta = \frac{\alpha^3}{3}$, $\Omega(\beta) = e^\beta E_1(\beta)$, and $E_1(\beta) = \int_\beta^\infty \frac{e^{-u}}{u} du$ denotes the exponential integral function. These follow by writing $\log(\alpha^2 + 3t) = \log 3 - \log \alpha + \log(\beta + \alpha t)$, where $\alpha T \sim \text{Gamma}(j, 1)$ under g_j , and integrating by parts, using $\int_0^\infty \log(\beta + s)e^{-s} ds = \log \beta + \Omega(\beta)$. Finally, we evaluate $E(Y^\delta)$. since the gamma mixture representation,

$$E(T) = kE_{g_2}(T) + (1 - k)E_{g_3}(T).$$

Using the well-known mean of the gamma distribution $E(T) = \frac{r}{\alpha}$ for $T \sim \text{Gamma}(r, \alpha)$ (see [33]), we obtain

$$E(T) = \frac{2k + 3(1 - k)}{\alpha} = \frac{3 - k}{\alpha}, \quad \text{so that} \quad \alpha E(Y^\delta) = \alpha E(T) = 3 - k.$$

Substituting the expressions for $E(\log Y)$, $E(Y^\delta)$, and $E[\log(\alpha^2 + 3Y^\delta)]$ into the entropy formula yields

$$H(Y) = -\log\left(\frac{\delta\alpha^3}{6 + \alpha^3}\right) - \left(\frac{2\delta - 1}{\delta}\right)[- \log(\alpha) + k\psi(2) + (1 - k)\psi(3)] \\ - [kA_2(\alpha) + (1 - k)A_3(\alpha)] + (3 - k).$$

As a check, Theorem 3 was compared with direct numerical evaluation of $-\int_0^\infty f \log f dy$. For example, $(\alpha, \delta) = (0.5, 0.6)$, $(1.3, 0.8)$ and $(2, 1.5)$ give $H(Y) = 4.1218$, 1.9149 and 0.6695 , respectively, and both methods agree to at least four decimals.

3.4 Stress-Strength Reliability of the PGD

Let $Y_1 \sim \text{PGD}(\alpha_1, \delta_1)$ and $Y_2 \sim \text{PGD}(\alpha_2, \delta_2)$ be two independent random variables, where Y_1 denotes the strength and Y_2 the stress (see [11]). The stress-strength reliability is defined by

$$R = P(Y_2 < Y_1) = P(Y_1 > Y_2). \quad (12)$$

For independent random variables, it follows that

$$R = \int_0^\infty f_{Y_1}(y)F_{Y_2}(y)dy = 1 - \int_0^\infty f_{Y_1}(y)S_{Y_2}(y)dy,$$

where $S_{Y_2}(y)$ denotes the survival function of Y_2 ,

$$S_{Y_i}(y) = \left[1 + \alpha_i y^{\delta_i} + b_i \alpha_i^2 y^{2\delta_i}\right] e^{-\alpha_i y^{\delta_i}}, \quad y > 0, \quad b_i = \frac{3}{6 + \alpha_i^3}, \quad i = 1, 2.$$

Theorem 4. Let $Y_1 \sim \text{PGD}(\alpha_1, \delta_1)$ and $Y_2 \sim \text{PGD}(\alpha_2, \delta_2)$ be independent random variables. Then the stress-strength reliability $R = P(Y_2 < Y_1)$ is given by

$$R = 1 - \frac{1}{6 + \alpha_1^3} \sum_{n=0}^\infty \frac{(-\lambda)^n}{n!} \left\{ \alpha_1^3 [\Gamma(n\rho + 2) + \lambda\Gamma((n+1)\rho + 2) + b_2\lambda^2\Gamma((n+2)\rho + 2)] \right. \\ \left. + 3[\Gamma(n\rho + 3) + \lambda\Gamma((n+1)\rho + 3) + b_2\lambda^2\Gamma((n+2)\rho + 3)] \right\}, \quad (13)$$

where $\rho = \frac{\delta_2}{\delta_1}$, $\lambda = \frac{\alpha_2}{\alpha_1^\rho}$ and $b_2 = \frac{3}{6 + \alpha_2^3}$. Throughout the paper, ρ and λ always have these meanings.

Proof. By definition,

$$R = P(Y_2 < Y_1) = 1 - \int_0^\infty f_{Y_1}(y)S_{Y_2}(y)dy.$$

Using the forms of $f_{Y_1}(y)$ and $S_{Y_2}(y)$, we obtain

$$R = 1 - \frac{\delta_1 \alpha_1^3}{6 + \alpha_1^3} \int_0^\infty y^{2\delta_1-1} (\alpha_1^2 + 3y^{\delta_1}) \left(1 + \alpha_2 y^{\delta_2} + b_2 \alpha_2^2 y^{2\delta_2}\right) e^{-\alpha_1 y^{\delta_1} - \alpha_2 y^{\delta_2}} dy.$$

Now consider the transformation $t = \alpha_1 y^{\delta_1}$, then

$$R = 1 - \frac{\delta_1 \alpha_1^3}{6 + \alpha_1^3} \int_0^\infty \left(\frac{1}{\delta_1} \alpha_1^{-2} t\right) \left(\alpha_1^2 + \frac{3t}{\alpha_1}\right) \left(1 + \alpha_2 \alpha_1^{-\rho} t^\rho + b_2 \alpha_2^2 \alpha_1^{-2\rho} t^{2\rho}\right) e^{-t - \alpha_2 \alpha_1^{-\rho} t^\rho} dt.$$

After simplification, we obtain

$$R = 1 - \frac{1}{6 + \alpha_1^3} \int_0^\infty (\alpha_1^3 t + 3t^2) (1 + \lambda t^\rho + b_2 \lambda^2 t^{2\rho}) e^{-t - \lambda t^\rho} dt,$$

where $\lambda = \frac{\alpha_2}{\alpha_1^\rho}$. Next, expand the factor $e^{-\lambda t^\rho}$ into its power series:

$$e^{-\lambda t^\rho} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} t^{n\rho}.$$

Substituting this series into the integral gives

$$R = 1 - \frac{1}{6 + \alpha_1^3} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_0^\infty (\alpha_1^3 t + 3t^2) (1 + \lambda t^\rho + b_2 \lambda^2 t^{2\rho}) e^{-t} t^{n\rho} dt.$$

Using the standard gamma identity $\int_0^\infty t^{v-1} e^{-t} dt = \Gamma(v)$, $v > 0$, then we have

$$R = 1 - \frac{1}{6 + \alpha_1^3} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \left\{ \alpha_1^3 [\Gamma(n\rho + 2) + \lambda \Gamma((n+1)\rho + 2) + b_2 \lambda^2 \Gamma((n+2)\rho + 2)] \right. \\ \left. + 3 [\Gamma(n\rho + 3) + \lambda \Gamma((n+1)\rho + 3) + b_2 \lambda^2 \Gamma((n+2)\rho + 3)] \right\}.$$

Remark (Convergence and truncation). The series in (13) comes from expanding $e^{-\lambda t^\rho}$. Since $\Gamma(n\rho + c)/n!$ behaves like $(n!)^{\rho-1}$ up to geometric and polynomial factors in n , the series converges absolutely for every $\lambda > 0$ when $\rho < 1$, converges for $\lambda < 1$ when $\rho = 1$, and diverges when $\rho > 1$. When $\delta_2 > \delta_1$, the same derivation is applied to the equivalent form $R = \int_0^\infty f_{Y_2}(y) S_{Y_1}(y) dy$, using the substitution $t = \alpha_2 y^{\delta_2}$ and the expansion of $e^{-\alpha_1 y^{\delta_1}}$. The resulting series has exponent ratio $1/\rho < 1$ and therefore converges. The terms alternate in sign, so the truncation error after N terms is bounded by the absolute value of the first omitted term. In our computations we stop when this term is below 10^{-8} and evaluate the gamma ratios on the log scale to avoid overflow. Table 1 reports the number of terms required for several parameter sets, together with a comparison with direct numerical integration of $\int_0^\infty f_{Y_1}(y) S_{Y_2}(y) dy$. For moderate λ about 12–70 terms are enough. When ρ is close to one and $\lambda > 1$, many more terms are needed and cancellation between the alternating terms can occur, so direct numerical quadrature of the integral is a safer alternative.

Table 1: Number of terms N needed in the series (13) for a truncation error below 10^{-8} , with R from the series and from direct numerical integration.

α_1	δ_1	α_2	δ_2	ρ	λ	N	R (series) = R (integration)
2.0	1.0	0.5	0.5	0.50	0.354	12	0.021294
1.0	2.0	0.5	1.2	0.60	0.500	16	0.076102
1.0	1.5	1.0	0.5	0.33	1.000	16	0.194519
0.8	1.5	1.2	1.0	0.67	1.392	37	0.532703
0.5	1.5	1.5	0.8	0.53	2.171	37	0.733678
0.6	1.2	2.5	0.7	0.58	3.368	69	0.930116
1.0	1.0	1.5	0.9	0.90	1.500	262	0.676173

Corollary 1. Setting $\delta_1 = \delta_2 = \delta$ in Theorem 4 gives $\rho = 1$ and $\lambda = \alpha_2/\alpha_1$. Therefore, the stress-strength reliability can be written as

$$R = 1 - \frac{1}{6 + \alpha_1^3} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \left\{ \alpha_1^3 [\Gamma(n+2) + \lambda \Gamma(n+3) + b_2 \lambda^2 \Gamma(n+4)] \right. \\ \left. + 3 [\Gamma(n+3) + \lambda \Gamma(n+4) + b_2 \lambda^2 \Gamma(n+5)] \right\}.$$

In this case, R does not depend on the common shape δ . Because the series above converges only for $\lambda < 1$, it is better to evaluate the integral directly. With $\delta_1 = \delta_2$, the substitution $x = y^\delta$ gives $1 - R = \frac{\alpha_1^3}{6 + \alpha_1^3} \int_0^\infty x(\alpha_1^2 + 3x)(1 + \alpha_2 x + c_2 x^2) e^{-(\alpha_1 + \alpha_2)x} dx$, which yields the closed form below, valid for all $\alpha_1, \alpha_2 > 0$.

$$R = 1 - \frac{\alpha_1^3}{6 + \alpha_1^3} \left[\frac{\alpha_1^2}{\kappa^2} + \frac{2(\alpha_1^2 \alpha_2 + 3)}{\kappa^3} + \frac{6(\alpha_1^2 c_2 + 3\alpha_2)}{\kappa^4} + \frac{72c_2}{\kappa^5} \right], \quad (14)$$

where $\kappa = \alpha_1 + \alpha_2$ and $c_2 = b_2 \alpha_2^2 = \frac{3\alpha_2^2}{6 + \alpha_1^3}$, with b_2 as in Theorem 4. (We write κ instead of k to avoid confusion with the mixing weight k in Section 3.3.) Expression (14) was verified numerically against direct integration. For example, $(\alpha_1, \alpha_2) = (0.3, 0.7)$, $(0.5, 2)$ and $(4, 4)$ give $R = 0.840396$, 0.957664 and 0.5 , respectively, with agreement to at least six decimals. The value $R = 0.5$ for $\alpha_1 = \alpha_2$ is the expected one, because Y_1 and Y_2 are then identically distributed.

4 Statistical Inference

This section presents maximum likelihood (ML) and Bayesian estimation of the PGD parameters (α, δ) and of the stress–strength reliability R from progressively Type-II censored samples, together with interval estimates.

4.1 Maximum likelihood estimation

MLE plays a crucial role in parameter estimation for reliability and lifetime data analysis, particularly under censoring schemes such as progressive Type-II censoring. Progressive Type-II censoring arises in life-testing experiments when units are removed from the study at predetermined failure times, either randomly or according to a specific plan, resulting in incomplete data. In such scenarios, classical statistical methods that assume complete observations may not perform well, and MLE provides a rigorous framework to estimate the underlying model parameters despite incompleteness of the data [34].

Let n identical units be placed in a life test and suppose that the experiment is conducted under a progressive Type-II censoring scheme. Let r denote the number of observed failures such that $1 \leq r \leq n$.

The censoring scheme is the vector of integers $\mathbf{M} = (M_1, M_2, \dots, M_r)$, where $M_i \geq 0$ is the number of surviving units randomly withdrawn from the test immediately after the i -th observed failure, $i = 1, \dots, r$, with $\sum_{i=1}^r M_i = n - r$. The special case $\mathbf{M} = (0, \dots, 0, n - r)$ is conventional Type-II censoring, and $\mathbf{M} = (0, \dots, 0)$ gives a complete sample. Let $y_1 < y_2 < \dots < y_r$ denote the observed failure times. The likelihood function for progressive Type-II censoring is given by

$$L(\alpha, \delta | \mathbf{y}) = C \prod_{i=1}^r \left[f(y_i; \alpha, \delta) (1 - F(y_i; \alpha, \delta))^{M_i} \right],$$

where the normalizing constant C does not depend on the parameters:

$$C = \prod_{i=1}^r \left(n - i + 1 - \sum_{j=1}^{i-1} M_j \right) = n(n-1-M_1)(n-2-M_1-M_2) \cdots \left(n-r+1 - \sum_{j=1}^{r-1} M_j \right).$$

Here M_i always denotes the number of removals, and C is the only product constant. By substituting Equations (3) and (4), we obtain

$$L(\alpha, \delta | \mathbf{y}) = C \prod_{i=1}^r \frac{\delta \alpha^3 y_i^{2\delta-1}}{6 + \alpha^3} (\alpha^2 + 3y_i^\delta) \left(\frac{3\alpha^2 y_i^{2\delta}}{6 + \alpha^3} + \alpha y_i^\delta + 1 \right)^{M_i} e^{-\alpha y_i^\delta (1+M_i)}. \quad (15)$$

The log-likelihood function is

$$\begin{aligned}\ell = & \ln C + r \ln \delta + 3r \ln \alpha - r \ln(6 + \alpha^3) + (2\delta - 1) \sum_{i=1}^r \ln y_i + \sum_{i=1}^r \ln(\alpha^2 + 3y_i^\delta) \\ & + \sum_{i=1}^r M_i \ln \left(\frac{3\alpha^2 y_i^{2\delta}}{6 + \alpha^3} + \alpha y_i^\delta + 1 \right) - \alpha \sum_{i=1}^r y_i^\delta (1 + M_i).\end{aligned}$$

The MLEs $\hat{\alpha}_L$ and $\hat{\delta}_L$ are the solutions of the likelihood equations $\partial \ell / \partial \alpha = 0$ and $\partial \ell / \partial \delta = 0$, where

$$\frac{\partial \ell}{\partial \alpha} = \frac{3r}{\alpha} - \frac{3r\alpha^2}{6 + \alpha^3} + \sum_{i=1}^r \frac{2\alpha}{\alpha^2 + 3y_i^\delta} + \sum_{i=1}^r M_i \frac{\frac{6\alpha y_i^{2\delta}}{6 + \alpha^3} - \frac{9\alpha^4 y_i^{2\delta}}{(6 + \alpha^3)^2} + y_i^\delta}{\frac{3\alpha^2 y_i^{2\delta}}{6 + \alpha^3} + \alpha y_i^\delta + 1} - \sum_{i=1}^r y_i^\delta (1 + M_i), \quad (16)$$

$$\frac{\partial \ell}{\partial \delta} = \frac{r}{\delta} + 2 \sum_{i=1}^r \ln y_i + \sum_{i=1}^r \frac{3y_i^\delta \ln y_i}{\alpha^2 + 3y_i^\delta} + \sum_{i=1}^r M_i \frac{\frac{6\alpha^2 y_i^{2\delta} \ln y_i}{6 + \alpha^3} + \alpha y_i^\delta \ln y_i}{\frac{3\alpha^2 y_i^{2\delta}}{6 + \alpha^3} + \alpha y_i^\delta + 1} - \alpha \sum_{i=1}^r y_i^\delta (1 + M_i) \ln y_i. \quad (17)$$

These equations have no closed-form solution, so ℓ is maximized numerically. To ensure positivity, we maximize over $(\log \alpha, \log \delta)$. The starting value is the best point of a coarse 41×41 grid on $\log \alpha \in [-4, 4]$ and $\log \delta \in [-3, 2.5]$, and the Nelder–Mead algorithm is then run from this point. This two-stage strategy avoids convergence to poor local solutions. A replicate is counted as non-convergent when the optimizer fails or the observed information matrix is not positive definite.

Asymptotic confidence intervals. Let $\mathbf{I}(\hat{\boldsymbol{\theta}}) = -\partial^2 \ell / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top$, evaluated at $\hat{\boldsymbol{\theta}} = (\hat{\alpha}_L, \hat{\delta}_L)$, be the observed Fisher information matrix. In this paper it is computed by central finite differences. Under the usual regularity conditions, $\hat{\boldsymbol{\theta}}$ is approximately normal with covariance $\mathbf{I}^{-1}(\hat{\boldsymbol{\theta}})$, and a $100(1 - \gamma)\%$ asymptotic confidence interval for θ_j is $\hat{\theta}_j \pm z_{\gamma/2} \sqrt{[\mathbf{I}^{-1}(\hat{\boldsymbol{\theta}})]_{jj}}$. We use $\gamma = 0.05$ throughout.

4.2 Bayesian estimation

MLE is a standard approach for parameter estimation; however, its reliability may weaken when applied to small or moderately sized samples, particularly when the data are subject to incompleteness such as progressive Type-II censoring. In such settings, Bayesian inference provides a valuable alternative, as it naturally incorporates uncertainty in the parameters and yields full posterior distributions instead of relying solely on point estimates.

Prior distributions. We assume independent improper priors

$$\pi(\alpha) \propto \alpha^{-3}, \quad \pi(\delta) \propto \delta, \quad \alpha, \delta > 0. \quad (18)$$

These are vague (diffuse) priors of power form, $\pi(\theta) \propto \theta^c$, rather than non-informative priors in a formal sense [35]. They carry no information about the location of the parameters, and their exponents were chosen so that the prior factors cancel the boundary behaviour of the likelihood. (i) As $\alpha \rightarrow 0$, the likelihood (15) behaves like α^{3r} , so α^{3r-3} remains integrable near zero for every $r \geq 1$, and the factor $e^{-\alpha \sum y_i^\delta (1 + M_i)}$ gives exponential decay as $\alpha \rightarrow \infty$. (ii) The factor $\delta^r \prod y_i^{2\delta} e^{-\alpha y_i^\delta}$ decays as $\delta \rightarrow \infty$, so the linear prior $\pi(\delta) \propto \delta$ keeps the posterior integrable while avoiding the strong shrinkage towards $\delta = 0$ that a $1/\delta$ prior would produce in small samples. Consequently, the joint posterior is proper for the samples considered here, and its normalizing constant was finite in every simulated replicate. We note that (18) is not the Jeffreys prior. To assess the effect of the prior, Section 5 also reports results for (a) vague independent gamma priors, $\text{Gamma}(0.001, 0.001)$ for both parameters, which are close to $1/\alpha$ and $1/\delta$, and (b) informative gamma priors whose means equal the true values and whose variances equal 0.5.

Posterior and Bayes estimators. The joint posterior is

$$\pi^*(\alpha, \delta | \mathbf{y}) = \frac{L(\alpha, \delta | \mathbf{y}) \pi(\alpha) \pi(\delta)}{\int_0^\infty \int_0^\infty L(\alpha, \delta | \mathbf{y}) \pi(\alpha) \pi(\delta) d\alpha d\delta},$$

with L given in (15). Under squared error loss, the Bayes estimators are the posterior means

$$\hat{\alpha}_B = \int_0^\infty \int_0^\infty \alpha \pi^*(\alpha, \delta | \mathbf{y}) d\alpha d\delta, \quad (19)$$

$$\hat{\delta}_B = \int_0^\infty \int_0^\infty \delta \pi^*(\alpha, \delta | \mathbf{y}) d\alpha d\delta. \quad (20)$$

The posterior does not belong to a known family, so these integrals have no closed form.

Computation. The integrals are evaluated by self-normalized importance sampling [36] on $\boldsymbol{\phi} = (\log \alpha, \log \delta)$:

1. Compute the MLE $\hat{\boldsymbol{\theta}}$ and the covariance matrix $\boldsymbol{\Sigma}_\phi = \mathbf{J} \mathbf{I}^{-1}(\hat{\boldsymbol{\theta}}) \mathbf{J}$ of $\hat{\boldsymbol{\phi}}$, where $\mathbf{J} = \text{diag}(1/\hat{\alpha}, 1/\hat{\delta})$.
2. Draw $\boldsymbol{\phi}^{(1)}, \dots, \boldsymbol{\phi}^{(K)}$, with $K = 4000$, from a bivariate Student- t proposal q with 5 degrees of freedom, location $\hat{\boldsymbol{\phi}}$ and scale $1.5\boldsymbol{\Sigma}_\phi$.
3. Compute the weights $w_k \propto L(\boldsymbol{\theta}^{(k)} | \mathbf{y}) \pi(\boldsymbol{\theta}^{(k)}) \alpha^{(k)} \delta^{(k)} / q(\boldsymbol{\phi}^{(k)})$, where $\alpha^{(k)} \delta^{(k)}$ is the Jacobian of the log transformation, and normalize them so that $\sum_k w_k = 1$.
4. Re-centre and re-scale the proposal at the weighted mean and covariance of the first-pass draws, then repeat Steps 2–3. This adaptive step makes the importance sampler efficient even when the posterior is far from normal, for example when r is small.
5. Estimate $\hat{\alpha}_B = \sum_k w_k \alpha^{(k)}$ and $\hat{\delta}_B = \sum_k w_k \delta^{(k)}$. The 95% equal-tailed credible intervals are given by the weighted 2.5% and 97.5% quantiles of the draws.

The effective sample size $1/\sum_k w_k^2$ was monitored. Its median over replicates was above 2300 (out of 4000) in all simulation settings.

4.3 Estimation of R based on MLE and Bayesian method

In this subsection, the stress–strength reliability measure is estimated using both the maximum likelihood and Bayesian approaches based on progressively Type-II censored data. Let $\mathbf{y}_1$ (strength, n units, r failures, scheme $\mathbf{M}^{(1)}$) and $\mathbf{y}_2$ (stress, m units, s failures, scheme $\mathbf{M}^{(2)}$) be independent progressively Type-II censored samples from PGD(α_1, δ_1) and PGD(α_2, δ_2), respectively. Table 2 summarizes the notation used for the four estimators of R .

Table 2: Notation for the estimators of the stress–strength reliability R .

Estimator	Case	Method	Formula
$\hat{R}_{1L}$	general, $\delta_1 \neq \delta_2$	ML (plug-in)	Eq. (21)
$\hat{R}_{1B}$	general, $\delta_1 \neq \delta_2$	Bayes (posterior mean)	Eq. (23)
$\hat{R}_{2L}$	equal shape, $\delta_1 = \delta_2$	ML (plug-in)	Eq. (22)
$\hat{R}_{2B}$	equal shape, $\delta_1 = \delta_2$	Bayes (posterior mean)	Eq. (24)

ML estimation, general case. By the invariance property, the MLE of R is obtained by substituting the MLEs $(\hat{\alpha}_{1L}, \hat{\delta}_{1L})$ and $(\hat{\alpha}_{2L}, \hat{\delta}_{2L})$, computed separately from each sample as in Section 4.1, into (13):

$$\hat{R}_{1L} = R(\hat{\alpha}_{1L}, \hat{\delta}_{1L}, \hat{\alpha}_{2L}, \hat{\delta}_{2L}), \quad (21)$$

that is, (13) with ρ , λ and b_2 replaced by $\hat{\rho}_L = \hat{\delta}_{2L}/\hat{\delta}_{1L}$, $\hat{\lambda}_L = \hat{\alpha}_{2L}/\hat{\alpha}_{1L}^{\hat{\rho}_L}$ and $\hat{b}_{2L} = 3/(6 + \hat{\alpha}_{2L}^3)$. When $\hat{\rho}_L \geq 1$, the exchanged form of Remark 3.4 or direct numerical integration is used.

ML estimation, equal-shape case. When $\delta_1 = \delta_2 = \delta$, the common shape is estimated jointly from both samples by maximizing $\ell_1(\alpha_1, \delta) + \ell_2(\alpha_2, \delta)$ over $(\alpha_1, \alpha_2, \delta)$. The estimates are then substituted into (14):

$$\hat{R}_{2L} = 1 - \frac{\hat{\alpha}_{1L}^3}{6 + \hat{\alpha}_{1L}^3} \left[\frac{\hat{\alpha}_{1L}^2}{\hat{\kappa}_L^2} + \frac{2(\hat{\alpha}_{1L}^2 \hat{\alpha}_{2L} + 3)}{\hat{\kappa}_L^3} + \frac{6(\hat{\alpha}_{1L}^2 \hat{c}_{2L} + 3\hat{\alpha}_{2L})}{\hat{\kappa}_L^4} + \frac{72\hat{c}_{2L}}{\hat{\kappa}_L^5} \right], \quad (22)$$

where $\hat{\kappa}_L = \hat{\alpha}_{1L} + \hat{\alpha}_{2L}$ and $\hat{c}_{2L} = 3\hat{\alpha}_{2L}^2/(6 + \hat{\alpha}_{2L}^3)$.

Asymptotic confidence interval for R . By the delta method, $\widehat{\text{Var}}(\hat{R}) = \nabla R(\hat{\boldsymbol{\theta}})^\top \mathbf{I}^{-1}(\hat{\boldsymbol{\theta}}) \nabla R(\hat{\boldsymbol{\theta}})$, where $\boldsymbol{\theta} = (\alpha_1, \delta_1, \alpha_2, \delta_2)$ in the general case, with a block-diagonal information matrix because the samples are independent, and $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \delta)$ in the equal-shape case. The gradient is computed numerically, and the 95% interval $\hat{R} \pm 1.96\{\widehat{\text{Var}}(\hat{R})\}^{1/2}$ is truncated to $[0, 1]$.

Bayesian estimation of R . Under squared error loss, the Bayes estimator of R is its posterior mean, $E[R(\boldsymbol{\theta}) | \mathbf{y}_1, \mathbf{y}_2]$. This is not the same as substituting the Bayes estimates of the parameters into R , because R is a nonlinear function of the parameters. In the general case, the two posteriors are independent. We draw 1500 values $\boldsymbol{\theta}_1^{(k)}$ and $\boldsymbol{\theta}_2^{(k)}$ from each posterior by importance resampling of the weighted draws of Section 4.2, and compute

$$\hat{R}_{1B} = \frac{1}{1500} \sum_{k=1}^{1500} R(\alpha_1^{(k)}, \delta_1^{(k)}, \alpha_2^{(k)}, \delta_2^{(k)}). \quad (23)$$

In the equal-shape case, the joint posterior of $(\alpha_1, \alpha_2, \delta)$ under the prior $\alpha_1^{-3} \alpha_2^{-3} \delta$ is sampled by the same importance-sampling scheme in three dimensions, and

$$\hat{R}_{2B} = \frac{1}{1500} \sum_{k=1}^{1500} R_{(14)}(\alpha_1^{(k)}, \alpha_2^{(k)}), \quad (24)$$

where $R_{(14)}$ denotes the closed form (14). The 95% credible intervals for R are given by the 2.5% and 97.5% quantiles of the simulated values of R . In practice, (13) is evaluated by Gauss-Laguerre quadrature (120 nodes) of the integral representation $R = 1 - E[S_{Y_2}(Y_1)]$, using the gamma-mixture form of $\alpha_1 Y_1^{\delta_1}$. This agrees with the truncated series to at least six decimals (Table 1) and remains stable when $\rho \geq 1$.

5 Monte Carlo simulation study

This section compares the ML and Bayesian estimators of Section 4 by Monte Carlo simulation. All computations were carried out in Python (NumPy/SciPy) using the algorithms of Section 4. The design is as follows.

- Data generation.** For given (n, r) and censoring scheme $\mathbf{M}$, a progressively Type-II censored sample $U_1 < \dots < U_r$ from the uniform distribution is generated by the algorithm of [37]. It is transformed to the PGD scale through $y_i = Q(U_i)$, where the quantile function is computed as described in the Remark following Theorem 1.
- Censoring schemes.** Unless stated otherwise, the $n - r$ removals are spread as evenly as possible over the r failures (Scheme III). Two further schemes are examined in Table 13: Scheme I, $\mathbf{M} = (n - r, 0, \dots, 0)$ (all removals at the first failure), and Scheme II, $\mathbf{M} = (0, \dots, 0, n - r)$ (conventional Type-II censoring).
- Parameter estimation** (Tables 3–6). We use $(\alpha, \delta) = (0.5, 0.6), (0.8, 1.2), (1.3, 0.8)$ and $(3.5, 2.1)$, with $(n, r) = (20, 5), (20, 7), (50, 20), (50, 30), (100, 50), (100, 60), (150, 120), (150, 130), (200, 150), (200, 160), (300, 250)$ and $(300, 260)$.
- Stress-strength reliability** (Tables 7–12). Two independent samples of equal size $m = n$ are generated with (r, s) failures. Tables 7–9 cover the general case $\delta_1 \neq \delta_2$, with true values $R = 0.533, 0.734$ and 0.930 . Tables 10–12 cover the equal-shape case, with $R = 0.840, 0.958$ and 0.500 . True values of R are computed from Theorem 4 and Corollary 1. Very small numbers of failures ($r \leq 4$) were not used, since two parameters per sample cannot be estimated reliably from so few failures.
- Estimators and criteria.** For each of $N = 1000$ replicates, the MLEs and asymptotic 95% confidence intervals, and the Bayes estimators under prior (18) with 95% credible intervals, are computed. We report the bias $N^{-1} \sum (\hat{\theta} - \theta)$, the mean squared error (MSE) $N^{-1} \sum (\hat{\theta} - \theta)^2$, the coverage probability (CP) of the 95% intervals, that is, the proportion of intervals containing the true value, and the average interval length (AL). The ML optimization failed to converge in at most 3 of the 1000 replicates (only for $n = 20, r = 5$), and these replicates were excluded.

Table 3: Bias, MSE, coverage probability (CP) and average length (AL) of the 95% intervals for the ML and Bayes estimators of α and δ (progressive Type-II censoring, scheme III; 1000 replicates), $\alpha = 0.5$, $\delta = 0.6$.

n	r	Par.	MLE				Bayes			
			Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	5	α	-0.0777	0.0508	0.831	0.761	-0.3053	0.1233	0.461	0.421
		δ	0.2165	0.1926	0.958	1.067	1.0306	2.3573	0.452	1.520
20	7	α	-0.0580	0.0445	0.839	0.719	-0.2500	0.0940	0.557	0.468
		δ	0.1321	0.0865	0.952	0.780	0.5432	0.6728	0.557	1.024
50	20	α	-0.0251	0.0153	0.919	0.472	-0.1066	0.0256	0.812	0.416
		δ	0.0410	0.0128	0.950	0.388	0.1241	0.0328	0.801	0.419
50	30	α	-0.0185	0.0131	0.908	0.426	-0.0801	0.0187	0.848	0.386
		δ	0.0266	0.0075	0.934	0.299	0.0783	0.0154	0.860	0.312
100	50	α	-0.0123	0.0078	0.923	0.328	-0.0507	0.0102	0.870	0.311
		δ	0.0188	0.0047	0.943	0.245	0.0519	0.0080	0.879	0.253
100	60	α	-0.0165	0.0067	0.925	0.304	-0.0480	0.0087	0.888	0.290
		δ	0.0168	0.0032	0.950	0.207	0.0411	0.0051	0.892	0.211
150	120	α	-0.0018	0.0041	0.933	0.241	-0.0209	0.0045	0.930	0.235
		δ	0.0044	0.0015	0.948	0.146	0.0174	0.0019	0.922	0.148
150	130	α	-0.0022	0.0034	0.950	0.239	-0.0208	0.0038	0.935	0.233
		δ	0.0042	0.0013	0.956	0.143	0.0167	0.0016	0.941	0.144
200	150	α	-0.0039	0.0030	0.943	0.210	-0.0186	0.0033	0.925	0.206
		δ	0.0038	0.0011	0.953	0.129	0.0137	0.0013	0.933	0.130
200	160	α	-0.0049	0.0029	0.936	0.209	-0.0194	0.0033	0.927	0.204
		δ	0.0060	0.0011	0.952	0.127	0.0158	0.0014	0.915	0.128
300	250	α	-0.0022	0.0018	0.956	0.170	-0.0118	0.0020	0.946	0.168
		δ	0.0027	0.0007	0.954	0.102	0.0089	0.0008	0.952	0.102
300	260	α	-0.0018	0.0018	0.954	0.169	-0.0112	0.0019	0.945	0.167
		δ	0.0029	0.0006	0.961	0.101	0.0090	0.0007	0.948	0.101

Table 4: Bias, MSE, coverage probability (CP) and average length (AL) of the 95% intervals for the ML and Bayes estimators of α and δ (progressive Type-II censoring, scheme III; 1000 replicates), $\alpha = 0.8$, $\delta = 1.2$.

n	r	Par.	MLE				Bayes			
			Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	5	α	-0.0815	0.0768	0.872	0.867	-0.3177	0.1800	0.624	0.682
		δ	0.4618	0.8308	0.954	2.131	1.4769	5.8348	0.583	2.886
20	7	α	-0.0532	0.0600	0.891	0.826	-0.2484	0.1243	0.714	0.695
		δ	0.2705	0.4341	0.943	1.518	0.7946	2.0158	0.695	1.844
50	20	α	-0.0304	0.0206	0.932	0.527	-0.1063	0.0326	0.862	0.504
		δ	0.0872	0.0501	0.941	0.755	0.2023	0.0996	0.849	0.797
50	30	α	-0.0133	0.0185	0.914	0.490	-0.0724	0.0244	0.878	0.472
		δ	0.0352	0.0242	0.952	0.564	0.1062	0.0404	0.898	0.583
100	50	α	-0.0096	0.0095	0.930	0.363	-0.0440	0.0118	0.904	0.356
		δ	0.0287	0.0162	0.953	0.468	0.0740	0.0232	0.902	0.478
100	60	α	-0.0097	0.0086	0.925	0.348	-0.0395	0.0103	0.903	0.342
		δ	0.0257	0.0116	0.950	0.394	0.0594	0.0158	0.912	0.400
150	120	α	-0.0080	0.0051	0.943	0.277	-0.0262	0.0058	0.930	0.274
		δ	0.0127	0.0054	0.947	0.279	0.0314	0.0066	0.929	0.282
150	130	α	-0.0081	0.0054	0.938	0.275	-0.0261	0.0061	0.922	0.272
		δ	0.0104	0.0052	0.946	0.272	0.0287	0.0062	0.922	0.275
200	150	α	-0.0026	0.0041	0.946	0.242	-0.0164	0.0044	0.934	0.240
		δ	0.0080	0.0040	0.954	0.246	0.0221	0.0046	0.939	0.248
200	160	α	-0.0063	0.0038	0.941	0.241	-0.0200	0.0042	0.935	0.239
		δ	0.0118	0.0041	0.945	0.241	0.0258	0.0048	0.932	0.243
300	250	α	-0.0044	0.0026	0.955	0.196	-0.0135	0.0028	0.949	0.195
		δ	0.0068	0.0027	0.938	0.194	0.0158	0.0030	0.931	0.195
300	260	α	-0.0004	0.0023	0.957	0.195	-0.0093	0.0024	0.956	0.194
		δ	0.0038	0.0022	0.956	0.191	0.0127	0.0025	0.946	0.192

Table 5: Bias, MSE, coverage probability (CP) and average length (AL) of the 95% intervals for the ML and Bayes estimators of α and δ (progressive Type-II censoring, scheme III; 1000 replicates), $\alpha = 1.3$, $\delta = 0.8$.

n	r	Par.	MLE				Bayes			
			Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	5	α	0.1074	0.3507	0.954	1.217	-0.0433	0.3071	0.806	1.169
		δ	0.3659	0.5768	0.945	1.570	0.6494	1.1364	0.743	1.715
20	7	α	0.0055	0.0870	0.923	0.934	-0.1351	0.1228	0.825	0.906
		δ	0.2153	0.2114	0.929	1.085	0.3979	0.4447	0.779	1.155
50	20	α	-0.0055	0.0243	0.922	0.564	-0.0585	0.0294	0.902	0.559
		δ	0.0544	0.0226	0.947	0.516	0.0995	0.0325	0.902	0.521
50	30	α	-0.0209	0.0214	0.936	0.533	-0.0661	0.0263	0.907	0.528
		δ	0.0365	0.0133	0.928	0.385	0.0658	0.0177	0.899	0.387
100	50	α	-0.0071	0.0097	0.941	0.379	-0.0309	0.0110	0.927	0.377
		δ	0.0204	0.0075	0.949	0.321	0.0393	0.0091	0.937	0.322
100	60	α	-0.0053	0.0096	0.946	0.377	-0.0274	0.0106	0.931	0.375
		δ	0.0168	0.0050	0.947	0.265	0.0304	0.0059	0.938	0.266
150	120	α	-0.0049	0.0060	0.955	0.303	-0.0188	0.0065	0.947	0.302
		δ	0.0069	0.0025	0.930	0.185	0.0146	0.0027	0.931	0.186
150	130	α	-0.0012	0.0057	0.958	0.300	-0.0148	0.0060	0.954	0.300
		δ	0.0057	0.0022	0.949	0.180	0.0132	0.0024	0.943	0.181
200	150	α	0.0018	0.0044	0.956	0.264	-0.0087	0.0045	0.950	0.264
		δ	0.0042	0.0019	0.942	0.164	0.0100	0.0020	0.939	0.164
200	160	α	0.0010	0.0043	0.960	0.263	-0.0094	0.0044	0.958	0.262
		δ	0.0040	0.0016	0.950	0.160	0.0097	0.0018	0.946	0.160
300	250	α	-0.0034	0.0031	0.943	0.213	-0.0102	0.0032	0.938	0.213
		δ	0.0030	0.0012	0.942	0.129	0.0067	0.0013	0.939	0.129
300	260	α	-0.0012	0.0031	0.940	0.212	-0.0080	0.0032	0.934	0.212
		δ	0.0032	0.0011	0.950	0.127	0.0069	0.0011	0.948	0.127

Table 6: Bias, MSE, coverage probability (CP) and average length (AL) of the 95% intervals for the ML and Bayes estimators of α and δ (progressive Type-II censoring, scheme III; 1000 replicates), $\alpha = 3.5$, $\delta = 2.1$.

n	r	Par.	MLE				Bayes			
			Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	5	α	18.1722	47716.2008	0.973	170.762	2.0315	20.0097	0.980	15.068
		δ	0.8589	2.7073	0.959	4.068	0.4727	0.7747	0.976	3.175
20	7	α	2.6680	109.3160	0.975	14.557	1.2797	10.5242	0.968	8.696
		δ	0.5486	1.3200	0.926	2.977	0.4544	0.7516	0.937	2.675
50	20	α	0.3514	0.9126	0.970	2.894	0.2751	0.7437	0.947	2.782
		δ	0.1428	0.1832	0.944	1.449	0.1599	0.1734	0.944	1.425
50	30	α	0.1475	0.3184	0.944	1.849	0.0929	0.2864	0.929	1.804
		δ	0.0837	0.0856	0.958	1.103	0.1095	0.0888	0.950	1.098
100	50	α	0.1114	0.1970	0.958	1.548	0.0912	0.1863	0.947	1.529
		δ	0.0530	0.0614	0.945	0.911	0.0666	0.0616	0.937	0.906
100	60	α	0.0721	0.1263	0.951	1.249	0.0463	0.1201	0.945	1.234
		δ	0.0490	0.0422	0.949	0.766	0.0622	0.0435	0.945	0.764
150	120	α	0.0328	0.0507	0.947	0.821	0.0182	0.0495	0.945	0.815
		δ	0.0231	0.0199	0.959	0.547	0.0334	0.0206	0.955	0.547
150	130	α	0.0259	0.0368	0.962	0.782	0.0124	0.0360	0.958	0.777
		δ	0.0230	0.0199	0.957	0.535	0.0334	0.0207	0.952	0.535
200	150	α	0.0182	0.0352	0.956	0.733	0.0068	0.0346	0.956	0.729
		δ	0.0171	0.0151	0.954	0.481	0.0247	0.0155	0.950	0.481
200	160	α	0.0217	0.0300	0.964	0.706	0.0108	0.0294	0.963	0.703
		δ	0.0190	0.0148	0.954	0.473	0.0268	0.0152	0.957	0.473
300	250	α	0.0120	0.0211	0.948	0.561	0.0048	0.0209	0.948	0.559
		δ	0.0090	0.0093	0.959	0.380	0.0141	0.0094	0.957	0.379
300	260	α	0.0143	0.0193	0.954	0.549	0.0075	0.0190	0.954	0.547
		δ	0.0109	0.0090	0.958	0.376	0.0161	0.0092	0.952	0.376

Table 7: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{1L}$, $\hat{R}_{1B}$; $\alpha_1 = 0.8$, $\delta_1 = 1.5$, $\alpha_2 = 1.2$, $\delta_2 = 1.0$; true $R = 0.5327$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	0.0234	0.01957	0.867	0.464	0.0177	0.02312	0.886	0.453
		10	10	0.0136	0.01402	0.912	0.424	0.0089	0.01667	0.917	0.415
		12	10	0.0178	0.01465	0.890	0.417	0.0184	0.01708	0.895	0.409
		15	13	0.0121	0.01203	0.907	0.385	0.0060	0.01367	0.899	0.378
50	50	20	20	0.0050	0.00693	0.924	0.307	0.0024	0.00743	0.929	0.303
		30	30	0.0048	0.00423	0.942	0.257	0.0022	0.00444	0.944	0.255
		25	22	0.0006	0.00626	0.928	0.287	-0.0025	0.00672	0.929	0.284
		35	33	0.0064	0.00433	0.935	0.249	0.0039	0.00452	0.938	0.247
100	100	70	70	0.0027	0.00204	0.939	0.175	0.0014	0.00208	0.944	0.174
		80	80	0.0023	0.00199	0.938	0.168	0.0010	0.00204	0.939	0.167
		85	75	-0.0010	0.00191	0.947	0.170	-0.0024	0.00197	0.947	0.169
		95	85	-0.0020	0.00181	0.949	0.163	-0.0036	0.00186	0.942	0.162
150	150	100	100	0.0007	0.00145	0.940	0.145	-0.0003	0.00148	0.940	0.144
		110	110	0.0018	0.00135	0.938	0.141	0.0010	0.00138	0.941	0.140
		125	120	-0.0018	0.00122	0.949	0.137	-0.0027	0.00125	0.940	0.136
		140	130	0.0017	0.00111	0.955	0.133	0.0008	0.00113	0.952	0.132
200	200	170	170	0.0002	0.00089	0.951	0.117	-0.0004	0.00090	0.951	0.117
		180	180	-0.0011	0.00086	0.949	0.115	-0.0018	0.00087	0.946	0.114
		185	170	0.0013	0.00092	0.936	0.116	0.0006	0.00093	0.937	0.115
		190	180	0.0007	0.00089	0.940	0.114	0.0000	0.00090	0.942	0.114
300	300	270	270	0.0005	0.00055	0.959	0.094	0.0001	0.00056	0.957	0.093
		280	280	0.0015	0.00056	0.945	0.093	0.0011	0.00057	0.948	0.092
		275	260	-0.0004	0.00059	0.941	0.095	-0.0009	0.00060	0.938	0.094
		285	275	0.0009	0.00060	0.933	0.093	0.0005	0.00061	0.932	0.092

Table 8: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{1L}$, $\hat{R}_{1B}$; $\alpha_1 = 0.5$, $\delta_1 = 1.5$, $\alpha_2 = 1.5$, $\delta_2 = 0.8$; true $R = 0.7337$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	0.0264	0.01577	0.837	0.407	0.0253	0.01717	0.883	0.417
		10	10	0.0248	0.01236	0.844	0.376	0.0272	0.01352	0.890	0.378
		12	10	0.0243	0.01168	0.857	0.378	0.0301	0.01263	0.909	0.376
		15	13	0.0162	0.00884	0.886	0.339	0.0170	0.00970	0.911	0.336
50	50	20	20	0.0105	0.00647	0.884	0.295	0.0099	0.00667	0.911	0.291
		30	30	0.0012	0.00389	0.923	0.235	0.0011	0.00404	0.932	0.234
		25	22	0.0056	0.00571	0.909	0.278	0.0053	0.00591	0.929	0.275
		35	33	0.0070	0.00363	0.923	0.223	0.0072	0.00375	0.934	0.222
100	100	70	70	0.0027	0.00143	0.953	0.156	0.0028	0.00147	0.950	0.156
		80	80	0.0020	0.00149	0.934	0.148	0.0024	0.00151	0.930	0.147
		85	75	0.0037	0.00172	0.924	0.151	0.0038	0.00175	0.931	0.150
		95	85	-0.0008	0.00142	0.948	0.144	-0.0006	0.00145	0.949	0.143
150	150	100	100	0.0028	0.00112	0.943	0.131	0.0028	0.00113	0.944	0.130
		110	110	-0.0001	0.00104	0.945	0.126	-0.0000	0.00105	0.945	0.125
		125	120	0.0000	0.00098	0.946	0.121	0.0002	0.00099	0.950	0.121
		140	130	0.0020	0.00088	0.941	0.117	0.0022	0.00089	0.945	0.116
200	200	170	170	0.0026	0.00068	0.947	0.102	0.0028	0.00068	0.943	0.102
		180	180	0.0005	0.00066	0.945	0.100	0.0008	0.00067	0.943	0.100
		185	170	0.0000	0.00072	0.938	0.102	0.0001	0.00072	0.934	0.102
		190	180	-0.0001	0.00065	0.950	0.100	0.0001	0.00066	0.950	0.100
300	300	270	270	0.0009	0.00049	0.928	0.082	0.0010	0.00049	0.927	0.082
		280	280	0.0001	0.00043	0.945	0.081	0.0002	0.00044	0.942	0.080
		275	260	-0.0007	0.00046	0.949	0.083	-0.0006	0.00046	0.948	0.083
		285	275	0.0010	0.00040	0.954	0.081	0.0011	0.00040	0.952	0.081

Table 9: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{1L}$, $\hat{R}_{1B}$; $\alpha_1 = 0.6$, $\delta_1 = 1.2$, $\alpha_2 = 2.5$, $\delta_2 = 0.7$; true $R = 0.9301$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	0.0068	0.00300	0.744	0.169	-0.0050	0.00315	0.918	0.257
		10	10	0.0046	0.00290	0.771	0.162	-0.0020	0.00292	0.924	0.221
		12	10	0.0040	0.00276	0.767	0.163	-0.0023	0.00272	0.920	0.219
		15	13	0.0033	0.00186	0.837	0.149	0.0005	0.00187	0.919	0.178
50	50	20	20	0.0018	0.00149	0.863	0.142	-0.0036	0.00155	0.933	0.165
		30	30	0.0004	0.00092	0.890	0.117	-0.0014	0.00093	0.947	0.122
		25	22	0.0038	0.00130	0.851	0.133	-0.0005	0.00132	0.937	0.151
		35	33	-0.0006	0.00085	0.906	0.112	-0.0018	0.00086	0.939	0.115
100	100	70	70	0.0005	0.00041	0.917	0.077	0.0001	0.00041	0.936	0.078
		80	80	0.0007	0.00035	0.924	0.073	0.0007	0.00035	0.946	0.073
		85	75	-0.0002	0.00038	0.918	0.075	-0.0004	0.00038	0.938	0.075
		95	85	0.0001	0.00033	0.920	0.071	0.0002	0.00033	0.933	0.071
150	150	100	100	0.0009	0.00029	0.930	0.065	0.0005	0.00029	0.947	0.065
		110	110	0.0010	0.00027	0.920	0.062	0.0009	0.00027	0.937	0.062
		125	120	0.0011	0.00023	0.930	0.059	0.0011	0.00023	0.940	0.059
		140	130	0.0010	0.00020	0.936	0.057	0.0011	0.00020	0.955	0.057
200	200	170	170	0.0005	0.00016	0.934	0.050	0.0006	0.00016	0.946	0.050
		180	180	0.0002	0.00016	0.944	0.049	0.0003	0.00016	0.948	0.049
		185	170	0.0002	0.00018	0.927	0.050	0.0003	0.00017	0.933	0.050
		190	180	0.0001	0.00017	0.938	0.049	0.0002	0.00017	0.946	0.049
300	300	270	270	0.0005	0.00010	0.938	0.040	0.0005	0.00010	0.945	0.040
		280	280	0.0001	0.00010	0.941	0.040	0.0003	0.00010	0.945	0.040
		275	260	0.0004	0.00011	0.938	0.041	0.0005	0.00011	0.946	0.041
		285	275	-0.0000	0.00010	0.945	0.040	0.0000	0.00010	0.948	0.040

Table 10: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{2L}$, $\hat{R}_{2B}$; $\alpha_1 = 0.3$, $\alpha_2 = 0.7$, $\delta_1 = \delta_2 = 2$; true $R = 0.8404$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	0.0136	0.00663	0.837	0.276	0.0545	0.00915	0.702	0.225
		10	10	0.0058	0.00581	0.868	0.268	0.0439	0.00756	0.762	0.222
		12	10	0.0073	0.00581	0.858	0.266	0.0433	0.00720	0.765	0.224
		15	13	0.0086	0.00419	0.900	0.244	0.0351	0.00525	0.836	0.213
50	50	20	20	0.0025	0.00290	0.906	0.198	0.0200	0.00328	0.874	0.182
		30	30	0.0020	0.00198	0.909	0.169	0.0135	0.00216	0.888	0.160
		25	22	0.0066	0.00227	0.923	0.186	0.0226	0.00274	0.884	0.172
		35	33	0.0004	0.00191	0.909	0.163	0.0110	0.00203	0.895	0.155
100	100	70	70	0.0004	0.00087	0.943	0.114	0.0056	0.00090	0.934	0.111
		80	80	0.0001	0.00082	0.930	0.109	0.0050	0.00085	0.927	0.106
		85	75	0.0018	0.00078	0.938	0.110	0.0068	0.00082	0.931	0.107
		95	85	0.0023	0.00076	0.930	0.104	0.0069	0.00080	0.922	0.102
150	150	100	100	0.0013	0.00066	0.925	0.095	0.0048	0.00068	0.920	0.093
		110	110	-0.0000	0.00059	0.932	0.092	0.0033	0.00060	0.929	0.090
		125	120	0.0012	0.00051	0.941	0.088	0.0044	0.00053	0.936	0.087
		140	130	0.0010	0.00044	0.957	0.085	0.0041	0.00045	0.952	0.084
200	200	170	170	-0.0001	0.00040	0.938	0.075	0.0023	0.00041	0.933	0.074
		180	180	-0.0008	0.00034	0.955	0.074	0.0016	0.00035	0.951	0.073
		185	170	0.0000	0.00037	0.957	0.075	0.0024	0.00037	0.955	0.074
		190	180	0.0006	0.00035	0.940	0.073	0.0029	0.00036	0.939	0.072
300	300	270	270	0.0009	0.00024	0.934	0.060	0.0024	0.00024	0.934	0.059
		280	280	0.0002	0.00023	0.945	0.059	0.0017	0.00023	0.947	0.059
		275	260	-0.0000	0.00024	0.947	0.061	0.0015	0.00024	0.943	0.060
		285	275	0.0009	0.00021	0.957	0.059	0.0025	0.00021	0.960	0.059

Table 11: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{2L}$, $\hat{R}_{2B}$; $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\delta_1 = \delta_2 = 0.3$; true $R = 0.9577$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	0.0044	0.00130	0.741	0.097	0.0117	0.00112	0.764	0.100
		10	10	0.0031	0.00103	0.769	0.096	0.0106	0.00091	0.790	0.097
		12	10	0.0019	0.00106	0.784	0.098	0.0086	0.00091	0.807	0.100
50	50	15	13	0.0015	0.00076	0.832	0.093	0.0069	0.00069	0.859	0.093
		20	20	0.0005	0.00053	0.870	0.083	0.0040	0.00049	0.898	0.081
		30	30	-0.0005	0.00042	0.878	0.073	0.0021	0.00040	0.898	0.069
100	100	25	22	-0.0002	0.00048	0.876	0.082	0.0031	0.00044	0.899	0.078
		35	33	-0.0006	0.00034	0.896	0.070	0.0018	0.00032	0.914	0.067
		70	70	-0.0000	0.00016	0.928	0.049	0.0012	0.00016	0.927	0.048
150	150	80	80	0.0005	0.00013	0.920	0.046	0.0018	0.00013	0.928	0.045
		85	75	-0.0003	0.00016	0.925	0.048	0.0009	0.00015	0.923	0.046
		95	85	-0.0001	0.00013	0.939	0.045	0.0010	0.00013	0.944	0.044
200	200	100	100	0.0003	0.00011	0.934	0.041	0.0011	0.00011	0.944	0.040
		110	110	-0.0003	0.00010	0.932	0.040	0.0005	0.00010	0.937	0.039
		125	120	0.0004	0.00009	0.930	0.038	0.0012	0.00009	0.936	0.037
300	300	140	130	-0.0002	0.00009	0.938	0.037	0.0006	0.00009	0.944	0.036
		170	170	-0.0003	0.00007	0.947	0.033	0.0003	0.00007	0.954	0.032
		180	180	0.0001	0.00006	0.940	0.032	0.0006	0.00006	0.946	0.031
		185	170	0.0000	0.00007	0.939	0.032	0.0006	0.00007	0.939	0.032
		190	180	-0.0002	0.00007	0.939	0.032	0.0004	0.00007	0.941	0.031
		270	270	-0.0001	0.00004	0.951	0.026	0.0003	0.00004	0.955	0.026
		280	280	0.0002	0.00004	0.948	0.026	0.0006	0.00004	0.953	0.025
		275	260	0.0003	0.00005	0.939	0.026	0.0006	0.00005	0.936	0.026
		285	275	0.0001	0.00004	0.940	0.026	0.0005	0.00004	0.949	0.025

Table 12: Bias, MSE, CP and AL of the ML and Bayes estimators of R ($\hat{R}_{2L}$, $\hat{R}_{2B}$; $\alpha_1 = 4$, $\alpha_2 = 4$, $\delta_1 = \delta_2 = 1.5$; true $R = 0.5000$; scheme III; $m = n$; 1000 replicates).

n	m	r	s	MLE				Bayes			
				Bias	MSE	CP	AL	Bias	MSE	CP	AL
20	20	8	8	-0.0007	0.01804	0.882	0.461	-0.0006	0.01330	0.964	0.453
		10	10	-0.0021	0.01443	0.901	0.421	-0.0020	0.01150	0.948	0.414
		12	10	0.0058	0.01286	0.908	0.412	0.0003	0.01059	0.956	0.404
50	50	15	13	0.0078	0.01109	0.911	0.377	0.0040	0.00970	0.949	0.370
		20	20	0.0008	0.00635	0.942	0.300	0.0007	0.00563	0.962	0.297
		30	30	-0.0012	0.00475	0.926	0.257	-0.0012	0.00447	0.936	0.254
100	100	25	22	0.0022	0.00585	0.922	0.280	-0.0000	0.00531	0.936	0.277
		35	33	-0.0009	0.00422	0.939	0.245	-0.0014	0.00401	0.949	0.243
		70	70	0.0034	0.00203	0.943	0.172	0.0033	0.00198	0.949	0.171
150	150	80	80	-0.0015	0.00180	0.940	0.163	-0.0014	0.00177	0.946	0.162
		85	75	-0.0006	0.00184	0.935	0.163	-0.0012	0.00180	0.939	0.162
		95	85	0.0027	0.00165	0.942	0.155	0.0022	0.00162	0.945	0.154
200	200	100	100	0.0012	0.00137	0.947	0.144	0.0012	0.00135	0.953	0.143
		110	110	-0.0012	0.00129	0.943	0.138	-0.0012	0.00127	0.952	0.137
		125	120	0.0005	0.00122	0.946	0.132	0.0004	0.00121	0.952	0.131
300	300	140	130	-0.0013	0.00106	0.950	0.126	-0.0016	0.00106	0.952	0.126
		170	170	-0.0002	0.00086	0.945	0.112	-0.0002	0.00085	0.947	0.112
		180	180	-0.0001	0.00076	0.956	0.110	-0.0001	0.00075	0.955	0.109
		185	170	-0.0003	0.00076	0.953	0.110	-0.0004	0.00075	0.952	0.110
		190	180	0.0005	0.00072	0.950	0.108	0.0004	0.00071	0.945	0.108
		270	270	-0.0010	0.00054	0.948	0.089	-0.0010	0.00054	0.953	0.089
		280	280	-0.0016	0.00050	0.956	0.088	-0.0016	0.00050	0.956	0.088
		275	260	-0.0000	0.00050	0.954	0.090	-0.0002	0.00050	0.955	0.089
		285	275	0.0003	0.00051	0.940	0.088	0.0003	0.00051	0.941	0.088

Table 13: Effect of the censoring scheme: MSE and CP of the ML and Bayes estimators for $\alpha = 0.8$, $\delta = 1.2$ (1000 replicates). Scheme I: $\mathbf{M} = (n-r, 0, \dots, 0)$; Scheme II: $\mathbf{M} = (0, \dots, 0, n-r)$; Scheme III: removals spread evenly over the r failures.

n	r	Scheme	α				δ			
			MSE		CP		MSE		CP	
			MLE	Bayes	MLE	Bayes	MLE	Bayes	MLE	Bayes
20	5	I	0.0788	0.1904	0.895	0.622	0.2583	1.7908	0.931	0.633
		II	0.0905	0.2216	0.832	0.539	1.7518	16.4255	0.969	0.506
		III	0.0768	0.1800	0.872	0.624	0.8308	5.8348	0.954	0.583
20	7	I	0.0698	0.1511	0.900	0.661	0.1502	0.7854	0.953	0.680
		II	0.0665	0.1626	0.879	0.625	0.5576	4.1105	0.973	0.619
		III	0.0600	0.1243	0.891	0.714	0.4341	2.0158	0.943	0.695
50	20	I	0.0245	0.0395	0.951	0.859	0.0330	0.0753	0.959	0.864
		II	0.0230	0.0404	0.929	0.820	0.0782	0.1868	0.949	0.827
		III	0.0206	0.0326	0.932	0.862	0.0501	0.0996	0.941	0.849
50	30	I	0.0215	0.0304	0.934	0.877	0.0261	0.0485	0.947	0.868
		II	0.0194	0.0303	0.929	0.853	0.0404	0.0817	0.954	0.852
		III	0.0185	0.0244	0.914	0.878	0.0242	0.0404	0.952	0.898
100	50	I	0.0129	0.0172	0.935	0.883	0.0142	0.0226	0.947	0.896
		II	0.0096	0.0126	0.936	0.910	0.0216	0.0337	0.943	0.910
		III	0.0095	0.0118	0.930	0.904	0.0162	0.0232	0.953	0.902
100	60	I	0.0112	0.0140	0.936	0.906	0.0124	0.0182	0.940	0.902
		II	0.0087	0.0114	0.953	0.901	0.0163	0.0243	0.943	0.906
		III	0.0086	0.0103	0.925	0.903	0.0116	0.0158	0.950	0.912
150	120	I	0.0056	0.0064	0.951	0.930	0.0058	0.0072	0.950	0.930
		II	0.0052	0.0060	0.944	0.934	0.0073	0.0091	0.939	0.927
		III	0.0051	0.0058	0.943	0.930	0.0054	0.0066	0.947	0.929
150	130	I	0.0055	0.0062	0.938	0.916	0.0055	0.0067	0.949	0.922
		II	0.0054	0.0061	0.938	0.925	0.0061	0.0074	0.948	0.917
		III	0.0054	0.0061	0.938	0.922	0.0052	0.0062	0.946	0.922
200	150	I	0.0045	0.0049	0.953	0.936	0.0046	0.0055	0.956	0.930
		II	0.0037	0.0041	0.959	0.954	0.0053	0.0063	0.955	0.940
		III	0.0041	0.0044	0.946	0.934	0.0040	0.0046	0.954	0.939
200	160	I	0.0043	0.0047	0.952	0.931	0.0042	0.0049	0.951	0.941
		II	0.0038	0.0042	0.954	0.944	0.0052	0.0062	0.949	0.938
		III	0.0038	0.0042	0.941	0.935	0.0041	0.0048	0.945	0.932
300	250	I	0.0026	0.0028	0.951	0.948	0.0024	0.0027	0.950	0.949
		II	0.0026	0.0028	0.950	0.944	0.0032	0.0036	0.947	0.933
		III	0.0026	0.0028	0.955	0.949	0.0027	0.0030	0.938	0.931
300	260	I	0.0028	0.0030	0.946	0.943	0.0028	0.0031	0.947	0.944
		II	0.0025	0.0026	0.954	0.948	0.0028	0.0031	0.969	0.950
		III	0.0023	0.0024	0.957	0.956	0.0022	0.0025	0.956	0.946

Table 14: Prior sensitivity of the Bayes estimators for $\alpha = 0.8$, $\delta = 1.2$ (scheme III, 1000 replicates): Bias and MSE under the prior (18) (P1), vague Gamma(0.001, 0.001) priors (P2) and informative gamma priors with means equal to the true values and variance 0.5 (P3).

n	r	Par.	P1		P2		P3	
			Bias	MSE	Bias	MSE	Bias	MSE
20	5	α	-0.3177	0.1800	-0.0786	0.0710	-0.0295	0.0364
		δ	1.4769	5.8348	0.4762	1.0982	0.2025	0.1847
20	7	α	-0.2484	0.1243	-0.0592	0.0533	-0.0415	0.0342
		δ	0.7946	2.0158	0.2685	0.3691	0.1734	0.1329
50	20	α	-0.1063	0.0326	-0.0182	0.0185	-0.0160	0.0175
		δ	0.2023	0.0996	0.0657	0.0475	0.0602	0.0387
50	30	α	-0.0724	0.0244	-0.0117	0.0167	-0.0105	0.0137
		δ	0.1062	0.0404	0.0395	0.0247	0.0353	0.0224
100	50	α	-0.0440	0.0118	-0.0108	0.0091	-0.0079	0.0085
		δ	0.0740	0.0232	0.0300	0.0159	0.0224	0.0147
100	60	α	-0.0395	0.0103	-0.0078	0.0075	-0.0014	0.0079
		δ	0.0594	0.0158	0.0230	0.0106	0.0121	0.0098
150	120	α	-0.0262	0.0058	-0.0044	0.0052	-0.0022	0.0050
		δ	0.0314	0.0066	0.0122	0.0054	0.0104	0.0051
150	130	α	-0.0261	0.0061	-0.0061	0.0047	-0.0028	0.0045
		δ	0.0287	0.0062	0.0112	0.0049	0.0101	0.0045
200	150	α	-0.0164	0.0044	-0.0023	0.0039	-0.0036	0.0039
		δ	0.0221	0.0046	0.0076	0.0041	0.0105	0.0042
200	160	α	-0.0200	0.0042	-0.0013	0.0039	-0.0020	0.0038
		δ	0.0258	0.0048	0.0064	0.0038	0.0071	0.0038
300	250	α	-0.0135	0.0028	0.0011	0.0026	-0.0018	0.0025
		δ	0.0158	0.0030	0.0032	0.0027	0.0043	0.0026
300	260	α	-0.0093	0.0024	0.0006	0.0026	-0.0035	0.0024
		δ	0.0127	0.0025	0.0031	0.0025	0.0066	0.0023

Discussion of the simulation results

Parameter estimation (Tables 3–6).

- Consistency.* For both methods and all parameter sets, the bias and MSE decrease steadily as n and r increase, and the MSE approaches zero. For instance, in Table 3 the MSE of $\hat{\alpha}_L$ decreases from 0.051 at $(n, r) = (20, 5)$ to 0.002 at $(300, 260)$. For fixed n , increasing the number of observed failures r also reduces the MSE, as expected, because more complete information is available.
- ML versus Bayes.* In 80 of the 96 (parameter, setting) combinations, the MLE has the smaller MSE. The Bayes estimators under prior (18) show a systematic bias in small samples: they underestimate α and overestimate δ , with a bias in δ of about 1.03 at $(n, r) = (20, 5)$ in Table 3. This is the effect of the prior factor $\alpha^{-3}\delta$, which pushes the posterior towards small α and large δ when the likelihood is flat. The difference between the methods vanishes for $n \geq 150$.
- When Bayes is preferable.* In Table 6 ($\alpha = 3.5$, $\delta = 2.1$), the likelihood is very flat in α for small samples, and the MLE of α is unstable, with MSE about 4.8×10^4 at $(20, 5)$ caused by a few very large estimates. The Bayes estimator is far more stable (MSE 20.0) and has smaller MSE for α at every sample size in this table. Thus Bayesian estimation is preferable when the sample is small and the MLE is poorly identified.
- Interval estimation.* The CP of the asymptotic ML intervals is close to the nominal 0.95 for $n \geq 100$, while it falls below the nominal level in small samples, at about 0.83–0.87 for α at $(20, 5)$ in Tables 3–4. The CP of the Bayes credible intervals under prior (18) is low in small samples, at about 0.45–0.62 at $(20, 5)$ in Tables 3–4, because of the bias described above, but it approaches 0.95 as n grows.

Effect of the censoring scheme (Table 13). The scheme matters mainly for the shape parameter δ and for small samples. Conventional Type-II censoring (Scheme II) gives the largest MSE for δ ; for example, at $(20, 5)$ the ML MSE is 1.75 under Scheme II, compared with 0.26 under Scheme I and 0.83 under Scheme III. Removing units early (Scheme I) gives the smallest MSE for δ , because the remaining failures are spread over a wider range of the distribution, which is informative about the tail. For α , the differences between schemes are small, and Scheme I is slightly worse at moderate sample sizes. For $n \geq 150$, the three schemes give practically the same results.

Effect of the prior (Table 14). The Bayes estimators are sensitive to the prior in small samples. The vague gamma prior (P2) gives much smaller bias and MSE than prior (18) (P1). At $(20, 5)$, the MSE of $\hat{\delta}_B$ is 1.10 under P2 versus 5.83 under P1, and under P2 the Bayes estimators become competitive with the MLE (smaller MSE for α at $r = 5, 7$ and for δ at $r = 7$). The informative prior (P3) gives the best results, as expected. For $n \geq 100$, the three priors lead to almost identical estimates. In practice, therefore, P2 or a weakly informative gamma prior is recommended for small samples.

Stress–strength reliability (Tables 7–12).

- In all tables, the bias and MSE of $\hat{R}$ decrease, and the CP approaches 0.95, as (n, m) and (r, s) increase. The biases are small: for the MLE they are below 0.03 in absolute value even for $n = 20$, while for the Bayes estimator they reach about 0.05 only in Table 10 at $n = 20$. All biases become negligible (of order 10^{-3}) for $n \geq 100$. The AL also decreases steadily with sample size.
- The ML and Bayes estimators of R have very similar MSE for $n \geq 50$. The MLE has slightly smaller MSE in Tables 7, 8 and 10, while the Bayes estimator has slightly smaller MSE in Tables 11 and 12 for small samples.
- The credible intervals for R have better coverage than the asymptotic ML intervals in small samples, especially when R is near the boundary. In Table 9 ($R = 0.930$), the ML coverage is only 0.74–0.84 for $n = 20$, while the Bayesian coverage is about 0.92, because the normal approximation for $\hat{R}$ is poor near one. The same pattern appears in Table 12, where the ML coverage is 0.88–0.91 and the Bayesian coverage is 0.95–0.96.
- Comparing equal and unequal numbers of failures, for example $(85, 75)$ versus $(80, 80)$, the results depend mainly on the total number of observed failures $r + s$, not on how the failures are split between the two samples.

6 Applications

In this section, the PGD distribution is applied to model two complete data sets. The first data set considered here was originally reported by Bhaumik et al. [38] and Usman and ul Haq [39]. The data set represents vinyl chloride data obtained from cleaning gradient monitoring wells in mg/l. Data are:

Data set 1. 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2.

The second data set represents the waiting times (in minutes) before service of 100 bank customers. Data were observed and reported by Ghitany et al. [40] and Ijaz et al. [41]. The data are:

Data set 2. 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

The PGD distribution is compared to seventeen other common distributions (for $y > 0$ and $\delta, \alpha, \theta > 0$), namely, the Lindley distribution (LD(δ)) [42], Shanker distribution (ShD(δ)) [43], Akash distribution (AkD(δ)) [44], Ishita distribution (IshD(δ)) [45], xgamma distribution (XgD(δ)) [46], Chris-Jerry distribution (CJD(δ)) [47], xLindley distribution (XLD(δ)) [48], Komal distribution (KD(δ)) [49], Zeghdoudi distribution (ZD(δ)) [50], power Zeghdoudi distribution (PZD(δ, α)) [12], power Lindley distribution (PLD(δ, α)) [1], power XLindley distribution (PXL(δ, α)), power exponentiated Lindley distribution (PELD(δ, α, θ)) [51], power Komal distribution (PKD(δ, α)) [14], power Chris-Jerry distribution (PCJD(δ, α)) [52], power length-biased new xLindley distribution (PLBNXLD(δ, α)) [17], and GD(δ).

To determine which distribution fits best, we compare the models using the negative maximized log-likelihood $-\hat{\ell}$ (column “ $-\hat{\ell}$ ” in Tables 15 and 16; smaller is better), the Kolmogorov–Smirnov (KS) statistic with its p-value (KS-pval), the Cramér–von Mises (CvM) statistic with its p-value (CvM-pval), and the Anderson–Darling (AD) statistic with its p-value (AD-pval). The model parameters are estimated by maximum likelihood, and the estimates with their standard errors (SE) are presented in Tables 15 and 16. In these tables, δ, α and θ denote the parameters of each competing model as defined in its original reference; for the PGD they are the shape and scale parameters of (4).

The PGD lifetime model provides a better fit than the LD, ShD, AkD, IshD, XgD, CJD, XLD, KD, ZD, PZD, PLD, PXL, PELD, PKD, PCJD and PLBNXLD distributions. For further investigation, the PGD distribution is also compared to the GD model. From Tables 15 and 16, the PGD attains the smallest $-\hat{\ell}$ and the smallest KS, CvM and AD statistics among all competing models, with the largest p-values for both data sets.

Comparison with the most relevant competitors. The most relevant competitors are the baseline GD and the two-parameter power models (PLD, PXL, PKD, PZD, PCJD and PLBNXLD), because they share the same construction. Two points emerge from Tables 15 and 16.

–Gain over the baseline. The power transformation improves the GD substantially: $-\hat{\ell}$ falls from 64.555 to 55.079 (Data set I) and from 322.465 to 317.077 (Data set II). For Data set I, the GD is rejected by all three tests at the 5% level. The estimated shapes, $\hat{\delta} = 0.643$ (SE 0.078) and 0.804 (SE 0.056), are both clearly below one. The data are therefore more right-skewed, with a heavier tail, than the GD allows, which is exactly the feature controlled by δ (Section 3).

–Comparison with other power models. Among the two-parameter power models, the PGD has the smallest $-\hat{\ell}$ and the smallest KS, CvM and AD statistics for both data sets. The PZD and PLBNXLD are very close, while PLD, PXL, PKD and PCJD fit less well. The advantage of the PGD comes from combining the gamma-mixture baseline with a shape parameter. A value $\delta < 1$ moves the mode towards the origin and lengthens the right tail. For Data set I, the fitted PGD density is almost reverse-J shaped, whereas the GD and ZD, whose densities vanish at the origin with a mode near one, cannot capture the many small observations.

Figures 5 and 8 show the histograms of the two data sets with the fitted densities of all competing models. Figures 6 and 9 show the empirical CDF with the fitted CDFs. Figures 7 and 10 show probability–probability (P–P) plots, that is, the fitted CDF $F(y_{(i)}; \hat{\theta})$ plotted against the empirical probabilities; points close to the 45-degree line indicate a good fit. For both data sets, the PGD curves follow the histogram and the empirical CDF closely, and its P–P points lie close to the diagonal. In contrast, the GD and ZD underestimate the frequency of small values in Data set I. These plots support the conclusions drawn from Tables 15 and 16.

Table 15: Goodness-of-fit measures and parameter estimates with standard errors (SE) for Data set I; $-\hat{\ell}$ is the negative maximized log-likelihood.

Model	$-\hat{\ell}$	KS KS-pval	CvM CvM-pval	AD AD-pval	$\hat{\delta}$ SE($\hat{\delta}$)	$\hat{\alpha}$ SE($\hat{\alpha}$)	$\hat{\theta}$ SE($\hat{\theta}$)
PGD	55.0788	0.0820 0.9763	0.0339 0.9639	0.2217 0.9833	0.6428 0.0775	1.8512 0.1788	- -
GD	64.5545	0.2352 0.0464	0.4974 0.0398	4.2629 0.0066	1.4796 0.1259	- -	- -
PLD	55.7599	0.0944 0.9226	0.0460 0.9028	0.3118 0.9286	0.8831 0.1095	0.9139 0.1452	- -
LD	56.3036	0.1327 0.5878	0.1185 0.5038	0.6875 0.5682	0.8238 0.1054	- -	- -
PXLD	55.6170	0.0935 0.9278	0.0446 0.9112	0.3003 0.9375	0.9516 0.1175	0.7447 0.1231	- -
XLD	55.7004	0.1081 0.8218	0.0643 0.7903	0.3903 0.8574	0.7145 0.0943	- -	- -
PKD	55.6837	0.0937 0.9266	0.0450 0.9088	0.3058 0.9333	0.9258 0.1123	0.8095 0.1239	- -
KD	55.8967	0.1162 0.7485	0.0799 0.6958	0.4761 0.7701	0.7627 0.0950	- -	- -
PZD	55.0840	0.0825 0.9747	0.0342 0.9625	0.2228 0.9829	1.8777 0.2213	0.6386 0.0798	- -
ZD	63.6745	0.2404 0.0393	0.5442 0.0302	4.3692 0.0059	1.3792 0.1405	- -	- -
PCJD	56.6873	0.1116 0.7909	0.0671 0.7728	0.4611 0.7855	1.3675 0.2014	0.8349 0.1016	- -
CJD	57.9268	0.1782 0.2303	0.2222 0.2287	1.2421 0.2518	1.1645 0.1321	- -	- -
PELD	83.7940	0.0907 0.9423	0.0437 0.9162	0.2863 0.9478	1.0121 0.2116	0.1561 0.2842	0.5787 0.2370
PLBNXLD	55.0949	0.0826 0.9744	0.0340 0.9635	0.2228 0.9823	0.6135 0.0790	2.0298 0.2590	- -
ShD	56.4564	0.1308 0.6059	0.1184 0.5043	0.7157 0.5447	0.8533 0.0948	- -	- -
AkD	57.5746	0.1565 0.3758	0.1567 0.3711	0.9440 0.3873	1.1656 0.1126	- -	- -
IshD	57.3029	0.1404 0.5142	0.1176 0.5077	0.7688 0.5031	1.1571 0.0962	- -	- -
XgD	56.4851	0.1384 0.5328	0.1096 0.5423	0.6293 0.6195	1.0313 0.1234	- -	- -

Table 16: Goodness-of-fit measures and parameter estimates with standard errors (SE) for Data set II; $-\hat{\ell}$ is the negative maximized log-likelihood.

Model	$-\hat{\ell}$	KS KS-pval	CvM CvM-pval	AD AD-pval	$\hat{\delta}$ SE($\hat{\delta}$)	$\hat{\alpha}$ SE($\hat{\alpha}$)	$\hat{\theta}$ SE($\hat{\theta}$)
PGD	317.0771	0.0378 0.9988	0.0199 0.9972	0.1383 0.9993	0.8039 0.0565	0.4912 0.0718	- -
GD	322.4647	0.1133 0.1532	0.2858 0.1483	2.1368 0.0775	0.3035 0.0175	- -	- -
PLD	318.3186	0.0520 0.9496	0.0458 0.9023	0.3029 0.9358	1.0832 0.0704	0.1530 0.0282	- -
LD	319.0374	0.0677 0.7487	0.0582 0.8265	0.4865 0.7602	0.1866 0.0133	- -	- -
PXLD	318.8048	0.0568 0.9038	0.0570 0.8337	0.3794 0.8686	1.1349 0.0695	0.1280 0.0228	- -
XLD	320.7617	0.0905 0.3862	0.1253 0.4751	0.9780 0.3688	0.1744 0.0125	- -	- -
PKD	318.4052	0.0533 0.9389	0.0485 0.8864	0.3182 0.9237	1.0927 0.0690	0.1482 0.0264	- -
KD	319.3398	0.0716 0.6840	0.0679 0.7658	0.5627 0.6835	0.1841 0.0130	- -	- -
PZD	317.2865	0.0400 0.9972	0.0247 0.9904	0.1656 0.9971	0.4149 0.0612	0.8514 0.0579	- -
ZD	320.4032	0.0989 0.2824	0.2062 0.2560	1.3985 0.2027	0.2909 0.0168	- -	- -
PCJD	319.8728	0.0561 0.9113	0.0613 0.8067	0.4554 0.7922	0.3210 0.0493	0.9420 0.0592	- -
CJD	320.3431	0.0749 0.6281	0.1218 0.4889	0.7193 0.5426	0.2792 0.0167	- -	- -
PELD	400.6617	0.0437 0.9911	0.0213 0.9958	0.1445 0.9990	1.3843 0.1937	0.2678 0.3857	0.1745 0.0472
PLBNXLD	317.4085	0.0416 0.9952	0.0278 0.9832	0.1865 0.9937	0.8946 0.0669	0.3518 0.0637	- -
ShD	317.6299	0.0423 0.9939	0.0294 0.9785	0.2072 0.9883	0.1983 0.0138	- -	- -
AkD	320.9646	0.1003 0.2669	0.2173 0.2365	1.3807 0.2077	0.2953 0.0168	- -	- -
IshD	321.8498	0.1090 0.1860	0.2600 0.1761	1.8342 0.1136	0.3016 0.0172	- -	- -
XgD	321.0203	0.0624 0.8304	0.0839 0.6709	0.6445 0.6065	0.2634 0.0162	- -	- -

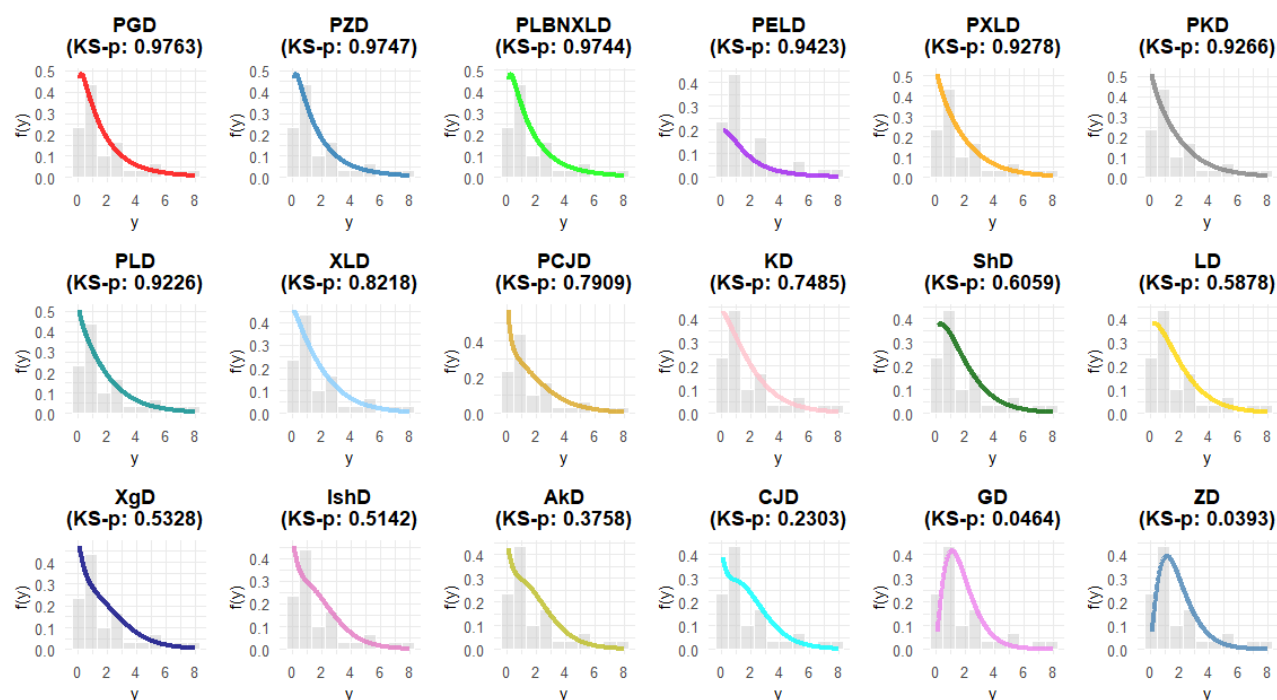


Fig. 5: Histogram of Data set I (vinyl chloride, mg/l) with the fitted PDFs of the PGD and the competing models.

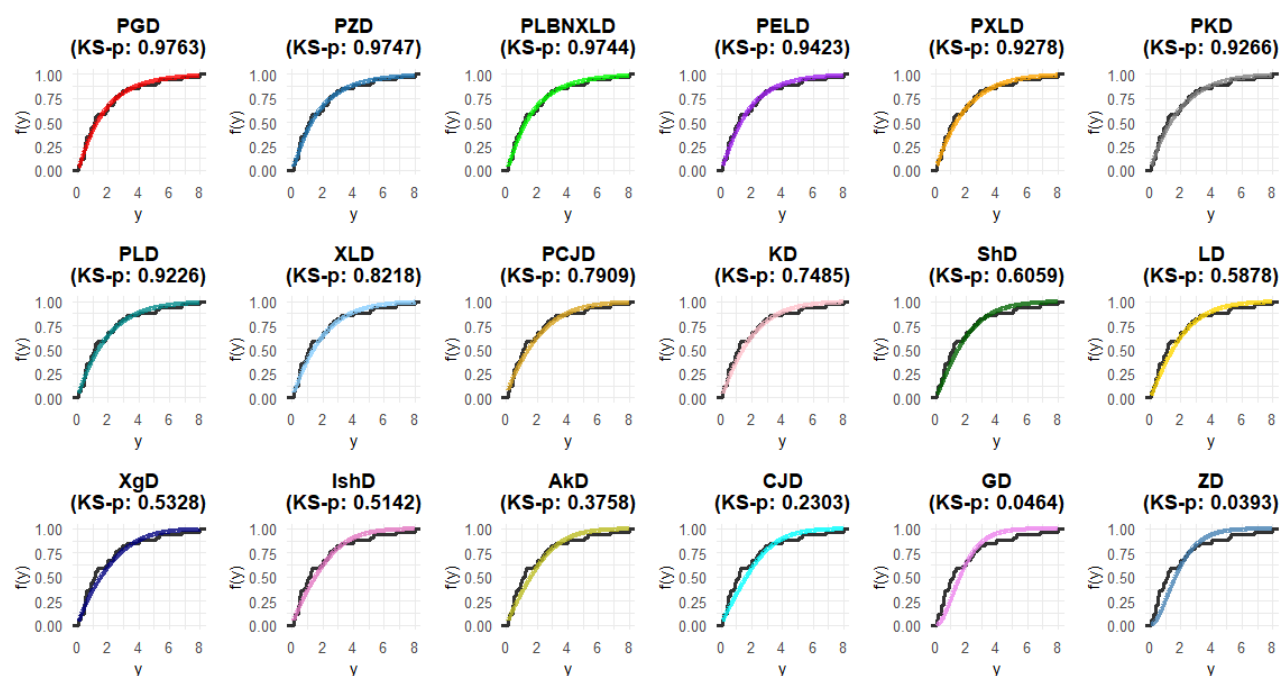


Fig. 6: Empirical CDF of Data set I with the fitted CDFs of the PGD and the competing models.

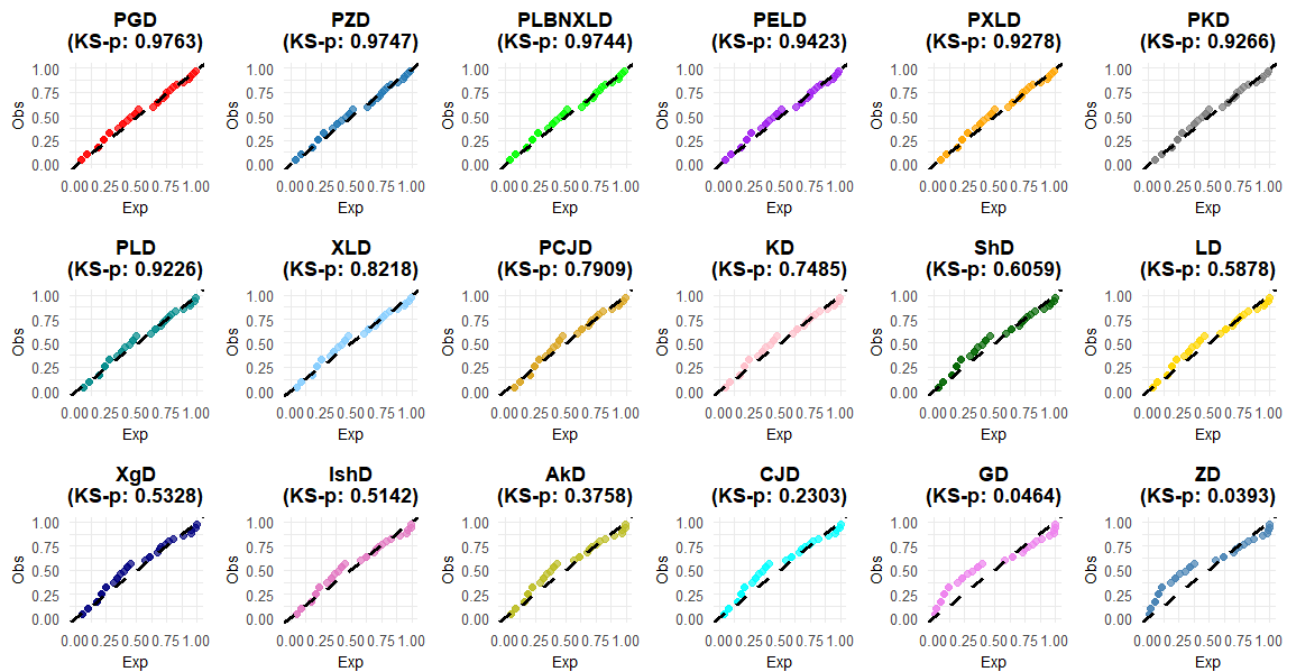


Fig. 7: P-P plots (fitted versus empirical probabilities) of the PGD and the competing models for Data set I.

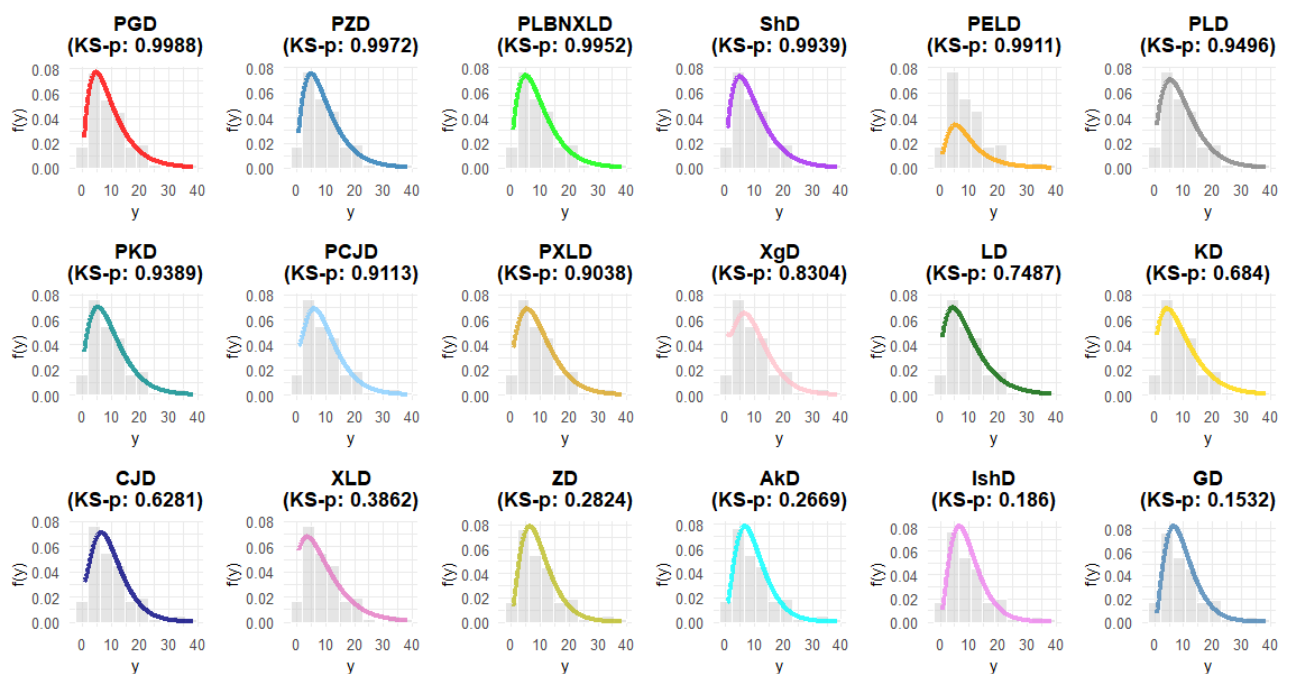


Fig. 8: Histogram of Data set II (bank waiting times, minutes) with the fitted PDFs of the PGD and the competing models.

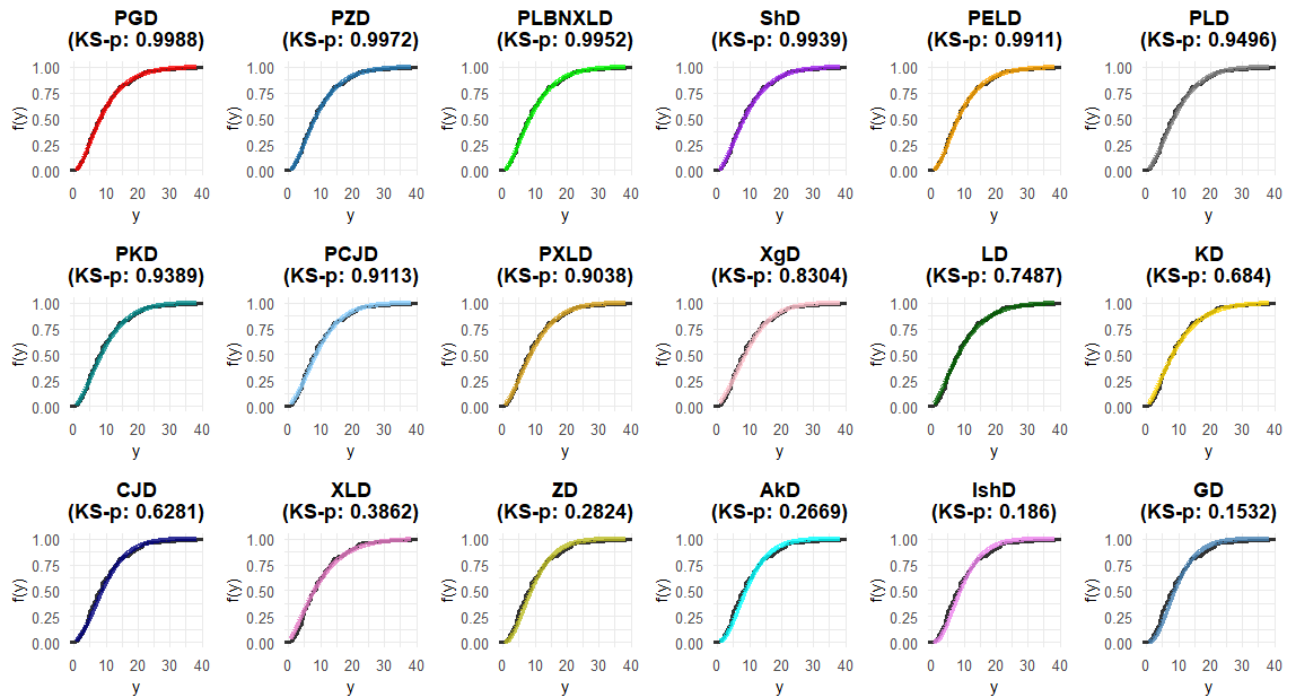


Fig. 9: Empirical CDF of Data set II with the fitted CDFs of the PGD and the competing models.

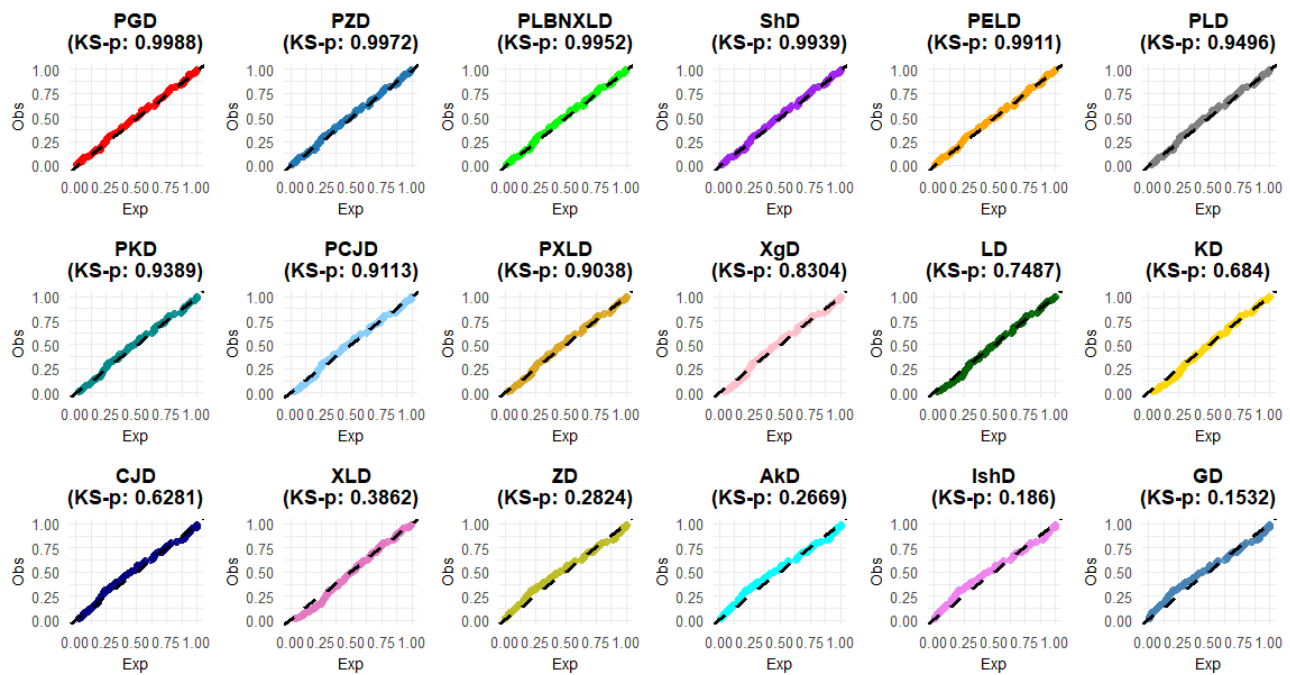


Fig. 10: P–P plots (fitted versus empirical probabilities) of the PGD and the competing models for Data set II.

7 Concluding Remarks

This article introduced the two-parameter power Garhy distribution (PGD), obtained by applying the power transformation to the Garhy distribution. The main findings are summarized below.

- Properties.* The PGD keeps the gamma-mixture structure of the GD after the transformation Y^δ . This gives closed-form raw moments and Shannon entropy, and a quantile function in terms of the generalized Lambert W function that can be evaluated by a simple, provably unique root search. The density can be decreasing or unimodal and the hazard rate is increasing. The shape parameter δ mainly controls skewness and tail weight, as shown by the Bowley and Moors measures.
- Stress–strength reliability.* $R = P(Y_2 < Y_1)$ was obtained as a series that converges for $\delta_2 < \delta_1$, with an exchanged form for $\delta_2 > \delta_1$, and in closed form when $\delta_1 = \delta_2$. Practical guidance on truncation was given.
- Inference.* Under progressive Type-II censoring, ML and Bayesian estimators of the parameters and of R , with asymptotic confidence and credible intervals, were developed. The simulation study shows that the MLE generally has smaller bias and MSE. Bayesian estimation is preferable for very small samples in which the likelihood is flat, and its credible intervals for R give better coverage near the boundary of $[0, 1]$. Bayesian results in small samples depend on the prior, and a vague gamma prior performed better than the power prior (18). Removing units early in the test (Scheme I) improves the estimation of the shape parameter.
- Applications.* For two real data sets, the PGD gave the largest likelihood and the best goodness-of-fit statistics among 18 models, and it clearly improved on the GD.

Future work. Natural extensions include objective (Jeffreys or reference) priors and Markov chain Monte Carlo methods for the joint posterior, bootstrap intervals for R , other censoring plans such as adaptive progressive and hybrid censoring, multicomponent stress–strength models, regression and quantile-regression models based on the PGD, and applications to censored reliability data.

Data Availability Statement

Both data sets analysed in Section 6 are publicly available in the cited sources [38–41] and are listed in full in Section 6. The code used for the simulation study is available from the corresponding author on request.

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Conflicts of Interest

The authors declare no conflict of interest.

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