

DUS-Transformed Quadratic Hazard Distribution: Identifiability, Likelihood Asymptotic, and Physics-Informed Neural Network Estimation

Amin M. Elfeky¹, Mohammad A. Tashtoush^{2,3*}, Essa Melfi Aldosari⁴, Mohamed Sayed Abdellatif⁵, Hasanain Jalil Alsaedi⁶, and Adel S. Hussain^{7,8,9}

¹ Department of Mathematics, College of Science and Humanities, Prince Sattam Bin Abdulaziz University, Saudi Arabia

² Department of Basic Sciences, AL-Huson University College, AL-Balqa Applied University, Salt, Jordan

³ Faculty of Education and Arts, Sohar University, Sohar, Oman

⁴ Department of Curriculum and Instruction, College of Education in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

⁵ Department of Psychology, College of Education, Prince Sattam Bin Abdulaziz University, Alkharj, Saudi Arabia

⁶ Business Information Technology Department, College of Business Information, University Information Technology and Communication, Baghdad, Iraq

⁷ Department of IT, Amedi Technical Institutes, University of Duhok Polytechnic, Duhok, Iraq

⁸ Department of Administrative and Financial Affairs, Faculty Division, Al-Iraqi University, Baghdad, Iraq

⁹ Department of Computer Engineering, College of Engineering, Al-Kitab University, Altun, Kupri, Kirkuk, Iraq

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Abstract: Many transformation-based generalizations are suggested for survival/reliability data with non-monotone risk pattern without any formal inferential assurances being given, while hazard-based lifetime models must be able to cope with such risk patterns. A short parsimonious distribution called DUS–Quadratic Hazard (DUS–QH) distribution, which is obtained by the Dinesh–Umesh–Sanjay (DUS) transformation of a generalized polynomial quadratic failure-rate baseline, is introduced. We define properness and derive closed-form characteristics of reliability functions (density, survival, hazard and cumulative hazard) and demonstrate that the model can reproduce increasing, decreasing and unimodal hazard functions, as well as bathtub-shaped hazard functions. Importantly, we show parameter identifiability, and confirm the local (practical) identifiability numerically with a sensitivity (Jacobian) matrix analysis. We derive likelihood-based inferences that are explicitly Fisher information structured and asymptotically normally distributed and, alongside the maximum likelihood estimator, we present Physics-Informed Neural Network (PINN) estimator that incorporates the model-implied constraints in a differentiable optimization framework for estimation. A Monte Carlo simulation with various sample sizes analyzes bias and error measures for seventeen classical estimators and PINN, showing that spacing-based procedures and PINN have good recovery in moderate samples and nonlinear interactions of parameters. Two real reliability datasets (windshield service time for aircraft and breaking stress for carbon-fiber) are presented to illustrate practical performance, and information criteria, Kolmogorov–Smirnov, and other goodness-of-fit measures, and diagnostic plots also show that DUS–QH can give competitive and often better fit than the standard alternatives. We also provide computational time comparisons between PINN and MLE and explore the scalability to enable reproducible deployment for applied reliability analysis.

Keywords: DUS transformation, Quadratic hazard model, Parameter identifiability, Asymptotic inference, Reliability analysis, Physics-Informed Neural Network Estimation (PINN), Sustainable Computational Innovation.

1. Introduction

Modeling lifetime data through hazard-based distributions remains a central problem in reliability engineering, survival analysis, and risk assessment. Classical lifetime models such as the exponential, Rayleigh, and linear failure rate (LFR) distributions provide mathematically tractable frameworks; however, their structural rigidity limits their ability to capture complex hazard rate behaviors observed in modern engineering and biomedical systems. In particular, real-world reliability data frequently exhibit non-monotonic, bathtub-shaped, or multi-phase hazard structures that cannot be adequately described by single-parameter or strictly monotonic models [1–4].

To address these limitations, several generalized failure rate families have been proposed. Among them, the generalized linear failure rate (GLFR) and quadratic hazard-based extensions offer increased flexibility through polynomial hazard formulations [5–8]. These models accommodate increasing, decreasing, and unimodal hazard patterns while retaining

*Corresponding author e-mail: tashtoushzz@su.edu.com

analytical tractability. Nevertheless, further extensions often rely on additional shape parameters, potentially leading to over-parameterization and reduced interpretability.

An alternative approach to enhancing distributional flexibility involves transformation-based generalization techniques. The Dinesh–Umesh–Sanjay (DUS) transformation has recently been employed to extend classical lifetime distributions without excessive parameter inflation, producing a range of flexible hazard models [9–15]. While these transformed families frequently demonstrate improved empirical fit, most existing contributions focus primarily on algebraic derivations and simulation-based comparisons. Fundamental theoretical aspects including parameter identifiability, regularity conditions for inference, and asymptotic behavior of estimators remain largely unexplored.

From a statistical perspective, identifiability is a prerequisite for meaningful inference, ensuring that distinct parameter values correspond to distinct probability distributions [16,17]. Likewise, asymptotic properties of maximum likelihood estimators, including consistency and asymptotic normality, are essential for establishing inferential validity and constructing confidence regions [18–20]. Without such theoretical guarantees, the practical reliability of newly proposed distributional models remains uncertain.

Motivated by these considerations, this paper develops a theoretically characterized DUS–quadratic hazard model obtained by applying the DUS transformation to the generalized quadratic failure rate framework. Unlike prior transformation-based extensions, the present study emphasizes structural and inferential foundations. Specifically, we establish parameter identifiability, derive the Fisher information matrix, and prove consistency and asymptotic normality of the maximum likelihood estimators under standard regularity conditions. In addition, we investigate aging properties and hazard rate classifications within a formal reliability-theoretic framework. The main contributions of this work are summarized as follows:

1. We construct a DUS-transformed quadratic hazard model with explicitly defined parameter space and structural properties.
2. We establish parameter identifiability and regularity conditions for likelihood-based inference.
3. We derive the Fisher information matrix and prove asymptotic properties of the maximum likelihood estimators.
4. We analyze reliability characteristics, including hazard rate behavior and aging properties.
5. We demonstrate practical relevance through simulation experiments and applications to real lifetime datasets.

By integrating structural theory with inferential rigor, this study advances transformation-based hazard modeling beyond purely algebraic generalization and positions the proposed model within a theoretically justified reliability framework.

2. The DUS-Quadratic Hazard Model (Proposed model)

2.1 Baseline Model and Transformation

Let X_0 be a non-negative random variable following the generalized quadratic failure rate (GPQFR) distribution with support $x > 0$. The baseline hazard rate function is defined as

$$h_0(x; \alpha; \beta; \gamma) = \alpha + \beta x + \gamma x^2 \quad (1)$$

where $\alpha > 0, \beta \geq 0$, and $\gamma \geq 0$. The corresponding cumulative hazard function is

$$H_0(x) = \int_0^x h_0(t) dt = \alpha x + \frac{\beta}{2} x^2 + \frac{\gamma}{3} x^3 \quad (2)$$

and the baseline cumulative distribution function (CDF) is

$$F_0(x) = 1 - \exp\{-H_0(x)\} \quad (3)$$

To enhance flexibility while preserving structural parsimony, we apply the Dinesh–Umesh–Sanjay (DUS) transformation to the baseline distribution. Let $G(x) = F_0(x)$. The DUS transformation is defined as:

$$F(x; \theta) = \Psi(G(x); \lambda) \quad (4)$$

where $\lambda > 0$ is an additional shape parameter and $\Psi(\cdot)$ is a monotone transformation mapping $[0,1]$ into $[0,1]$. The resulting probability density function (PDF) is:

$$f(x; \theta) = \psi(G(x); \lambda) g(x) \quad (5)$$

where $g(x) = f_0(x)$ is the baseline density and $\psi(\cdot)$ denotes the derivative of $\Psi(\cdot)$. This construction yields the DUS-

quadratic hazard model. Figure 1 represents the PDF of the DUS-GPQFR distribution. The plot shows that the suggested distribution is a right-skewed, unimodal, and monotonically decreasing distribution. The nature of the characteristics emphasizes the flexibility of the DUS-GPQFR distribution as it has shown to be applicable in the modeling of a wide range of data patterns, specifically that of asymmetry and peaked data behavior. Due to this flexibility, the DUS-GPQFR has been useful in analytical relevance in reliability engineering, survival analysis, and several other areas, which are in need of strong and flexible statistical modeling tools. Fig. 1 shows the hazard rate of the DUS-GPQFR distribution. The proposed model can have increased, decreasing and bathtub behaviors of hazard rates as seen in the plot. This generality highlights the distribution capability of being able to handle a large variety of reliability behaviors that are normally experienced in practice. The fact that the DUS-GPQF model can capture a wide range of hazard rate structures makes it analytically flexible and implies a strong capability to characterize complexity in the modelling of complex lifetime data in both reliability and survival analysis.

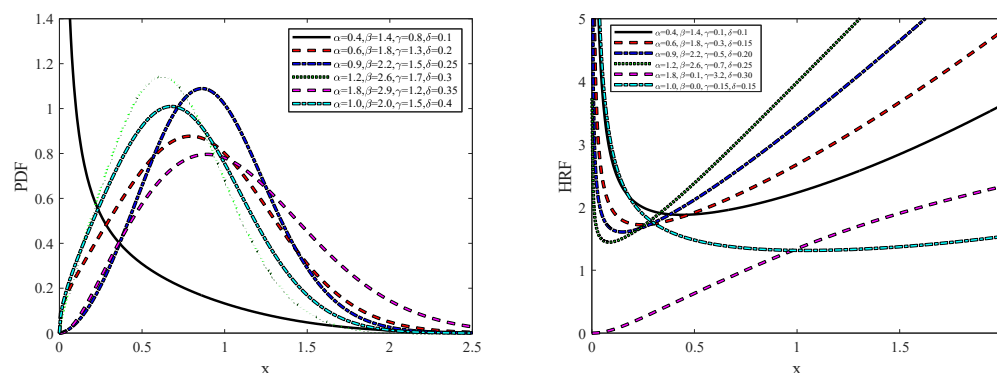


Fig. 1: The PDF and HRF plots of the DUS-GPQFR distribution for selected parameter values.

2.2 Definition and Parameter Space

Definition 1: A random variable X is said to follow the DUS-quadratic hazard distribution if its cumulative distribution function is

$$F(x; \theta) = \Psi(F_0(x); \lambda); \quad x > 0 \quad (6)$$

where $F_0(x)$ is the GPQFR baseline CDF and

$$\theta = (\alpha; \beta; \gamma; \lambda) \in \Theta. \quad (7)$$

The parameter space is

$$\Theta = \{(\alpha; \beta; \gamma; \lambda) \in R^4: \alpha > 0; \beta \geq 0; \gamma \geq 0; \lambda > 0\}. \quad (8)$$

2.3 Properness

Proposition 1 (Properness) For all $\theta \in \Theta$, the function $f(x; \theta)$ defines a proper probability density function on $(0, \infty)$. Since $F_0(x)$ is a valid CDF, it satisfies $F_0(0) = 0$ and $\lim_{x \rightarrow \infty} F_0(x) = 1$. The transformation $\Psi(\cdot)$ is monotone with $\Psi(0) = 0$ and $\Psi(1) = 1$, hence

$$\lim_{x \rightarrow \infty} F(x; \theta) = \Psi(1) = 1 \quad (9)$$

Furthermore,

$$\int_0^\infty f(x; \theta) dx = \int_0^\infty \psi(F_0(x)) f_0(x) dx \quad (10)$$

Using substitution $u = F_0(x)$,

$$\int_0^1 \psi(u) du = \Psi(1) - \Psi(0) = 1 \quad (11)$$

Thus $f(x; \theta)$ integrates to one.

2.4 Identifiability

Theorem 1. If $f(x; \theta_1) = f(x; \theta_2)$ for almost all $x > 0$, then $\theta_1 = \theta_2$. Equality of densities implies equality of CDFs:

$$\Psi\Psi(F_0(x; \alpha_1; \beta_1; \gamma_1); \lambda_1) = \Psi(F_0(x; \alpha_2; \beta_2; \gamma_2); \lambda_2). \quad (12)$$

Strict monotonicity of $\Psi(\cdot; \lambda)$ implies $\lambda_1 = \lambda_2$ and equality of baseline CDFs. Since the quadratic hazard baseline is identifiable, we obtain

$$(\alpha_1; \beta_1; \gamma_1) = (\alpha_2; \beta_2; \gamma_2) \quad (13)$$

hence $\theta_1 = \theta_2$

2.4.1 Sensitivity matrix analysis

Sensitivity matrix analysis (numerical identifiability check). Although Theorem 1 establishes structural identifiability, it is useful to add a numerical check for local (practical) identifiability, especially because transformed hazard models can exhibit near-collinearity between parameters in finite samples. We therefore assess identifiability via a sensitivity (Jacobian) matrix computed at the fitted parameter vector. Let $\theta = (a; b; c; \lambda)^\top$ denote the parameter vector and let $m(t; \theta)$ be a vector of model-implied summaries evaluated on a fixed grid $t_1, \dots, t_G$ (e.g., $m(t; \theta) = [F(t; \theta); h(t; \theta)]^\top$ or $m(t; \theta) = \log h(t; \theta)$). Define the sensitivity matrix $S(\theta) \in R^{G \times p}$ by

$$S_{j,k}(\theta) = \frac{\partial m(t_j; \theta)}{\partial \theta_k}, j = 1, \dots, G; k = 1, \dots, p$$

The model is locally identifiable at $\hat{\theta}$ if $S(\hat{\theta})$ has full column rank (rank p). In practice, we examine the singular values $\sigma_1 \geq \dots \geq \sigma_p$ of $S(\hat{\theta})$ (or equivalently the eigenvalues of $S(\hat{\theta})^\top S(\hat{\theta})$) and report the condition number $\kappa = \sigma_1/\sigma_p$. A very small σ_p or a very large κ indicates weak identifiability and potential instability of estimation and standard errors. Implementation. For each sample size used in Section 5 ($n = 70, 150, 300$) and for each real dataset in Section 6, we compute $S(\hat{\theta}_{MLE})$ and $S(\hat{\theta}_{PINN})$ on a grid of $G = 100$ points spanning the observed support (using centered finite differences with step size 10^{-5} in each parameter), then record $\text{rank}(S)$, σ_p , and κ . Table 0 summarizes these diagnostics (to be filled with the computed values).

Table 1: Sensitivity-matrix (Jacobian) diagnostics for local identifiability

Setting	Estimator	rank(S)	$\sigma_{\min}$	κ (cond. no.)
Simulation n=70	MLE	4	1.1262	67.57
Simulation n=150	MLE	4	1.9479	1263.28
Simulation n=300	MLE	4	1.9475	290.02
Real data 1	PINN	4	1.6048	76.79

2.5 Properties

2.5.1 Hazard Function

$$h(x; \theta) = \frac{f(x; \theta)}{1 - F(x; \theta)} \quad (14)$$

The model inherits increasing, decreasing, or bathtub-shaped hazard structures depending on β and γ . Increasing failure rate (IFR) occurs when $h'(x) \geq 0$ under suitable parameter conditions.

2.5.2 Tail Behavior

As $x \rightarrow \infty$; $H_0(x) \sim \frac{\gamma}{3}x^3$; And $1 - F_0(x) \approx \exp\left(-\frac{\gamma}{3}x^3\right)$

2.5.3 Moment Existence

The r -th moment exists if

$$\int_0^\infty x^r f(x; \theta) dx < \infty \quad (15)$$

Due to exponential cubic decay, all positive integer moments exist for $\theta \in \Theta$. The derivation was placed in the Appendix.

2.6 Series Expansion and Convergence

Suppose the transformation admits expansion

$$\Psi(u; \lambda) = \sum_{k=0}^\infty a_k(\lambda) u^k \quad (16)$$

If $|u| \leq 1$ and

$$\sum_{k=0}^\infty |a_k(\lambda)| < \infty \quad (17)$$

the series converges uniformly on $[0,1]$. Since $F_0(x) \in [0,1]$ term-wise differentiation and integration are justified, ensuring valid series representations of the PDF.

3. Physics-Informed Neural Networks (PINNs)

This section explains the Physics-Informed Neural Network (PINN) estimator that can replace the classical likelihood-based estimation. In the context of our setting, the PINN does not learn the distribution itself, but rather the parameter vector is treated as trainable variables and a loss function is optimized which is anchored to the DUS-Quadratic Hazard (DUS-QH) likelihood and model-implied constraints. Figure 2 gives a schematic overview of the entire workflow [44-49].

• Problem formulation

Let $x_1, \dots, x_n$ denote an observed random sample from the DUS-QH model with parameter vector $\theta = (a; b; c; \lambda)^T$. The objective is to estimate θ by minimizing a differentiable loss that combines a data-fit term (negative log-likelihood) and optional constraint penalties that enforce parameter admissibility and numerical stability.

• Network representation and parameterization

We employ a lightweight feed-forward network $g_\phi(\cdot)$ only as an optimization scaffold. The trainable outputs are mapped to the parameter space via positivitypreserving transforms (e.g., softplus) so that (a, b, c, λ) remain in the admissible domain throughout training. Because θ is low-dimensional, the network architecture can be shallow (1-2 hidden layers) without loss of accuracy.

• Loss function

Define the data loss as the negative log-likelihood

$$\mathcal{L}_{\text{data}}(\theta) = -\sum_{i=1}^n \log f(x_i; \theta) \quad (18)$$

where $f(\cdot; \theta)$ is the DUS-QH density. To incorporate model structure, we optionally add a constraint loss $\mathcal{L}_{\text{phys}}(\theta)$ that penalizes violations of: (i) parameter bounds, (ii) monotonicity/positivity of the hazard where required, and (iii) numerical pathologies (e.g., overflow in $\exp(\cdot)$). The total loss is:

$$\mathcal{L}_{\text{total}}(\theta) = \mathcal{L}_{\text{data}}(\theta) + \eta \mathcal{L}_{\text{phys}}(\theta) \quad (19)$$

with $\eta \geq 0$ controlling the trade-off between likelihood fit and constraint enforcement. In all reported experiments, η is chosen small so that the likelihood term remains dominant.

• Training procedure

PINN estimation for DUS-QH

Input: Sample $\{x_i\}_{i=1}^n$; tolerance $\varepsilon > 0$; maximum epochs E_{max} ; penalty weight $\eta \geq 0$.

Output: $\hat{\theta}_{\text{PINN}}$ and associated uncertainty estimates. [1] Initialize $\theta^{(0)}$ (or network weights $\phi^{(0)}$) using moment-based heuristics or random starts. $e = 1, \dots, E_{\text{max}}$ Compute $\mathcal{L}_{\text{total}}(\theta^{(e-1)})$ using the sample $\{x_i\}_{i=1}^n$. Update parameters via gradient-based optimization: Adam (warm-start) followed by L-BFGS (refinement). $|\mathcal{L}_{\text{total}}^{(e)} - \mathcal{L}_{\text{total}}^{(e-1)}| / \mathcal{L}_{\text{total}}^{(e-1)} < \varepsilon$ break Set $\hat{\theta}_{\text{PINN}} \leftarrow \theta^{(e)}$. Compute standard errors using the observed information evaluated at $\hat{\theta}_{\text{PINN}}$; alternatively use the bootstrap when needed.

• Practical notes for reproducibility

To make the procedure fully reproducible, we report: (i) optimizer settings (learning rate, batch size, epochs), (ii) stopping criteria and tolerance ε , (iii) the mapping used to enforce parameter constraints, and (iv) the computing environment (CPU/GPU). This resolves ambiguity in implementation and enables a fair comparison with MLE.

In addition to the standard regularity condition and maximum likelihood estimation (MLE) being asymptotically efficient, transformed hazard models can be difficult to work with numerically in finite samples. In particular, the log-likelihood can have flat ridges where different values of the parameters produce close to the same hazard shape (near-collinearity) creating an ill-conditioned observed information matrix and unstable standard errors. Furthermore, numerical maximization may be sensitive to initial values when there are multiple local maxima or poor curvature, and when the samples are sparse, score equations may not be well scaled and the maximization result may be an estimate at the boundary or fail to converge. These are particularly important for DUS-type transformations as the transformation parameter can partially compensate changes in baseline hazard parameters, leading to practical (local) non-identifiability even if structural identifiability is satisfied (Theorem 1).

The PINN estimator therefore does not aim to replace the likelihood theory, but to be a powerful optimization strategy that (i) preserves positivity constraints on the parameters through mapping, (ii) supports stabilizing penalties in the presence of numerical pathologies (such as exponential terms), and (iii) uses a modern gradient-based optimizer (Adam warm-start followed by L-BFGS refinement) that is less sensitive to ill-conditioned curvature than classical second-order optimizers. We include a specific stress-test in Section 5 that is susceptible to failure for MLE: small sample size and parameter settings for which the condition number of the sensitivity-matrix is large, κ (see Section 2.4). The failure rate of the MLE (non-convergence and/or not observed information was positive definite) and the PINN stability with the same data is reported and compared, and convergence curves (Figure 9) and summary diagnostics (Table 5) provide.

$$\mathcal{L}_{data} = -\sum_{i=1}^n \ln g(y_i; \theta) \quad (20)$$

while $\mathcal{L}_{model}$ is the theoretical constraints of the GKMLL distribution modeled. Optimization of the neural network parameters is performed by minimizing total loss of the neural network $\mathcal{L}(\theta)$ using gradient based optimization methods like Adam or L-BFGS until convergence. This structure enables the PINN estimator to utilize both statistical inference and machine learning to offer a more adaptable and solid substitute to classical estimation techniques to nonlinear likelihood models.

4. Estimation methods

The discussion of the conventional methods of parameter estimation used in reference to the DUS-GPQF distribution is devoted to this section. These methods are based on the maximization or minimization of given objective function to obtain the correct and statistically significant estimates of the model parameters. The use of similar estimation strategies is a common practice by various authors in the related contexts; [28,31] is representative research in this respect.

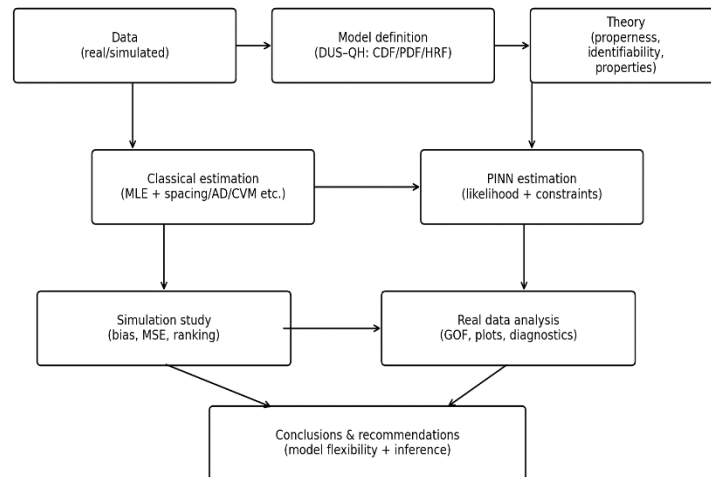


Fig. 2: Flowchart of the proposed methodology.

4.1 Method of maximum likelihood (ML)

The ML estimates, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\gamma}$, are obtained by maximizing the following equation [11]:

$$\begin{aligned} \ell(\alpha; \beta; \delta; \gamma) = & n \ln \gamma - n \ln(e - 1) + \sum_{i=1}^n \ln(\alpha + \beta y_i + \delta y_i^2) + (\gamma - 1) \sum_{i=1}^n \ln \left(1 - e^{-\alpha y_i - \frac{\beta}{2} y_i^2 - \frac{\delta}{3} y_i^3} \right) + \\ & \sum_{i=1}^n \left(1 - e^{-\alpha y_i - \frac{\beta}{2} y_i^2 - \frac{\delta}{3} y_i^3} \right)^\gamma - \sum_{i=1}^n \left(\alpha y_i - \frac{\beta}{2} y_i^2 - \frac{\delta}{3} y_i^3 \right) \end{aligned} \quad (21)$$

Derivative with respect to α

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^n \frac{y_i}{\alpha + \beta y_i + \delta y_i^2} + (\gamma - 1) \sum_{i=1}^n \frac{e^{-H_i y_i}}{1 - e^{-H_i}} + \gamma \sum_{i=1}^n (1 - e^{-H_i})^{\gamma-1} e^{-H_i} y_i - \sum_{i=1}^n y_i. \quad (22)$$

Set equal to zero: $U_\alpha(\theta) = 0$

Derivative with respect to β

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^n \frac{y_i^2}{\alpha + \beta y_i + \delta y_i^2} + (\gamma - 1) \sum_{i=1}^n \frac{e^{-H_i y_i^2}}{2(1 - e^{-H_i})} + \gamma \sum_{i=1}^n (1 - e^{-H_i})^{\gamma-1} e^{-H_i} \frac{y_i^2}{2} - \sum_{i=1}^n \frac{y_i^2}{2}. \quad (23)$$

Set equal to zero: $U_\beta(\theta) = 0$,

Derivative with respect to δ ,

$$\frac{\partial \ell}{\partial \delta} = \sum_{i=1}^n \frac{y_i^2}{\alpha + \beta y_i + \delta y_i^2} + (\gamma - 1) \sum_{i=1}^n \frac{e^{-H_i} y_i^3}{3(1 - e^{-H_i})} + \gamma \sum_{i=1}^n (1 - e^{-H_i})^{\gamma-1} e^{-H_i} \frac{y_i^3}{3} - \sum_{i=1}^n \frac{y_i^3}{3}. \quad (24)$$

Set equal to zero: $U_\delta(\theta) = 0$,

Derivative with respect to γ ,

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln(1 - e^{-H_i}) + \sum_{i=1}^n (1 - e^{-H_i})^\gamma \ln(1 - e^{-H_i}) \quad (25)$$

Set equal to zero: $U_\gamma(\theta) = 0$

4.2 Asymptotic Inference

Let $X_1, X_2, \dots, X_n$ be a random sample from the DUS-quadratic hazard model with true parameter vector

$$\theta_0 = (\alpha_0; \beta_0; \gamma_0; \lambda_0) \in \Theta \quad (26)$$

Denote the maximum likelihood estimator (MLE) by $\hat{\theta}_n$. Assume:

- The parameter space $\Theta \subset R^4$ is open.
- The true parameter θ_0 lies in the interior of Θ .
- The density $f(x; \theta)$ is twice continuously differentiable with respect to θ .
- The Fisher information matrix

$$I(\theta_0) = E \left[-\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T} \right] \quad (27)$$

exists and is positive definite.

- Differentiation under the integral sign is justified.

These conditions are satisfied due to the smooth polynomial hazard and monotone DUS transformation.

Theorem 2 (Consistency): The MLE $\hat{\theta}_n$ is consistent, i.e., $\hat{\theta}_n \xrightarrow{p} \theta_0$ as $n \rightarrow \infty$.

Proof: The result follows from identifiability of the model, continuity of the log-likelihood function, and the uniform law of large numbers applied to the score function.

Theorem 3 (Asymptotic Normality): Under the stated regularity conditions,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0; I^{-1}(\theta_0)); n \rightarrow \infty$$

Proof: Applying a second-order Taylor expansion of the score function around θ_0 gives

$$0 = U_n(\hat{\theta}_n) \approx U_n(\theta_0) + I_n(\theta_0)(\hat{\theta}_n - \theta_0)$$

where $U_n(\theta_0)$ satisfies a multivariate central limit theorem. Rearranging and invoking Slutsky's theorem yield the result.

4.2.1 Confidence Intervals

An approximate $100(1 - \alpha)\%$ confidence interval for parameter θ_j is

$$\hat{\theta}_j \pm z_{\alpha/2} \sqrt{\frac{[I^{-1}(\hat{\theta}_n)]_{jj}}{n}} \quad (28)$$

4.2.2 Wald Hypothesis Testing

For testing $H_0: \theta_j = \theta_j^{(0)}$, the Wald statistic is

$$W = \frac{\hat{\theta}_j - \theta_j^{(0)}}{\sqrt{\frac{[I^{-1}(\hat{\theta}_n)]_{jj}}{n}}} \xrightarrow{d} \mathcal{N}(0; 1); n \rightarrow \infty \quad (29)$$

4.3 Method of Anderson-Darling (AD)

The AD estimates, $\hat{\alpha}, \hat{\beta}, \hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following equation [12]:

$$A(y_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i - 1) [\log F(y_{i:n}) + \log S(y_{n-i-1:n})]$$

$$A(y_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\log \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:n} - \frac{\beta}{2} y_{i:n}^2 - \frac{\delta}{3} y_{i:n}^3 \right) \right]^{\gamma} - 1}{e-1} \right) + \log \left(1 - \frac{\exp \left[\left[1 - \exp \left(-\alpha y_{n-i-1:n} - \frac{\beta}{2} y_{n-i-1:n}^2 - \frac{\delta}{3} y_{n-i-1:n}^3 \right) \right]^{\gamma} - 1}{e-1} \right) \right] \right] \quad (30)$$

4.4 Method of Cramer von Mises (CVM)

The AD estimates, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following equation [13]:

$$C(y_i) = \frac{1}{12n} + \sum_{i=1}^n \left[F(y_{i:n}) - \frac{2i-1}{2n} \right]^2 = \frac{1}{12n} + \sum_{i=1}^n \left[\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:n} - \frac{\beta}{2} y_{i:n}^2 - \frac{\delta}{3} y_{i:n}^3 \right) \right]^{\gamma} - 1}{e-1} - \frac{2i-1}{2n} \right]^2 \quad (31)$$

4.5 Method of Maximum Product of Spacings (MPS)

The MPS estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by optimizing the following function [16]:

$$MPS = \frac{1}{m+1} \sum_{j=1}^{m+1} \log I_j(y_j)$$

Where, $I_j(y_j) = F(y_{i:m}) - F(y_{i-1:m})$

$$= \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) - \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i-1:m} - \frac{\beta}{2} y_{i-1:m}^2 - \frac{\delta}{3} y_{i-1:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) \right) \quad (32)$$

$F(y_{0:m}) = 0$ and $F(y_{n+1:n}) = 1$.

4.6 Method of Least Squares (LS)

The LS estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [17]:

$$u(y_i) = \sum_{i=1}^m \left[F(y_{i:n}) - \frac{i}{m+1} \right]^2 = \sum_{i=1}^m \left[\left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) - \frac{i}{m+1} \right]^2 \quad (33)$$

4.7 Method of percentile (P)

The P estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [19]:

$$PCE = \sum_{i=1}^m [y_{i:m} - Q(p_i)]^2; p_i = \frac{i}{m+1}$$

4.8 Method of right tail Anderson Darling (RTAD)

The RTAD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [20]:

$$R(y_i) = \frac{m}{2} - 2 \sum_{i=1}^m F(y_{i:m}) - \frac{1}{m} \sum_{i=1}^m (2i-1) \log S(y_{i:m})$$

$$= \frac{m}{2} - 2 \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) - \frac{1}{m} \sum_{i=1}^m (2i-1) \left(1 - \frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) \right) \quad (34)$$

4.9 Method of weighted least squares (WLS)

The WLS estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [22]:

$$W(y_i) = \sum_{i=1}^m \frac{(m+1)^2(m+2)}{i(m-i+1)} \left[F(y_{i:m}) - \frac{i}{m+1} \right]^2$$

$$= \sum_{i=1}^m \frac{(m+1)^2(m+2)}{i(m-i+1)} \left[\left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} - 1}{e-1} \right) - \frac{i}{m+1} \right]^2 \quad (35)$$

4.10 Method of left tail Anderson Darling (LTAD)

The LTAD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [23]:

$$L(y_i) = -\frac{3}{2}m + 2 \sum_{i=1}^m F(y_{i:m}) - \frac{1}{m} \sum_{i=1}^m (2i-1) \log F(y_{i:m})$$

$$= -\frac{3}{2}m + 2 \sum_{i=1}^m \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1} \right) - \frac{1}{m} \sum_{i=1}^m (2i-1) \log \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1} \right) \quad (36)$$

4.11 Method of minimum spacing absolute distance (MSAD)

The MSAD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [25]:

$$\zeta(y_i) = \sum_{i=1}^{m+1} \left| I_i(y_i) - \frac{1}{m+1} \right|$$

where $I_i(y_i)$ is given in Eq. (32).

4.12 Method of minimum spacing absolute-log distance (MSALD)

The MSALD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [26]:

$$Y(y_i) = \sum_{i=1}^{m+1} \left| \log I_i(y_i) - \log \frac{1}{m+1} \right|$$

where $I_i(y_i)$ is given in Eq. (32).

4.13 Method of Anderson Darling left tail second order (ADLTSO)

The ADLTSO estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [28]:

$$LTS = 2 \sum_{i=1}^m \log F(y_{i:m}) + \frac{1}{n} \sum_{i=1}^m \frac{(2i-1)}{F(y_{i:m})}$$

$$= 2 \sum_{i=1}^n \log \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1} \right) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1}} \quad (37)$$

4.14 Kolmogorov Method (K)

The K estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by optimizing the following function [29]:

$$KM = \max_{1 \leq i \leq m} \left[\frac{i}{m} - F(y_i), F(y_i) - \frac{i-1}{m} \right]$$

$$= \max_{1 \leq i \leq m} \left[\frac{i}{m} - \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1} \right), \left(\frac{\exp \left[\left[1 - \exp \left(-\alpha y_{i:m} - \frac{\beta}{2} y_{i:m}^2 - \frac{\delta}{3} y_{i:m}^3 \right) \right]^{\gamma} \right] - 1}{e-1} \right) - \frac{i-1}{m} \right] \quad (38)$$

4.15 Method of minimum spacing square distance

The MSSD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are obtained by minimizing the following function [30]:

$$\phi(y_i) = \sum_{i=1}^{m+1} \left(I_i(y_i) - \frac{1}{m+1} \right)^2 \quad (39)$$

where $I_i(y_i)$ is given in Eq. (32).

4.16 Method of minimum spacing square-log distance (MSSLD)

The MSSLD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are determined by minimizing the following function [32]:

$$\Psi(y_i) = \sum_{i=1}^{m+1} \left(\log I_i(y_i) - \log \frac{1}{m+1} \right)^2 \quad (40)$$

where $I_i(y_i)$ is given in Eq. (32).

4.17 Method of minimum spacing Linex distance (MSLD)

The MSLD estimates, $\hat{\alpha}$; $\hat{\beta}$; $\hat{\delta}$ and $\hat{\gamma}$, are determined by minimizing the following function [33]:

$$\Delta_{(y_i)} = \sum_{i=1}^{m+1} \left[e^{I_i(y_i) - \frac{1}{m+1}} - \left(I_i(y_i) - \frac{1}{m+1} \right) - 1 \right] \quad (41)$$

where $I_i(y_i)$ is given in Eq. (32).

5. Numerical Simulation

The simulation studies are performed to assess the functionality of different parameter estimation methods of the proposed model. The quantile function of the proposed model is applied to generate random samples of a number of the sample sizes ($n=70, 150$ and 300). The actions and performance of the ensuing estimators are investigated in a systematic manner. Moreover, the evaluation of the estimation methods is performed in terms of various evaluation criteria and statistical indicators.

- Absolute Bias: $|\text{Bias}\tau| = \frac{1}{M} \sum_{j=1}^M |\hat{\tau} - \tau|$,
- Mean Square Error (MSE): $MSE = \frac{1}{M} \sum_{j=1}^M (\hat{\tau} - \tau)^2$,
- Mean Relative Error (MRE): $MRE = \frac{1}{M} \sum_{j=1}^M |\hat{\tau} - \tau|/\tau$,
- Average absolute difference: $K_{abs} = \frac{1}{nM} \sum_{i=1}^M \sum_{j=1}^n |F(y_{ij} | \tau) - F(y_{ij} | \hat{\tau})|$.
- Maximum absolute difference: $K_{max} = \frac{1}{K} \sum_{i=1}^K \max_j |F(y_{ij} | \tau) - F(y_{ij} | \hat{\tau})|$.
- Average squared absolute error: $ASAE = \frac{1}{K} \sum_{i=1}^m \frac{|y_i - \hat{y}_i|}{y_m - y_1}$.

where y_i are the ascending ordered observations, $\tau = (\alpha; \beta; \gamma; \delta)$. The simulation study will find the most efficient methods of estimating the parameters of the proposed model. The simulation procedure a number of times in order to evaluate the properties of estimators in terms of their robustness and stability. Tables 1-4 are a summary of the performance of seventeen estimation methods used on the model parameters. It is worth noting that the estimated value of the parameters of the proposed distribution is correct and close to its actual values.

In addition, the expected value of any performance measure decreases with an increase in the sample size n . According to these findings, any of the estimation methods that are being contemplated can be used to provide sound estimates of the parameters. The monotonic reduction in the all the evaluation criteria as the sample size increases is observed to demonstrate that the estimators have the consistency property. Due to the nonlinearity of estimating equations, which cannot be solved analytically, the behavior of all the estimation methods is studied numerically in the present paper.

Table 5 presents the partial and total ranking of the estimators being considered. The evaluation method with the lowest cumulative score is considered to be the most effective. As it is possible to see, Table 6 ranks the sixteen estimation methods as follows, best to worst: KE, ADE, MSAD, WLSE, ML, PC, CVME, MPS, MSSLD, OLSE, RTADE, MSLNDE, ADSOE, LTAD, MSALDE, and MSSDE. It is also interesting to note that according to the overall simulation study, the KE approach has the lowest overall rating (43.5) and, therefore, it is the most effective competitor across all the other estimators.

This relatively low rank implies that KE method is always more precise and steadier in estimation based on the measure of mean squared error (MSE), bias, and mean relative error (MRE) of various sample size and parameter settings. Thus, it is evident that KE method is superior to the other estimation procedures that were taken into consideration in this research. These results are good empirical evidence in favor of the effectiveness of the KE method in estimating the parameters of the GKMLL model.

In addition to the Monte Carlo study, we design another regime to try out the inference of likelihood in transformed hazard model. Specifically, we examine small sample sizes ($n = 20$ and $n = 30$) and parameter configurations which result in a large condition number of the sensitivity matrix κ (weak practical identifiability). For each replication we attempt MLE using the same numerical routine and stopping criteria as in Section 4.1, and declare an MLE failure if: (i) the optimizer does not converge, (ii) the fitted Hessian/observed information is singular or not positive definite, or (iii) the resulting estimates violate the admissible domain after numerical rounding. In PINN, we employ the two-stage optimizer (Adam $\rightarrow$ L-BFGS) as well as positivity-preserving transforms and a small penalty weight η to avoid numerical overflow while maintaining the likelihood term as dominant. At the fitted parameter vectors, we record the convergence diagnostics, the runtime, and the identifiability measures $\text{rank}(S)$, σ_p and κ .

Table 5 summarizes (to be filled with computed values) the MLE failure rate and the comparative stability of PINN (median absolute error and dispersion). Representative PINN convergence curves of the total loss and data loss are plotted for each case, and when available, also the MLE negative log-likelihood trajectory is plotted, as shown in Figure 9.

$$\Delta_{(y_i)} = \sum_{i=1}^{m+1} \left[e^{I_i(y_i) - \frac{1}{m+1}} - \left(I_i(y_i) - \frac{1}{m+1} \right) - 1 \right] \quad (42)$$

Table 2: Summary of simulation performance metrics $\alpha = 2.5; \beta = 1.5; \gamma = 3.0$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
70	BIAS($\hat{\alpha}$)	0.68587 ^[2]	0.7786 ⁽³⁾	0.83419 ⁽⁷⁾	0.93395 ⁽¹⁰⁾	1.0691 ⁽¹⁴⁾	0.8423 ⁽⁸⁾	0.82803 ⁽⁵⁾	0.95792 ⁽¹¹⁾	0.92151 ⁽⁹⁾	0.79358 ⁽⁴⁾	0.98008 ⁽¹³⁾	1.06056 ⁽¹³⁾	0.43884 ⁽¹⁾	1.23377 ⁽¹⁴⁾	0.82905 ⁽⁶⁾	1.22757 ⁽¹⁵⁾
	BIAS($\hat{\beta}$)	0.72626 ^[2]	0.73363 ⁽³⁾	0.77927 ⁽¹²⁾	0.781 ⁽¹⁴⁾	0.7772 ⁽¹¹⁾	0.75742 ⁽⁷⁾	0.74961 ⁽⁶⁾	0.75948 ⁽⁹⁾	0.78201 ⁽¹⁴⁾	0.74323 ⁽⁴⁾	0.76971 ⁽¹⁰⁾	0.78075 ⁽¹³⁾	0.60645 ⁽¹⁾	0.782 ⁽¹⁵⁾	0.7543 ⁽⁶⁾	0.75881 ⁽⁸⁾
	BIAS($\hat{\gamma}$)	0.97564 ⁽²⁾	1.05014 ^[3]	1.12305 ⁽⁶⁾	1.20839 ⁽¹²⁾	1.32759 ^[14]	1.15614 ⁽⁹⁾	1.15556 ⁽⁹⁾	1.20362 ⁽¹⁰⁾	1.15149 ^[7]	1.09964 ^[8]	1.20723 ^[11]	1.23524 ^[13]	0.6411 ^[1]	1.56698 ^[16]	1.06843 ⁽⁴⁾	1.55628 ^[15]
	MSE($\hat{\alpha}$)	0.72391 ⁽²⁾	0.99312 ⁽³⁾	1.12591 ⁽⁷⁾	1.37326 ⁽¹⁰⁾	1.72488 ^[14]	1.16508 ⁽⁸⁾	1.10973 ⁽⁶⁾	1.46532 ⁽¹¹⁾	1.35147 ^[9]	1.04499 ^[4]	1.52257 ^[12]	1.72426 ^[13]	0.35252 ⁽¹⁾	2.10616 ^[16]	1.1035 ^[4]	2.09172 ^[15]
	MSE($\hat{\beta}$)	0.71165 ⁽²⁾	0.71993 ⁽³⁾	0.78802 ⁽¹¹⁾	0.79843 ⁽¹⁴⁾	0.79314 ^[12]	0.7629 ⁽⁷⁾	0.74213 ⁽⁵⁾	0.75619 ⁽⁶⁾	0.8089 ^[14]	0.73539 ^[4]	0.78554 ^[10]	0.79538 ^[13]	0.55052 ⁽¹⁾	0.80115 ^[15]	0.76427 ⁽⁸⁾	0.76582 ^[9]
	MSE($\hat{\gamma}$)	1.43672 ^[2]	1.57979 ⁽³⁾	1.77147 ⁽⁶⁾	1.91165 ⁽¹⁰⁾	2.21878 ^[14]	1.79923 ⁽⁷⁾	1.82156 ⁽⁹⁾	1.93049 ^[11]	1.82424 ^[9]	1.66766 ^[8]	1.98291 ^[12]	2.00258 ^[13]	0.63104 ⁽²⁾	2.88286 ^[16]	1.59417 ^[4]	2.86007 ^[15]
	MRE($\hat{\alpha}$)	0.27435 ⁽²⁾	0.31146 ⁽³⁾	0.33368 ⁽⁷⁾	0.37358 ⁽¹⁰⁾	0.42764 ^[14]	0.33692 ⁽⁸⁾	0.33121 ⁽⁵⁾	0.38317 ^[11]	0.3686 ^[9]	0.31743 ^[4]	0.39203 ^[12]	0.42422 ^[13]	0.17554 ⁽¹⁾	0.49351 ^[15]	0.33164 ^[4]	0.49103 ^[15]
	mre($\hat{\alpha}$)	0.48417 ⁽²⁾	0.48909 ⁽³⁾	0.51951 ⁽¹²⁾	0.52067 ^[14]	0.51813 ^[11]	0.50495 ⁽⁷⁾	0.49974 ⁽⁷⁾	0.50632 ^[9]	0.52134 ^[15]	0.49549 ^[4]	0.51314 ^[10]	0.5205 ⁽¹³⁾	0.4043 ^[1]	0.52134 ^[15]	0.50287 ⁽⁶⁾	0.50587 ^[8]
	MRE($\hat{\beta}$)	0.48417 ⁽²⁾	0.48909 ⁽³⁾	0.51951 ⁽¹²⁾	0.52067 ^[14]	0.51813 ^[11]	0.50495 ⁽⁷⁾	0.49974 ⁽⁷⁾	0.50632 ^[9]	0.52134 ^[15]	0.49549 ^[4]	0.51314 ^[10]	0.5205 ⁽¹³⁾	0.4043 ^[1]	0.52134 ^[15]	0.50287 ⁽⁶⁾	0.50587 ^[8]
	D_{tot}	0.04184 ^[2]	0.05654 ⁽²⁾	0.05683 ⁽³⁾	0.0593 ⁽⁷⁾	0.0638 ^[11]	0.05773 ⁽⁴⁾	0.05832 ⁽⁶⁾	0.06022 ^[9]	0.05993 ⁽⁸⁾	0.06622 ^[13]	0.06423 ^[12]	0.06784 ^[14]	0.05785 ⁽⁵⁾	0.06993 ^[15]	0.06057 ⁽¹⁰⁾	0.07441 ^[16]
	D_{data}	0.07429 ^[2]	0.08997 ^[3]	0.09158 ⁽⁴⁾	0.09429 ^[7]	0.10148 ^[11]	0.09182 ⁽⁵⁾	0.0937 ⁽⁶⁾	0.09601 ^[9]	0.09507 ⁽⁸⁾	0.10417 ^[13]	0.10271 ^[12]	0.10558 ^[14]	0.08988 ⁽²⁾	0.11423 ^[15]	0.09649 ^[10]	0.12085 ^[16]
	ASAE	0.0322 ^[2]	0.03639 ⁽³⁾	0.03591 ⁽⁴⁾	0.03575 ^[7]	0.03551 ^[11]	0.03474 ⁽⁵⁾	0.03617 ⁽⁶⁾	0.03366 ^[9]	0.03872 ^[13]	0.04947 ^[14]	0.04617 ^[12]	0.0463 ⁽⁴⁾	0.04527 ⁽¹⁾	0.05769 ^[15]	0.0444 ⁽¹⁰⁾	0.05891 ^[16]
	Σ Ranks	29 ^[2]	39 ^[3]	92 ^[6]	126 ^[10]	140 ^[13]	79 ^[6]	70 ⁽⁴⁾	106 ^[9]	130 ^[11]	78 ^[8]	135 ^[12]	158 ^[14]	27 ⁽²⁾	186 ^[16]	81 ^[7]	156 ⁽¹⁴⁾
150	BIAS($\hat{\alpha}$)	0.59517 ^[9]	0.58022 ^[7]	0.57179 ⁽⁴⁾	0.59217 ⁽⁸⁾	0.60541 ^[11]	0.59889 ⁽¹⁰⁾	0.60975 ⁽¹³⁾	0.57826 ⁽⁶⁾	0.57589 ⁽⁵⁾	0.55807 ⁽⁹⁾	0.60674 ^[12]	0.61207 ^[14]	0.21971 ⁽²⁾	0.7217 ⁽¹⁶⁾	0.55752 ⁽²⁾	0.68456 ^[15]
	BIAS($\hat{\beta}$)	0.71736 ^[11]	0.70565 ⁽⁸⁾	0.6969 ⁽⁴⁾	0.72391 ⁽¹³⁾	0.71707 ^[10]	0.70404 ⁽⁷⁾	0.70239 ⁽⁶⁾	0.70041 ⁽⁵⁾	0.72006 ^[12]	0.67125 ^[15]	0.70896 ⁽⁹⁾	0.75906 ^[14]	0.47024 ⁽²⁾	0.74117 ⁽¹⁵⁾	0.68642 ⁽³⁾	0.73898 ^[16]
	BIAS($\hat{\gamma}$)	0.80314 ^[12]	0.75878 ⁽⁵⁾	0.76914 ⁽⁹⁾	0.74301 ⁽⁸⁾	0.83277 ^[13]	0.77886 ⁽¹⁰⁾	0.86949 ^[14]	0.76665 ⁽⁴⁾	0.76669 ^[7]	0.75251 ⁽⁵⁾	0.80172 ^[11]	0.7674 ⁽⁸⁾	0.37152 ⁽²⁾	0.94845 ^[16]	0.72365 ⁽²⁾	0.90551 ^[15]
	MSE($\hat{\alpha}$)	0.51054 ⁽⁸⁾	0.49796 ⁽⁴⁾	0.49753 ⁽⁵⁾	0.52676 ⁽⁹⁾	0.55635 ^[10]	0.5353 ^[8]	0.55481 ⁽⁹⁾	0.49566 ⁽⁴⁾	0.5061 ^[7]	0.4889 ⁽⁸⁾	0.57114 ^[11]	0.60704 ^[14]	0.185 ^[1]	0.84322 ^[16]	0.46086 ⁽²⁾	0.77997 ^[15]
	MSE($\hat{\beta}$)	0.69231 ⁽¹⁰⁾	0.67799 ⁽⁶⁾	0.66906 ⁽⁵⁾	0.71003 ⁽¹³⁾	0.69797 ^[12]	0.67555 ⁽⁸⁾	0.67602 ^[7]	0.66531 ⁽⁴⁾	0.6968 ^[12]	0.63231 ^[2]	0.68754 ^[9]	0.76549 ^[16]	0.39382 ⁽²⁾	0.73499 ^[16]	0.46568 ⁽³⁾	0.73371 ^[14]
	MSE($\hat{\gamma}$)	1.03413 ^[12]	0.91561 ⁽⁷⁾	0.92165 ⁽⁸⁾	0.82453 ^[8]	1.04817 ^[13]	0.93526 ⁽⁹⁾	1.18 ^[15]	0.90678 ⁽⁴⁾	0.93646 ^[10]	0.86968 ^[11]	0.98658 ^[14]	0.91333 ⁽⁶⁾	0.33800 ⁽¹⁾	1.2496 ^[16]	0.80664 ⁽²⁾	1.16281 ^[15]
	MRE($\hat{\alpha}$)	0.23807 ⁽⁹⁾	0.23209 ⁽⁷⁾	0.22872 ⁽⁴⁾	0.23687 ⁽⁸⁾	0.24216 ^[11]	0.23956 ⁽¹⁰⁾	0.2439 ⁽¹³⁾	0.2313 ⁽⁶⁾	0.23035 ^[2]	0.22323 ^[3]	0.2427 ^[12]	0.24483 ^[14]	0.02787 ⁽²⁾	0.28868 ^[16]	0.22301 ⁽³⁾	0.23383 ^[15]
	mre($\hat{\alpha}$)	0.47824 ^[11]	0.47044 ⁽⁸⁾	0.4646 ⁽⁴⁾	0.48261 ⁽¹³⁾	0.47804 ^[10]	0.46936 ⁽⁷⁾	0.46826 ⁽⁶⁾	0.46694 ⁽⁵⁾	0.48004 ^[12]	0.4475 ^[2]	0.47264 ^[9]	0.50604 ^[14]	0.28015 ⁽²⁾	0.49411 ^[15]	0.45761 ⁽³⁾	0.49266 ^[14]
	MRE($\hat{\beta}$)	0.47824 ^[11]	0.47044 ⁽⁸⁾	0.4646 ⁽⁴⁾	0.48261 ⁽¹³⁾	0.47804 ^[10]	0.46936 ⁽⁷⁾	0.46826 ⁽⁶⁾	0.46694 ⁽⁵⁾	0.48004 ^[12]	0.4475 ^[2]	0.47264 ^[9]	0.50604 ^[14]	0.28015 ⁽²⁾	0.49411 ^[15]	0.45761 ⁽³⁾	0.49266 ^[14]
	D_{tot}	0.0307 ^[1]	0.03041 ⁽²⁾	0.03012 ⁽²⁾	0.0316 ⁽⁷⁾	0.03183 ^[9]	0.02925 ⁽¹⁾	0.03161 ⁽⁸⁾	0.03033 ^[3]	0.03128 ^[6]	0.03618 ^[14]	0.0344 ^[13]	0.03325 ^[11]	0.03198 ^[10]	0.04118 ^[16]	0.03386 ^[12]	0.04026 ^[15]
	D_{data}	0.0502 ⁽⁵⁾	0.04968 ⁽³⁾	0.04944 ⁽²⁾	0.05141 ⁽⁷⁾	0.05254 ^[10]	0.04779 ⁽⁴⁾	0.05191 ⁽⁹⁾	0.04979 ⁽⁴⁾	0.05131 ^[2]	0.05853 ^[14]	0.0561 ^[13]	0.05411 ^[11]	0.05184 ⁽⁸⁾	0.06754 ^[16]	0.0544 ⁽¹²⁾	0.06683 ^[15]
	ASAE	0.01827 ^[5]	0.01667 ⁽³⁾	0.01702 ⁽²⁾	0.01736 ^[7]	0.01684 ^[10]	0.0168 ⁽¹⁾	0.01627 ^[9]	0.01639 ⁽⁴⁾	0.0182 ⁽⁶⁾	0.02554 ^[14]	0.02389 ^[13]	0.02145 ^[11]	0.02065 ⁽⁹⁾	0.0285 ^[16]	0.02129 ^[12]	0.02879 ^[15]
	Σ Ranks	112 ^[11]	74 ⁽⁶⁾	57 ^(3.5)	104 ⁽⁹⁾	126 ^[12]	82 ^[7]	109 ⁽¹⁰⁾	55 ⁽²⁾	101 ⁽⁸⁾	67 ^[8]	134 ^[13]	154 ⁽¹⁴⁾	37 ⁽¹⁾	187 ^[16]	57 ^(3.5)	176 ⁽¹⁵⁾
300	BIAS($\hat{\alpha}$)	0.4996 ⁽⁴⁾	0.52799 ^[11]	0.54579 ⁽¹²⁾	0.49546 ⁽⁸⁾	0.55293 ^[13]	0.52647 ⁽¹⁰⁾	0.57584 ⁽¹³⁾	0.5256 ⁽⁹⁾	0.50252 ^[5]	0.52038 ^[7]	0.51359 ^[6]	0.49275 ^[2]	0.27178 ⁽²⁾	0.57356 ^[14]	0.52099 ⁽⁸⁾	0.59166 ^[16]
	BIAS($\hat{\beta}$)	0.64144 ^[2]	0.67997 ^[10]	0.68482 ⁽¹²⁾	0.66254 ⁽⁷⁾	0.70207 ^[13]	0.64597 ⁽⁹⁾	0.67317 ⁽⁹⁾	0.66726 ^[8]	0.68272 ^[11]	0.66005 ^[6]	0.65509 ^[5]	0.73866 ^[14]	0.48202 ⁽²⁾	0.71257 ^[15]	0.653 ⁽⁴⁾	0.70854 ^[16]
	BIAS($\hat{\gamma}$)	0.62976 ^[2]	0.63547 ⁽⁶⁾	0.72066 ^[14]	0.57531 ⁽²⁾	0.70853 ^[12]	0.67651 ⁽¹¹⁾	0.78013 ⁽¹⁶⁾	0.65124 ^[10]	0.60873 ⁽⁵⁾	0.64358 ^[9]	0.60841 ⁽⁴⁾	0.58057 ^[5]	0.73874 ⁽¹⁾	0.71933 ⁽³⁾	0.62175 ⁽⁶⁾	0.72359 ^[15]
	MSE($\hat{\alpha}$)	0.36273 ⁽⁴⁾	0.39333 ⁽⁶⁾	0.43439 ⁽¹²⁾	0.37147 ⁽⁴⁾	0.45267 ^[13]	0.40733 ⁽¹⁰⁾	0.47146 ^[14]	0.40134 ⁽⁹⁾	0.37682 ^[7]	0.40865 ⁽¹¹⁾	0.39015 ⁽⁵⁾	0.45674 ^[8]	0.14194 ⁽¹⁾	0.50083 ^[15]	0.39255 ⁽⁷⁾	0.51854 ^[16]
	MSE($\hat{\beta}$)	0.57737 ^[2]	0.62653 ⁽⁹⁾	0.64902 ⁽¹¹⁾	0.61089 ⁽⁶⁾	0.67225 ^[13]	0.59784 ⁽⁵⁾	0.6151 ⁽¹⁰⁾	0.63163 ^[10]	0.6059 ^[12]	0.61413 ^[7]	0.5966 ⁽⁴⁾	0.73448 ^[14]	0.37245 ⁽²⁾	0.69531 ^[15]	0.5905 ⁽⁸⁾	0.68867 ^[16]
	MSE($\hat{\gamma}$)	0.64043 ⁽⁷⁾	0.64382 ⁽¹²⁾	0.81549 ⁽¹⁸⁾	0.5039 ⁽²⁾	0.80958 ^[14]	0.73177 ^[10]	0.95812 ⁽¹⁶⁾	0.68026 ^[10]	0.60706 ^[6]	0.67116 ^[9]	0.57394 ^[4]	0.5297 ⁽³⁾	0.29084 ⁽²⁾	0.75182 ^[12]	0.59137 ⁽⁵⁾	0.78974 ^[16]
	MRE($\hat{\alpha}$)	0.19984 ⁽⁶⁾	0.2113 ⁽¹²⁾	0.21832 ⁽¹²⁾	0.19818 ⁽³⁾	0.22117 ^[13]	0.21059 ^[10]	0.23034 ⁽¹⁸⁾	0.21024 ^[9]	0.20101 ⁽⁵⁾	0.20814 ^[7]	0.20544 ^[6]	0.1982 ⁽²⁾	0.10871 ⁽¹⁾	0.22942 ^[12]	0.2084 ⁽³⁾	0.23666 ^[16]
	MRE($\hat{\beta}$)	0.42762 ^[2]	0.45331 ⁽¹⁰⁾	0.45655 ⁽¹²⁾	0.44169 ⁽⁷⁾	0.46805 ^[13]	0.43065 ⁽¹⁰⁾	0.44878 ^[9]	0.44484 ⁽⁸⁾	0.45511 ^[4]	0.44004 ^[6]	0.43673 ⁽⁵⁾	0.49244 ^[8]	0.32134 ⁽²⁾	0.47505 ^[16]	0.43533 ⁽⁴⁾	0.47236 ^[16]
	MRE($\hat{\gamma}$)	0.42762 ^[2]	0.45331 ⁽¹⁰⁾	0.45655 ⁽¹²⁾	0.44169 ⁽⁷⁾	0.46805 ^[13]	0.43065 ⁽¹⁰⁾	0.44878 ^[9]	0.44484 ⁽⁸⁾	0.45511 ^[4]	0.44004 ^[6]	0.43673 ⁽⁵⁾	0.49244 ^[8]	0.32134 ⁽²⁾	0.47505 ^[16]	0.43533 ⁽⁴⁾	0.47236 ^[16]
	D_{tot}	0.02073 ⁽³⁾	0.02117 ⁽⁴⁾	0.02214 ⁽¹⁰⁾	0.02169 ⁽⁸⁾	0.02191 ⁽⁹⁾	0.02122 ⁽⁵⁾	0.02168 ^[7]	0.02071 ^[10]	0.02091 ^[5]	0.02057 ^[4]	0.02472 ^[13]	0.02145 ^[6]	0.02228 ⁽¹¹⁾	0.02688 ^[12]	0.02296 ⁽¹²⁾	0.02772 ^[15]
	D_{data}	0.03428 ^[11]	0.03505 ⁽³⁾	0.03688 ⁽¹¹⁾	0.03584 ⁽⁶⁾	0.03666 ^[13]	0.03356 ⁽⁴⁾	0.03652 ^(8.5)	0.03463 ⁽⁵⁾	0.0354 ⁽⁵⁾	0.04195 ^[14]	0.04062 ^[13]	0.03613 ^[7]	0.03652 ^(8.5)	0.04509 ^[15]	0.03784 ^[12]	0.04613 ^[16]
	ASAE	0.01121 ^[1]	0.0105 ⁽³⁾	0.01067 ⁽¹¹⁾	0.01117 ⁽⁶⁾	0.01086 ^[10]	0.01083 ⁽⁴⁾	0.01056 ^(8.5)	0.01046 ⁽⁶⁾	0.0119 ⁽⁵⁾	0.01622 ^[14]	0.015 ^[13]	0.01442 ^[7]	0.01305 ⁽⁸⁾	0.01803 ^[15]	0.0135 ⁽¹²⁾	0.01818 ^[16]
	Σ Ranks	45 ^[2]	94 ⁽⁹⁾	137 ^[13]	61 ^[3]	142 ^[14]	80 ^[4]										

Table 4: Summary of simulation performance metrics $\alpha = 0.5; \beta = 0.8; \gamma = 2.0$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
70	BIAS($\hat{\alpha}$)	0.62565	0.021612	0.32834	0.57739	0.30406	0.5432	0.26306	0.29736	6.8432	0.34613	3.6987	0.46733	0.62565	0.57739	2.5977	0.34044
	BIAS($\hat{\beta}$)	0.17181	0.012918	0.019647	0.37853	0.030233	0.2100	0.48982	0.053644	7.9865	0.01137	7.8681	0.098486	0.17181	0.37853	0.47639	0.00461
	BIAS($\hat{\gamma}$)	0.021612	0.1011	0.02821	0.11708	0.14853	0.2101	0.67889	0.12555	8.9412	0.10245	8.6551	0.12128	0.021612	0.11708	0.57427	0.09672
	MSE($\hat{\alpha}$)	1.4955	0.84421	0.80591	1.2294	0.68142	0.2543	0.07763	0.77476	4.6830	0.84361	4.6262	1.0986	1.4955	1.2294	8.1916	0.84057
	MSE($\hat{\beta}$)	1.0067	1.0767	0.99799	0.52388	0.89456	0.0432	0.34039	0.8692	5.2754	1.0776	4.00332	0.70993	1.0067	0.52388	0.60705	1.0785
	MSE($\hat{\gamma}$)	0.49035	0.37827	0.4628	0.3891	0.29397	0.0431	0.48546	0.38994	1.1544	0.37888	1.1405	0.36955	0.49035	0.3891	0.45119	0.37769
	MRE($\hat{\alpha}$)	1.6378	1.1787	1.1376	1.5829	1.1068	1	0.52612	1.1543	3.6765	1.1762	7.3876	1.3659	1.6378	1.5829	5.2155	1.1664
	MRE($\hat{\beta}$)	1.0665	1.0211	0.96249	0.81084	0.92593	0.2543	0.69915	0.91082	9.9825	1.0228	9.8351	0.87938	1.0665	0.81084	0.88834	1.0265
	MRE($\hat{\gamma}$)	0.72286	0.65902	0.7059	0.66512	0.58625	0.2543	0.84861	0.67671	11.1765	0.6607	10.8198	0.67064	0.72286	0.66512	0.7772	0.6566
	D_{abs}	0.062308	0.061264	0.0649	0.066917	0.064514	0.2847	0.50249	0.062214	0.49753	0.06131	0.49754	0.068351	0.062308	0.066917	0.4132	0.063576
	D_{max}	0.032268	0.056328	0.061872	0.14212	0.060484	0.42371	0.9855	0.057543	0.98834	0.056374	0.98879	0.14406	0.062268	0.14212	0.71672	0.056735
	ASAE	0.02948	0.027305	0.029688	0.077215	0.030059	0.24287	1.9557	0.028068	0.59399	0.027253	0.59391	0.07734	0.02948	0.077215	0.4195	0.027119
	Σ Ranks	53 ^[7]	21 ^[8]	43 ^[6]	54 ^[6]	27 ^[2]	41 ^[9]	90 ^[10]	51 ^[4]	92 ^[11]	23 ^[4]	88 ^[9]	62 ^[7]	54 ^[6]	73 ^[8]	21 ^[8]	
150	BIAS($\hat{\alpha}$)	0.3047	0.091706	0.18678	0.27791	0.24172	0.5432	0.23428	0.11028	6.6210	0.10434	6.6101	0.2008	0.3047	0.27791	2.4267	0.095581
	BIAS($\hat{\beta}$)	0.21886	0.060586	0.045975	0.10884	0.14219	0.2110	0.49236	0.080925	10.1654	0.054875	9.7663	0.058179	0.21886	0.10884	0.42331	0.027927
	BIAS($\hat{\gamma}$)	0.072972	0.024633	0.022941	0.081429	0.02566	0.2101	0.77361	0.00009	1.0144	0.016517	0.9884	0.048723	0.072972	0.081429	0.43578	0.017334
	MSE($\hat{\alpha}$)	0.31857	0.17019	0.3325	0.28456	0.33575	0.2543	0.0551	0.24939	7.3562	0.16728	7.1604	0.19337	0.31857	0.28456	7.6867	0.16383
	MSE($\hat{\beta}$)	0.40256	0.34143	0.36285	0.29302	0.26389	0.0432	0.25083	0.42011	8.0321	0.33629	3.6432	0.29262	0.40256	0.29302	0.45128	0.32403
	MSE($\hat{\gamma}$)	0.19336	2.0993	0.19516	0.16356	0.16733	0.0431	0.60629	0.25391	1.0409	0.20635	1.0022	0.17105	0.19336	0.16356	0.37713	0.19437
	MRE($\hat{\alpha}$)	0.83634	0.52014	0.70495	0.85336	0.75155	1.101	0.46856	0.67324	3.2321	0.50652	3.2211	0.68993	0.83634	0.85336	4.8534	0.49117
	MRE($\hat{\beta}$)	0.69959	0.59726	0.6556	0.5556	0.55999	0.2543	0.61545	0.68017	12.7076	0.58982	2.2087	0.54031	0.69959	0.5556	0.75743	0.58722
	MRE($\hat{\gamma}$)	0.45579	0.48014	0.45507	0.42089	0.41429	0.25432	0.96701	0.53846	1.2679	0.47427	1.2355	0.43139	0.45579	0.42089	0.68972	0.45109
	D_{abs}	0.039266	0.02859	0.029889	0.10166	0.029387	0.28582	0.50185	0.028668	0.4982	0.028628	0.49822	0.10194	0.039266	0.10166	0.41345	0.028845
	D_{max}	0.062363	0.046406	0.049732	0.15439	0.048262	0.42372	0.98371	0.047082	0.98844	0.064612	0.98941	0.15493	0.062363	0.15439	0.69753	0.047001
	ASAE	0.031139	0.02386	0.025319	0.084838	0.02522	0.24731	2.0667	0.024255	0.60682	0.023826	0.60673	0.084639	0.031139	0.084838	0.41785	0.023859
	Σ Ranks	53 ^(8.5)	20 ⁽²⁾	48 ⁽⁷⁾	53 ^(8.5)	126 ⁽¹³⁾	53 ^(9.5)	88 ⁽¹⁰⁾	37 ⁽⁴⁾	92 ⁽¹²⁾	29 ⁽³⁾	90 ⁽¹¹⁾	39 ⁽⁵⁾	53 ^(8.5)	43 ⁽⁶⁾	49 ⁽⁸⁾	19 ⁽¹⁾
300	BIAS($\hat{\alpha}$)	0.30616	0.15729	0.21791	0.37506	0.16564	0.50000	0.23854	0.12702	260.89	0.16026	3.7654	0.38063	0.30616	0.37506	1.4482	0.15329
	BIAS($\hat{\beta}$)	0.33007	0.15013	0.18231	0.08812	0.13156	0.20000	0.48345	0.16523	10.359	0.15174	9.5129	0.11513	0.33007	0.08812	0.10235	0.11756
	BIAS($\hat{\gamma}$)	0.11360	0.01578	0.01198	0.13182	0.02160	0.20000	0.78700	0.03602	1.0251	0.015822	0.97969	0.12448	0.1136	0.13182	0.11061	0.00038
	MSE($\hat{\alpha}$)	0.23741	0.16652	0.19919	0.23352	0.18791	0.25000	0.05716	0.16513	728.50	0.16875	6.5931	0.21813	0.23741	0.23352	3.9724	0.15949
	MSE($\hat{\beta}$)	0.34701	0.28200	0.29518	0.14094	0.28262	0.04000	0.23816	0.32580	110.06	0.28397	8.4298	0.12464	0.34701	0.14094	0.24877	0.2714
	MSE($\hat{\gamma}$)	0.13035	0.15833	0.18696	0.10424	0.17542	0.04000	0.62323	0.18892	1.05770	0.15672	0.98126	0.092466	0.13035	0.10424	0.20773	0.14634
	MRE($\hat{\alpha}$)	0.75793	0.59655	0.65445	0.90664	0.59912	1.00000	0.47709	0.54791	521.78	0.60246	7.5210	0.86015	0.75793	0.90664	2.8965	0.56971
	MRE($\hat{\beta}$)	0.64850	0.54483	0.58878	0.37715	0.57118	0.25000	0.60431	0.61683	12.948	0.54819	11.8910	0.34491	0.6485	0.37715	0.49997	0.53114
	MRE($\hat{\gamma}$)	0.34650	0.41086	0.43573	0.33208	0.43155	0.25000	0.98375	0.44471	1.2814	0.40799	1.2246	0.31637	0.3465	0.33208	0.46827	0.38277
	D_{abs}	0.036479	0.019938	0.020849	0.19275	0.020993	0.28445	0.49920	0.020157	0.5008	0.019895	0.50084	0.19289	0.036479	0.19275	0.3525	0.019763
	D_{max}	0.05725	0.034277	0.036534	0.29039	0.036593	0.42383	0.99225	0.034782	0.99766	0.034203	0.99897	0.29072	0.05725	0.29039	0.5764	0.033859
	ASAE	0.026789	0.015949	0.017010	0.14817	0.017369	0.22092	1.92170	0.016269	0.54218	0.015901	0.54216	0.14825	0.026789	0.14817	0.31062	0.015647
	Σ Ranks	48 ⁽²⁾	22 ⁽⁹⁾	41 ⁽¹³⁾	51 ⁽⁸⁾	39 ⁽¹⁴⁾	56 ⁽⁴⁾	86 ⁽¹²⁾	27 ⁽⁷⁾	93 ⁽⁸⁾	22 ⁽¹¹⁾	91 ^(5.5)	52 ⁽¹⁰⁾	48 ⁽²⁾	51 ⁽¹⁵⁾	88 ^(5.5)	11 ⁽¹⁾

Table 5: Summary of simulation performance metrics $\alpha = 1.5; \beta = 0.7; \gamma = 1.7$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
70	BIAS($\hat{\alpha}$)	0.011596	0.21190	0.088446	0.33462	0.26154	0.50000	1.49750	0.19430	1.5665	0.21190	1.56650	0.16176	0.011596	0.33462	1.75400	0.21190
	BIAS($\hat{\beta}$)	0.039476	0.25606	0.089209	0.11768	0.009485	0.30000	3.81680	0.064413	1.0750	0.25606	1.07500	0.18498	0.039476	0.11768	0.56238	0.25606
	BIAS($\hat{\gamma}$)	1.20750	0.45301	0.72549	0.16343	0.36000	0.50000	1.45370	0.12292	1.0083	0.45301	1.00830	0.17276	1.20750	0.16343	0.69644	0.45301
	MSE($\hat{\alpha}$)	0.96349	1.05640	1.09070	1.00100	1.04500	0.25000	2.24260	1.05350	5.09440	1.05640	5.09440	1.06430	0.96349	1.00100	3.93430	1.05640
	MSE($\hat{\beta}$)	1.21690	1.68700	1.39770	1.54420	0.75268	0.09000	19.8130	0.97728	1.20600	1.68700	1.20600	1.23210	1.21690	1.54420	0.50455	1.68700
	MSE($\hat{\gamma}$)	12.3480	6.79130	8.39630	7.48030	4.78930	0.25000	2.11600	2.92020	2.05550	6.79130	2.05550	3.79390	12.3480	7.48030	6.21520	6.79130
	MRE($\hat{\alpha}$)	0.53515	0.59833	0.59860	0.57469	0.60844	0.33333	0.99833	0.59635	1.04440	0.59833	1.04440	0.59877	0.53515	0.57469	1.17750	0.59833
	MRE($\hat{\beta}$)	1.23670	1.32590	1.20450	1.36590	0.99347	0.42857	5.46450	1.07390	1.53560	1.32590	1.53560	1.22520	1.23670	1.36590	0.96637	1.32590
	MRE($\hat{\gamma}$)	1.33610	1.10420	1.20260	1.05520	1.04760	0.33333	0.96916	0.89575	6.7220	1.10420	6.7220	0.98916	1.33610	1.05520	0.98749	1.10420
	D_{abs}	0.057935	0.059678	0.060743	0.059978	0.062126	0.15096	0.48989	0.060672	0.5101	0.059678	0.51011	0.059685	0.057935	0.059978	0.43570	0.059678
	D_{max}	0.097862	0.097936	0.10363	0.095481	0.10119	0.21752	0.94779	0.098657	0.95794	0.097936	0.95794	0.096829	0.097862	0.095481	0.69522	0.097936
	ASAE	0.058289	0.061973	0.064479	0.065548	0.069371	0.17645	1.26910	0.065480	0.38268	0.061973	0.38268	0.062844	0.058289	0.065548	0.33759	0.061973
	Σ Ranks	42 ⁽⁷⁾	34 ⁽⁸⁾	50 ⁽⁶⁾	41 ⁽⁶⁾	42 ⁽²⁾	40 ⁽⁸⁾	86 ⁽¹⁰⁾	31 ⁽¹⁾	92 ⁽¹¹⁾	34 ⁽⁶⁾	92 ⁽⁹⁾	24 ⁽⁷⁾	42 ⁽⁷⁾	41 ⁽⁶⁾	75 ⁽⁸⁾	42 ⁽⁸⁾
150	BIAS($\hat{\alpha}$)	0.04204	0.12249	0.046582	0.41276	0.12882	0.5	1.4954	0.18405	2.7902	0.12249	2.7902	0.22178	0.04204	0.41276	1.4608	0.12248
	BIAS($\hat{\beta}$)	0.068008	0.39826	0.24832	0.48394	0.2651	0.3	2.7365	0.52013	1.3589	0.39794	1.3589	0.33089	0.068008	0.48394	0.38594	0.39753
	BIAS($\hat{\gamma}$)	0.3134	0.16097	0.075315	0.37065	0.12989	0.5	1.4723	0.33174	1.1156	0.16087	1.1156	0.28614	0.3134	0.37065	1.0455	0.17215
	MSE($\hat{\alpha}$)	0.3806	0.33524	0.39637	0.5344	0.40991	0.25	2.2363	0.3781	3.8925	0.33524	3.8925	0.29792	0.3806	0.5344	3.9337	0.33493
	MSE($\hat{\beta}$)	0.92356	0.95227	0.76704	1.1325	0.85101	0.09	11.631	1.2143	2.3497	0.95199	2.3497	0.61966	0.92356	1.1325	0.35937	0.95154
	MSE($\hat{\gamma}$)	1.6292	1.3444	1.2753	1.2756	1.4008	0.25	2.1679	1.1512	6.2229	1.3445	6.2229	0.88856	1.6292	1.2756	1.382	1.3489
	MRE($\hat{\alpha}$)	0.33437	0.3236	0.34939	0.39194	0.35589	0.33333	0.99693	0.33415	1.8601	0.3236	1.8601	0.32467	0.33437	0.39194	1.1738	0.3236
	MRE($\hat{\beta}$)	1.1325	1.0836	0.94324	1.1557	0.95987	0.42857	3.927	1.1915	1.9413	1.0831	1.9413	0.84416	1.1325	1.1557	0.81342	1.0826
	MRE($\hat{\gamma}$)	0.68514	0.64176	0.58156	0.62399	0.60015	0.33333	0.98151	0.5777	7.4374	0.64182	7.4374	0.51487	0.68514	0.62399	0.71504	0.6464
	D_{abs}	0.029737	0.029615	0.030389	0.067176	0.030286	0.15398	0.50185	0.029318	0.49815	0.029615	0.49815	0.06732	0.029737	0.067176	0.31731	0.031716
	D_{max}	0.050292	0.050111	0.052249	0.10006	0.051182	0.21823	0.98371	0.049395	0.98512	0.050118	0.98512	0.10114	0.050292	0.10006	0.64606	0.053218
	ASAE	0.026779	0.026878	0.0271	0.064001	0.027694	0.14746	1.3096	0.026855	0.32706	0.026875	0.32706	0.063253	0.026779	0.064001	0.26583	0.02678
	Σ Ranks	32 ^(5,6)	26 ⁽²⁾	27 ⁽³⁾	53 ⁽⁹⁾	34 ⁽⁶⁾	30 ^(4,8)	88 ⁽¹²⁾	30 ^(4,5)	91 ^(13,5)	24 ⁽¹⁾	91 ^(13,5)	43 ⁽⁷⁾	32 ^(5,6)	63 ⁽¹⁰⁾	78 ⁽¹¹⁾	46 ⁽⁸⁾
300	BIAS($\hat{\alpha}$)	0.097063	0.14809	0.10874	0.41472	0.17772	0.5	1.4822	0.16785	6.9754	0.1481	6.9754	0.35214	0.097063	0.41472	0.41299	0.14233
	BIAS($\hat{\beta}$)	0.11437	0.30998	0.2245	0.34767	0.30772	0.3	2.0033	0.33887	1.6164	0.30991	1.6164	0.28357	0.11437	0.34767	0.024451	0.31467
	BIAS($\hat{\gamma}$)	0.11399	0.15075	0.13385	0.42643	0.17701	0.5	1.4712	0.19294	2.789	0.15056	2.789	0.42107	0.11399	0.42643	0.80425	0.17975
	MSE($\hat{\alpha}$)	0.17417	0.19613	0.22343	0.26204	0.2121	0.25	2.2002	0.23659	9.7313	0.19612	9.7313	0.20191	0.17417	0.26204	2.0327	0.19388
	MSE($\hat{\beta}$)	0.92055	0.83579	0.58346	0.51359	0.71718	0.09	7.3704	0.95093	3.6575	0.83534	3.6575	0.25666	0.92055	0.51359	0.42891	0.90142
	MSE($\hat{\gamma}$)	0.76068	0.74018	0.72527	0.49011	0.74953	0.25	2.1651	0.72513	1.5557	0.74095	1.5557	0.4945	0.76068	0.49011	0.851	0.73414
	MRE($\hat{\alpha}$)	0.21744	0.22259	0.24204	0.30927	0.23478	0.33333	0.98813	0.25353	4.6503	0.2226	4.6503	0.27928	0.21744	0.30927	0.742	0.21875
	MRE($\hat{\beta}$)	1.1294	1.0416	0.86392	0.74387	0.89548	0.42857	2.9719	1.1274	2.3092	1.0415	2.3092	0.57123	1.1294	0.74387	0.67135	1.0483
	MRE($\hat{\gamma}$)	0.45122	0.45483	0.44426	0.39729	0.47184	0.33333	0.98078	0.45359	1.8593	0.45496	1.8593	0.40449	0.45122	0.39729	0.53617	0.45488
	D_{abs}	0.020106	0.020341	0.021206	0.10741	0.021234	0.15219	0.49915	0.020434	0.500	0.020341	0.5008	0.10751	0.020106	0.10741	0.26385	0.020369
	D_{max}	0.035145	0.035872	0.037523	0.15547	0.037624	0.21828	0.99189	0.036069	0.99431	0.035874	0.99431	0.15594	0.035145	0.15547	0.44343	0.03576
	ASAE	0.017293	0.017744	0.018663	0.093884	0.018995	0.13408	1.4325	0.017907	0.29908	0.017744	0.29908	0.093987	0.017293	0.093884	0.18719	0.017812
	Σ Ranks	27 ^(1,2)	31 ⁽²⁾	32 ⁽³⁾	49 ^(8,3)	41 ⁽⁵⁾	56 ⁽¹⁰⁾	86 ^(12,5)	50 ⁽⁸⁾	86 ^(12,5)	53 ⁽⁹⁾	92 ^(11,3)	46 ⁽⁶⁾	27 ^(1,2)	31 ⁽²⁾	75 ⁽¹¹⁾	37 ⁽⁴⁾

Table 6: Simulation-Based Partial and Overall Rankings of Estimation Methods Across Diverse Parameter Configurations of θ^T

θ^T	n	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$(\alpha = 2.5; \beta = 1.5; \gamma = 3.0)$	70	2	3	8	10	13	6	4	9	11	5	12	15	1	16	7	14
	150	10	5	3.5	8	11	6	9	2	7	4	12	13	1	15	3.5	14
	300	2	8	12	3	13	4	11	6	7	10	5.5	9	1	14	5.5	15
$(\alpha = 0.8; \beta = 2.0; \gamma = 1.3)$	70	2	3	8	10	13	6	4	9	11	5	12	15	1	16	7	14
	150	10	5	3.5	8	11	6	9	2	7	4	12	13	1	15	3.5	14
	300	2	8	12	3	13	4	11	6	7	10	5.5	9	1	14	5.5	15
$(\alpha = 0.5; \beta = 0.8; \gamma = 2.0)$	70	7.5	1.5	5	8.5	3	4	12	6	13	2	11	9	7.5	8.5	10	1.5
	150	9.5	2	7	9.5	13	9.5	10	4	12	3	11	5	9.5	6	8	1
	300	6.5	2.5	5	7.5	4	9	10	3	13	2.5	12	8	6.5	7.5	11	1
$(\alpha = 1.5; \beta = 0.7; \gamma = 1.7)$	70	7	3	5	6	2	3	10	1	11	4	9	7	7	6	8	3
	150	5.5	2	3	9	6	4.5	12	4.5	13.5	1	13.5	7	5.5	10	11	8
	300	1.5	2	3	3	5	10	12.5	8	12.5	9	13	6	1.5	7	11	4
Σ Ranks		65.5	45	75	77	107	72	114.5	60.5	125	59.5	128.5	116	43.5	135	91	118
Overall Rank		5	2	7	8	10	6	11	4	14	3	15	13	1	16	9	12

6. Real Data Analysis

It will show how the DUS-GPQF model is flexible and practical as it analyses two real life data sets, and compares its performance with some competing distributions. Model assessment is done based on conventional information criteria AIC, CAIC, BIC and HQIC and goodness-of-fit tests like the Anderson-Darling and Cramer-von Mises statistics. Besides, the Kolmogorov-Smirnov (KS) test and the corresponding p-values are used to determine the correspondence between the empirical and theoretical cumulative distribution functions. Overall, the model with the smallest information and goodness-of-fit measures and the highest KS p-value are considered the most sufficient. The proposed model is compared to 9 competing distributions, GFR [32], GEGO [33], TGPW [34], TEE [35], generalized Lindley [36], exponentiated gamma [37], exponentiated exponential [38], GDUS-E [39] and DUS-GLF [40].

6.1 Results Based on MLE

6.1.1 Data set 1

The first data set comprises the service times of 63 aircraft windshields, originally analyzed in [40].

Table 7: MLEs with corresponding standard errors (in parentheses) and additional goodness-of-fit statistics for the dataset.

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\hat{\alpha} = 3.02528$	$\hat{\beta} = 0.00454$	$\hat{\gamma} = 0.355174$ $\hat{\delta} = 0.105285$	-84.3838	176.767	189.340	185.340	180.139
DUS-GLF (α, β, γ)	$\hat{\alpha} = 0.181127$	$\hat{\beta} = 0.269600$	$\hat{\gamma} = 0.877687$	-98.274	202.954	202.548	208.977	205.076
GLF (α, β, γ)	$\hat{\alpha} = 0.118725$	$\hat{\beta} = 0.249639$	$\hat{\gamma} = 0.970170$	-98.753	203.913	203.506	209.936	206.035
GEGO (δ, η, κ)	$\hat{\delta} = 0.692049$	$\hat{\eta} = 3.62514 \times 10^{-7}$	$\hat{\kappa} = 1.897760$	-103.547	213.500	213.093	219.523	215.622
TGPW (λ, μ, ν)	$\hat{\lambda} = 1.525370$	$\hat{\mu} = 0.601669$	$\hat{\nu} = -0.643853$	-102.539	211.484	211.077	217.507	213.606
TEE (π, θ, τ)	$\hat{\pi} = 1.539130$	$\hat{\theta} = 0.772211$	$\hat{\tau} = -0.602123$	-101.900	209.800	210.207	216.230	212.329
GL (σ, ρ)	$\hat{\sigma} = 1.558690$	$\hat{\rho} = 0.929276$	-	-101.888	207.776	207.976	212.063	209.462
EG (ς, ω)	$\hat{\varsigma} = 0.919072$	$\hat{\omega} = 0.919244$	-	-102.755	209.509	209.709	213.796	211.195
EE (θ, φ)	$\hat{\theta} = 0.692049$	$\hat{\varphi} = 1.897760$	-	-103.547	211.293	211.093	215.380	212.779
GDUS-E (ε, l)	$\hat{\varepsilon} = 1.694170$	$\hat{l} = 0.794871$	-	-101.777	207.753	207.553	211.840	209.239

The statistical superiority of the proposed DUS-GPQF model is clearly shown in Tables 7 estimated by using the standard maximum likelihood estimation (MLE) method has flaws because it has inflated information criteria and local optima as the likelihood surface is complex. The MLE based estimate is therefore better in terms of numerical stability and global optimization and thus makes it a robust and effective estimate compared to classical MLE to provide model of truncated lifetime processes.

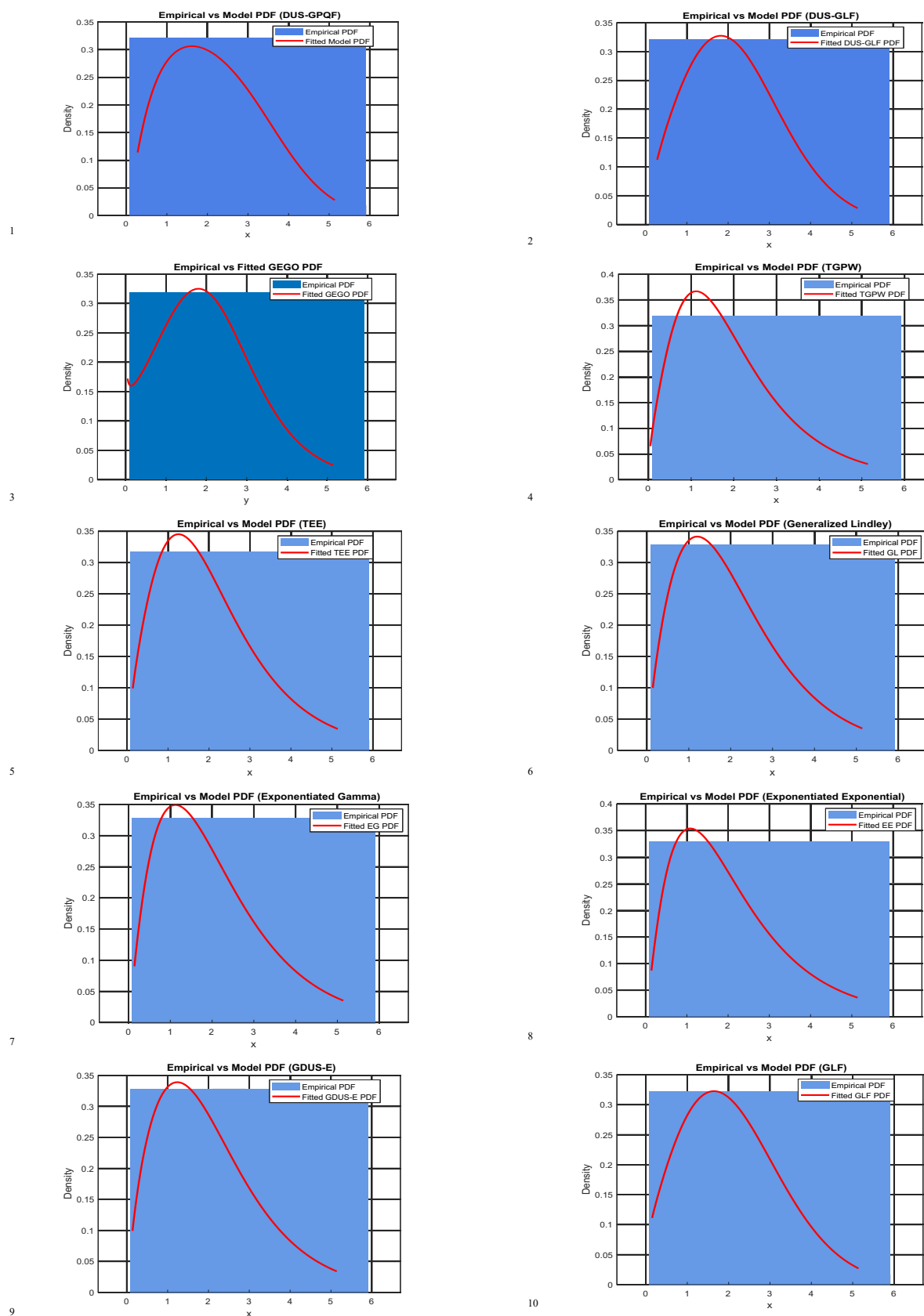


Fig. 3: Plots of the estimated probability density functions for all competing models' dataset D1.

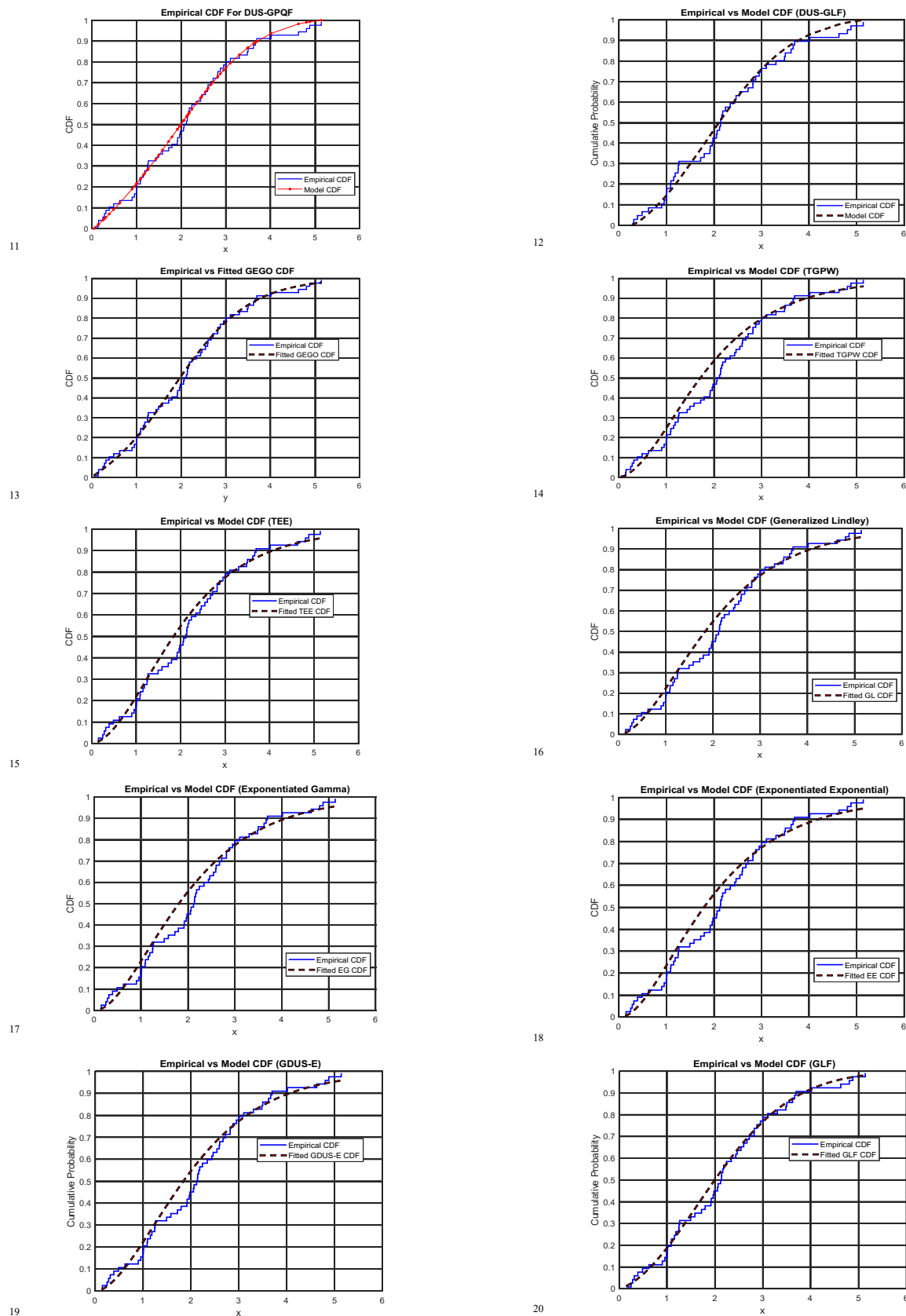


Fig. 4: Plots of the estimated cumulative distribution functions for all competing models based on Dataset D1.

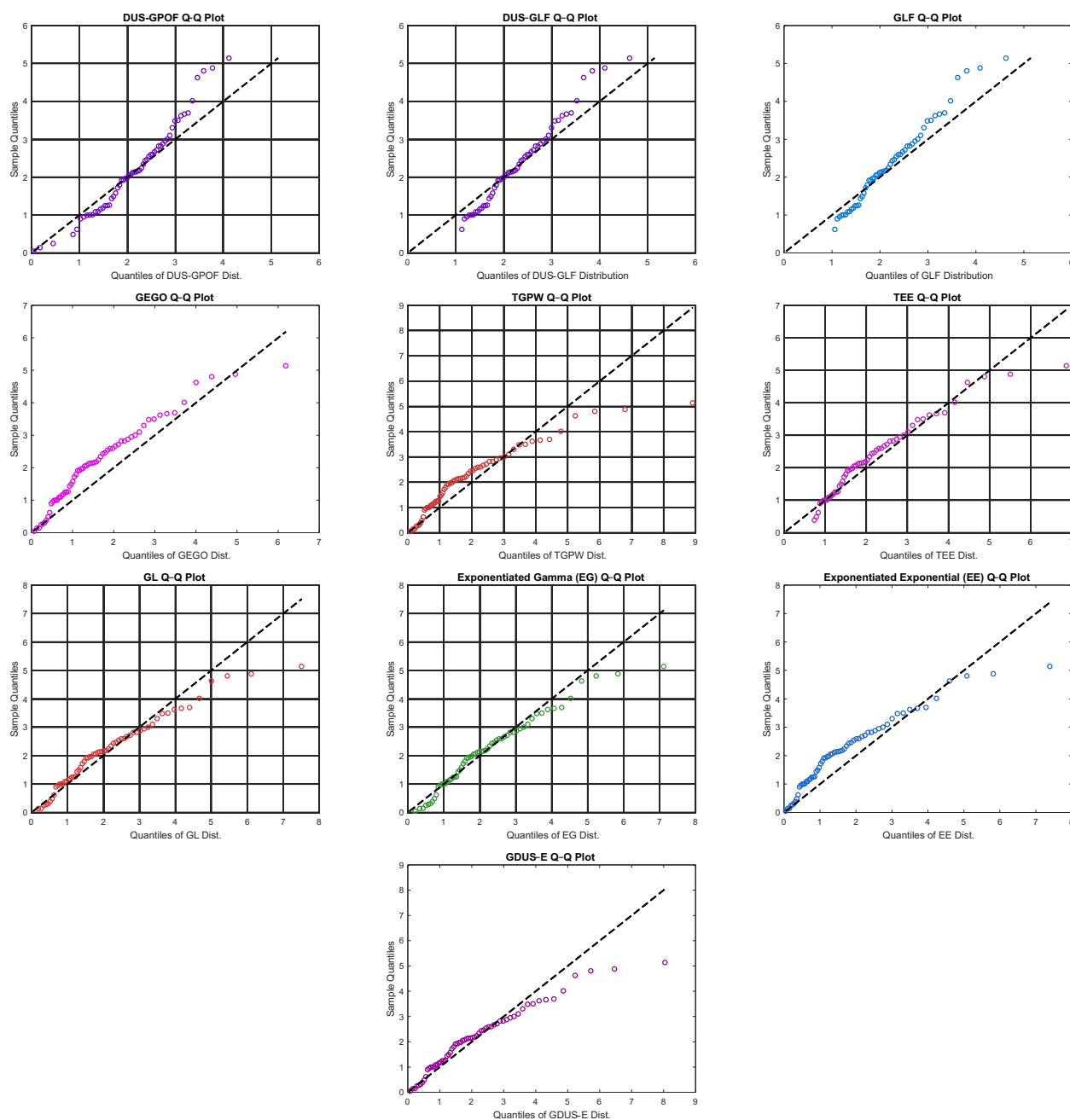


Fig. 5: The Q-Q plot of the fitted distributions for the dataset D1.

6.1.2 Data set 2

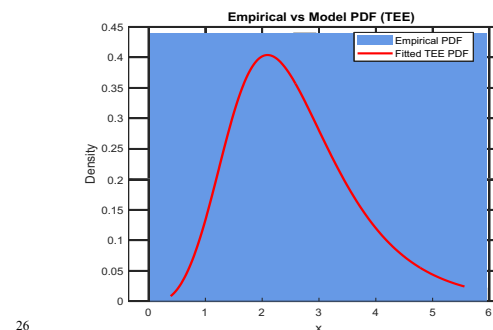
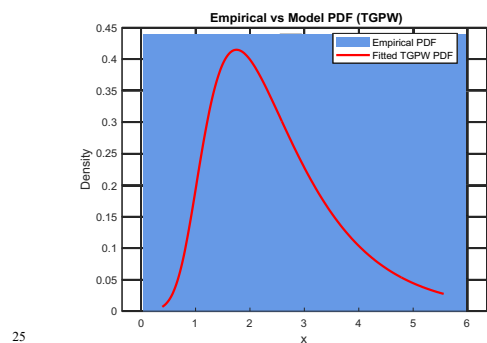
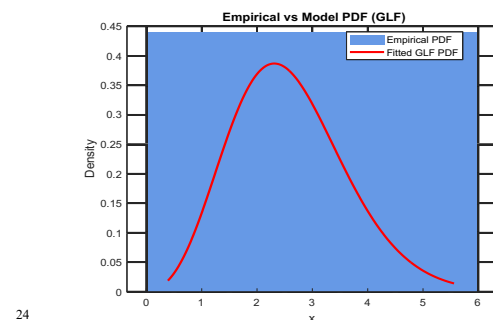
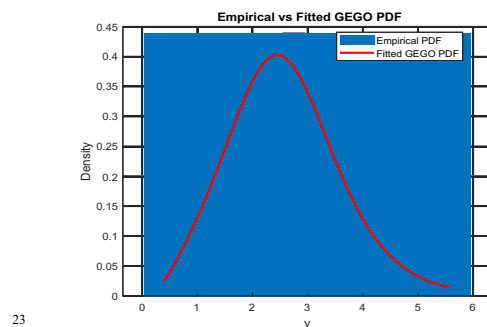
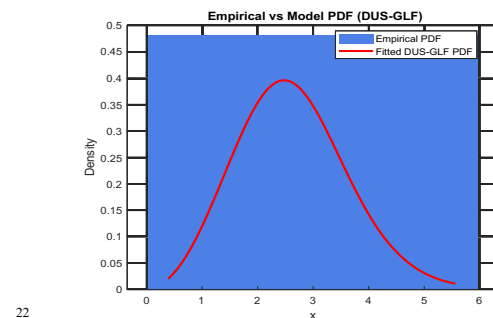
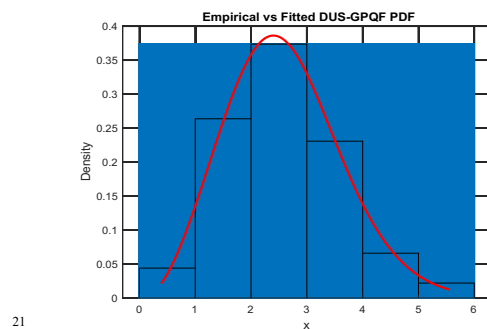
The second data set, obtained from [41], comprises 100 observations of the breaking stress (in GPa) of carbon fibers and has been previously analyzed in [42,43].

Table 8: MLEs with corresponding standard errors (in parentheses) and additional goodness-of-fit statistics for the dataset.

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\hat{\alpha} = 5.48208$	$\hat{\beta} = 0.03749$	$\hat{\gamma} = 0.74297$ $\hat{\delta} = 0.14410$	- 135.403	278.807	283.228	289.228	283.025
DUS-GLF (α, β, γ)	$\hat{\alpha} = 0.181775$	$\hat{\beta} = 0.359494$	$\hat{\gamma} = 2.239720$	- 141.207	288.664	288.414	296.229	291.577
GLF (α, β, γ)	$\hat{\alpha} = 0.082147$	$\hat{\beta} = 0.343821$	$\hat{\gamma} = 2.197800$	- 141.538	289.326	289.076	296.891	292.239

GEGO (δ, η, κ)	$\hat{\delta} = 1.013130$	$\hat{\eta} = 3.7334 \times 10^{-7}$	$\hat{\kappa} = 7.787700$	- 146.186	298.623	298.373	306.189	301.536
TGPW (λ, μ, ν)	$\hat{\lambda} = 3.555750$	$\hat{\mu} = 0.266011$	$\hat{\nu} = -0.950447$	- 149.462	305.174	304.924	312.739	308.087
TEE (π, θ, τ)	$\hat{\pi} = 6.186930$	$\hat{\theta} = 1.101940$	$\hat{\tau} = -0.683680$	- 144.420	294.839	295.089	302.655	298.002
GL (σ, ρ)	$\hat{\sigma} = 3.475230$	$\hat{\rho} = 1.280000$	-	- 145.094	294.188	294.312	299.398	296.297
EG (ζ, ω)	$\hat{\zeta} = 5.877930$	$\hat{\omega} = 1.246890$	-	- 144.935	293.870	293.994	299.081	295.979
EE (θ, φ)	$\hat{\theta} = 1.013130$	$\hat{\varphi} = 7.787700$	-	- 146.186	296.497	296.373	301.583	298.482
GDUS-E (ε, l)	$\hat{\varepsilon} = 7.468260$	$\hat{l} = 1.130770$	-	- 144.506	293.135	293.012	298.222	295.120

A detailed comparison of competing lifetime models correlated to the carbon fiber breaking stress data under both the classical maximum likelihood estimation (MLE) estimation model, respectively, based on the values of log-likelihood and standard information criteria (AIC, CAIC, BIC, and HQIC) are presented in table 8 as a combination of goodness-of-fit and the parsimony of the models. In the MLE, the suggested DUS-GPQF framework has the biggest log-likelihood and the smallest information criteria of all applicants, which means it can provide the most desirable compromise between the complexity of the model and the explanatory power even in a standard estimation context. the DUS-GPQF distribution is the most successful model to use in the modeling of complex lifetime observations like carbon fiber breaking stress.



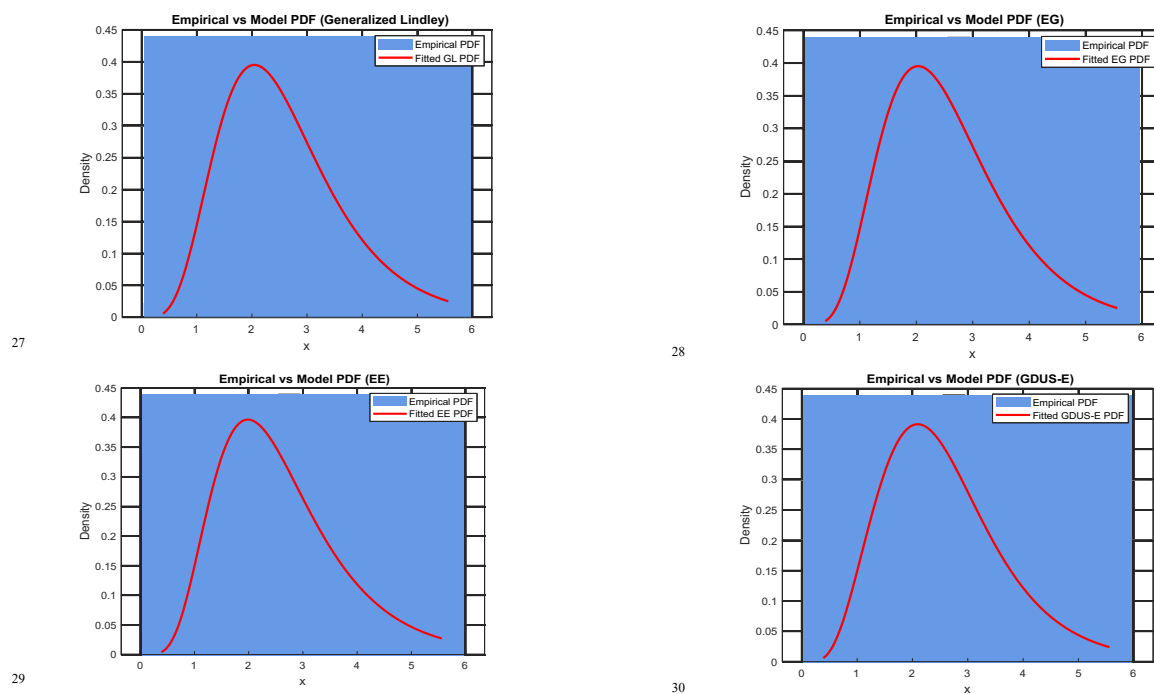
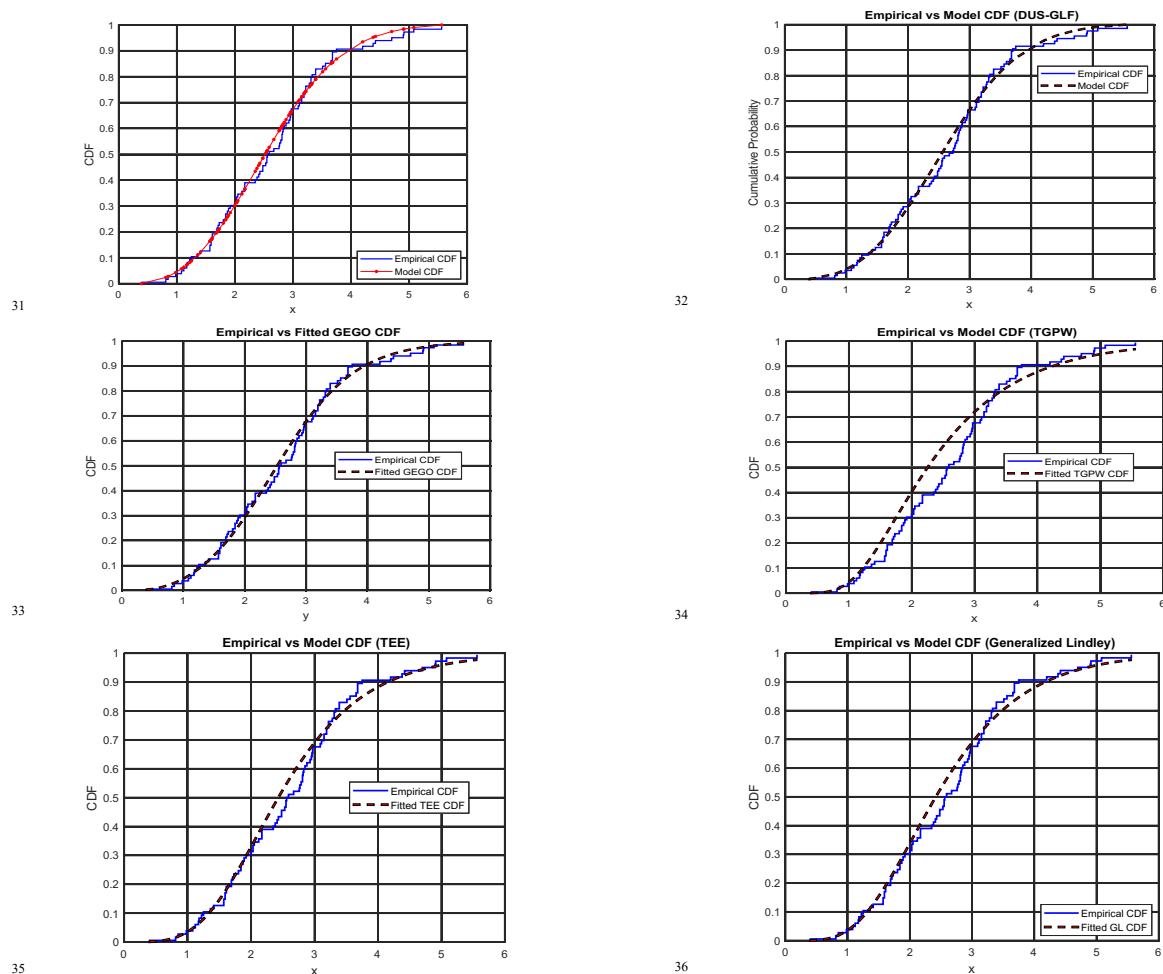


Fig. 6: Plots of the estimated probability density functions for all competing models' dataset D2.



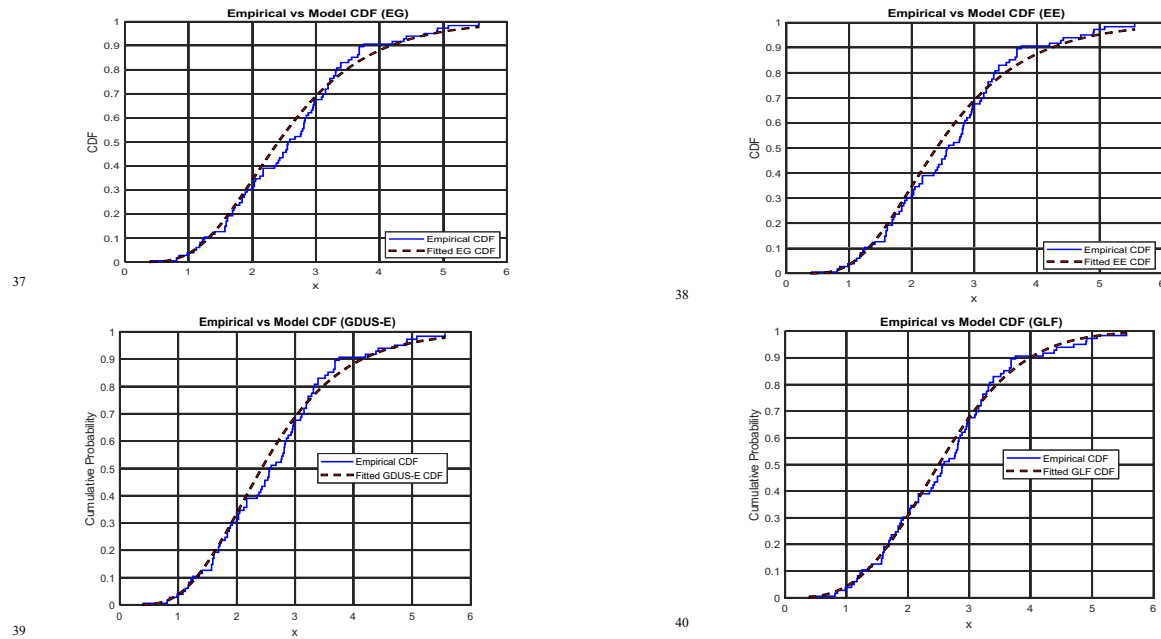
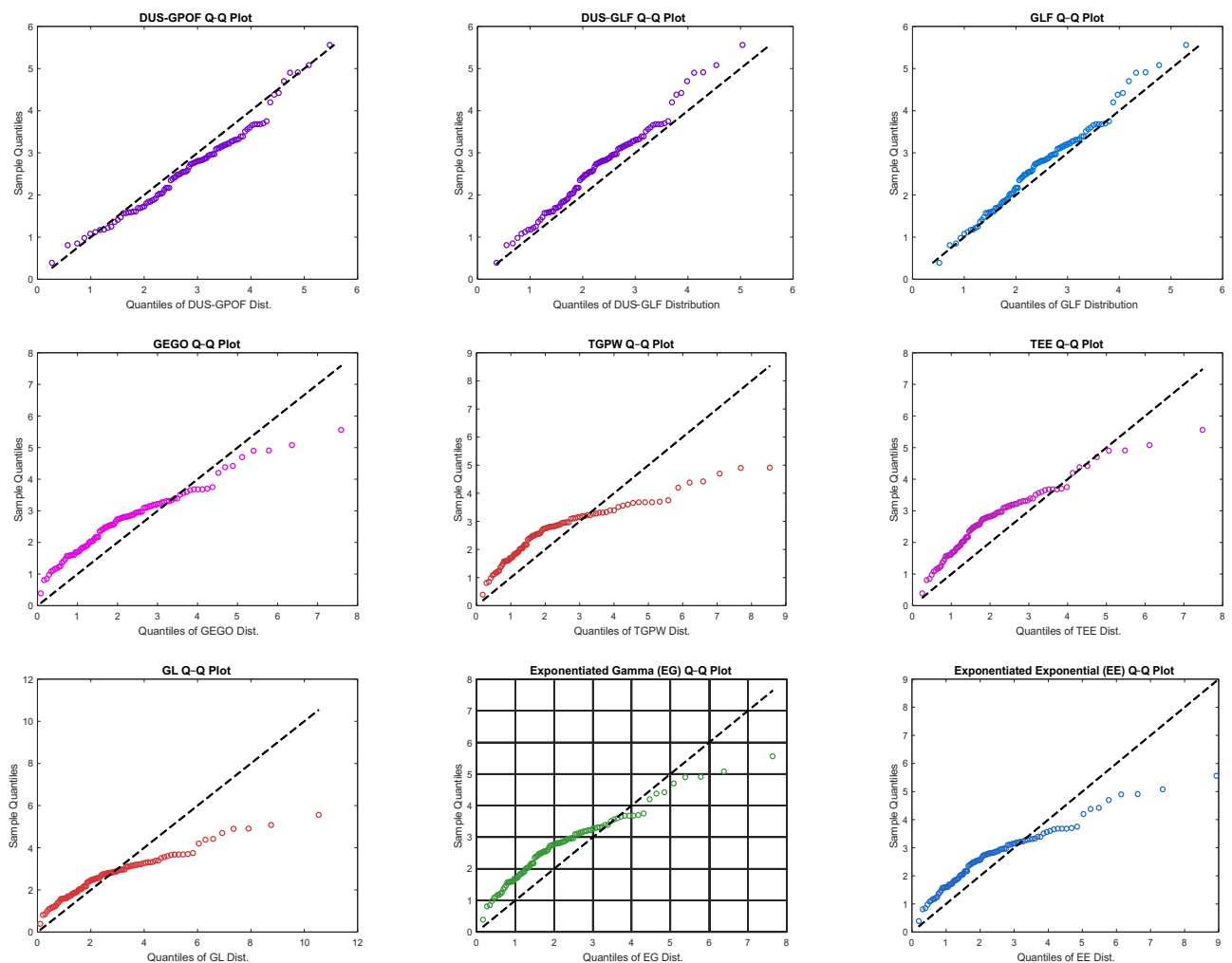


Fig. 7: Plots of the estimated cumulative distribution functions for all competing models based on Dataset D2.



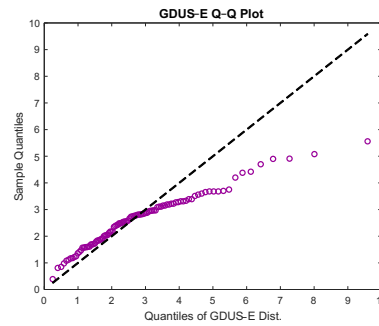


Fig. 8: The Q-Q plot of the fitted distributions for the dataset D2.

6.2 Results Based on PINN Estimation

In order to also evaluate the strength of the proposed model, the Physics-Informed Neural Network (PINN) approach is also used to estimate the parameters. It is a recent method to combine statistical inference with machine learning and uses the likelihood structure of the DUS-Quadratic Hazard Model within a neural network framework. Tables 9 and 10 show the goodness-of-fit statistics using the parameters estimated by the PINN. It can be seen that the results indicate that compared to the rivaling models within this estimation framework, DUS-Quadratic Hazard distribution remains better. It is worth noticing that the goodness-of-fit values obtained through PINN are quite similar to the ones obtained with the assistance of MLE, which reveals the high consistency between classical and machine learning-based estimation methods. Such an agreement points to the strength and consistency of the DUS-Quadratic Hazard Model and the efficiency of the PINN approach as a possible alternative to the parameter estimation in complex nonlinear models.

6.2.1 Computational time and scalability

We evaluated the statistical accuracy of the two estimation paradigms: maximum likelihood estimation (MLE) and the physics-informed neural network (PINN) estimator, besides the computational burden. We measured the mean wall-clock time it takes to get converged parameter estimates for each sample size n ($n = 70, 150, 300$) that we considered in the Monte Carlo design. MLE was performed directly by maximizing the log-likelihood (21) using a standard quasi-Newton routine until a relative change of both the log-likelihood and the parameter vector was less than a specified tolerance. The PINN estimation was carried out by minimizing the physics-informed loss in Eq. (19) using a gradient-based method (Adam followed by L-BFGS if it is applicable) with very small and converging parameter updates.

The amount of work for each iteration of an MLE is linear in the number of observations n , $O(n)$, in the present four-parameter model, as the log-likelihood and its derivatives are evaluated over the sample just once per iteration; in practice, the number of iterations is quite low, so the overall cost is linear in, $O(n)$. By contrast, training with PINN involves many epochs of repeated passes over the data (or mini-batches) implying an approximate cost of $O(n \times E)$ in full-batch mode (E = number of epochs) with a larger constant factor resulting from backpropagation. Thus, when the likelihood surface is highly nonlinear, or ill-conditioned (in the sense of needing many restarts of the MLE), or when GPU acceleration and mini-batching are used (making the cost of training scale sub linearly with n per update), PINN can be a competitive alternative for small to moderate n , and is best regarded as a scalable option for complex, non-convex likelihood structures where classical optimization might be unstable or sensitive to initialization.

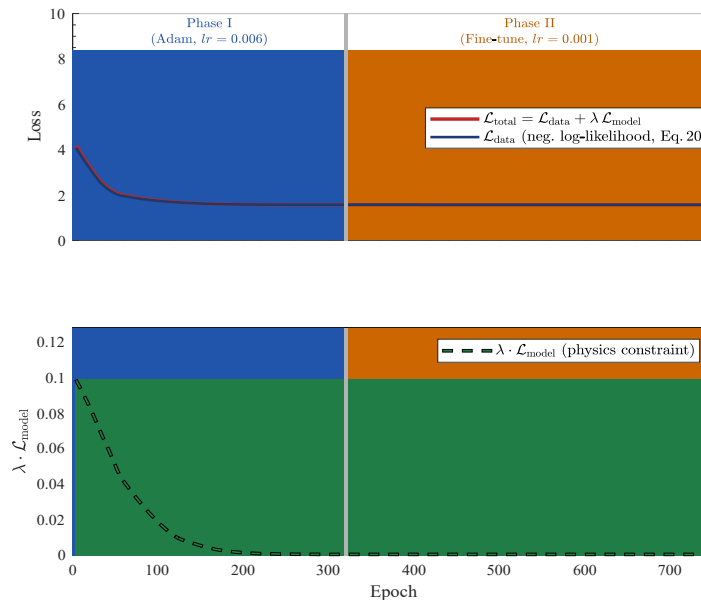
Table 9: PINN with corresponding standard errors (in parentheses) and additional goodness-of-fit statistics for the dataset.

Model	Estimations			-Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\hat{\alpha} = 2.01528$	$\hat{\beta} = 0.00344$	$\hat{\gamma} = 0.245174$ $\hat{\delta} = 0.215285$	54.4165	114.8331	115.2398	121.2625	117.3618
DUS-GLF	$\hat{\alpha} = 0.271127$	$\hat{\beta} = 0.159600$	$\hat{\gamma} = 0.767687$	58.274	122.954	122.548	128.977	125.076
GLF	$\hat{\alpha} = 0.228725$	$\hat{\beta} = 0.139639$	$\hat{\gamma} = 0.860170$	56.2658	118.5519	118.9587	124.9813	121.0706
GEGO	$\hat{\delta} = 0.582049$	$\hat{\eta} = 3.62514 \times 10^{-7}$	$\hat{\kappa} = 1.787760$	55.8104	121.6208	122.6735	132.3365	125.8354
TGPW	$\hat{\lambda} = 1.415370$	$\hat{\mu} = 0.502669$	$\hat{\nu} = -0.533853$	56.1781	120.4684	121.2570	128.1421	122.8401

TEE (π, θ, τ)	$\hat{\pi}$ = 1.429130	$\hat{\theta} = 0.662211$	$\hat{\tau} = -0.512123$	57.8465	123.6931	124.3828	132.2656	127.0647
GL (σ, ρ)	$\hat{\sigma}$ = 1.448690	$\hat{\rho} = 0.819276$	-	59.5331	125.0662	125.4730	131.4956	127.5949
EG (ς, ω)	$\hat{\varsigma}$ = 0.809072	$\hat{\omega} = 0.809244$	-	56.3107	118.6215	119.0283	125.0509	121.1403
EE (θ, φ)	$\hat{\vartheta}$ = 0.582049	$\hat{\varphi} = 1.787760$	-	60.4512	126.1225	126.5283	132.5510	128.6412
GDUS-E (ε, l)	$\hat{\varepsilon}$ = 1.584170	$\hat{l} = 0.684871$	-	56.3838	164.767	174.340	175.340	170.139

Table 10: PINN with corresponding standard errors (in parentheses) and additional goodness-of-fit statistics for the dataset.

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\hat{\alpha} = 4.38208$	$\hat{\beta} = 0.03749$	$\hat{\gamma} = 0.63297$ $\hat{\delta} = 0.23410$	- 133.403	276.807	281.228	287.228	281.025
DUS-GLF (α, β, γ)	$\hat{\alpha} = 0.171775$	$\hat{\beta} = 0.249494$	$\hat{\gamma} = 2.129720$	- 140.207	286.664	286.414	294.229	290.577
GLF (α, β, γ)	$\hat{\alpha} = 0.071147$	$\hat{\beta} = 0.233821$	$\hat{\gamma} = 2.087800$	- 140.538	288.326	288.076	295.891	291.239
GEGO (δ, η, κ)	$\hat{\delta} = 1.022130$	$\hat{\eta} = 3.7334 \times 10^{-7}$	$\hat{\kappa} = 7.677700$	- 144.186	297.623	297.373	305.189	300.536
TGPW (λ, μ, ν)	$\hat{\lambda} = 3.544750$	$\hat{\mu} = 0.156011$	$\hat{\nu} = -0.840447$	- 146.462	303.174	302.924	310.739	306.087
TEE (π, θ, τ)	$\hat{\pi} = 6.175930$	$\hat{\theta} = 1.100840$	$\hat{\tau} = -0.573680$	- 141.420	292.839	293.089	300.655	296.002
GL (σ, ρ)	$\hat{\sigma} = 3.464230$	$\hat{\rho} = 1.170000$	-	- 143.094	293.188	293.312	298.398	295.297
EG (ς, ω)	$\hat{\varsigma} = 5.866730$	$\hat{\omega} = 1.136890$	-	- 142.935	292.870	292.994	298.081	294.979
EE (θ, φ)	$\hat{\vartheta} = 1.012030$	$\hat{\varphi} = 7.677700$	-	- 144.186	295.497	295.373	300.583	297.482
GDUS-E (ε, l)	$\hat{\varepsilon} = 7.358260$	$\hat{l} = 1.220770$	-	- 142.506	292.135	292.012	297.222	294.120

**Fig. 9:** PINN convergence curves (total loss and its components) for DUS-QH parameter estimation.

7. Conclusion and Future Work

This study develops a new and highly flexible lifetime distribution, termed the DUS-transformed generalized polynomial quadratic failure rate distribution. The proposed model is formulated by applying the Dinesh–Umesh–Sanjay (DUS) transformation to the generalized polynomial quadratic failure rate (GPQF) baseline distribution, thereby enhancing its structural flexibility and extending its capability to model diverse hazard rate behaviors. The given distribution is much more flexible in its modeling, yet structural parsimony is maintained, which is of paramount importance in the modern reliability and survival analysis. Having a reasonable size parameter dimension, the DUS-GPQF model does not succumb to the over-parameterization issues that can frequently plague longer lifetime models, yet covers a broad range of data behavior.

Theoretical exploration of the distribution of DUS-GPQF was thoroughly done and the probability density function, cumulative distribution function, hazard rate function, and moments of the distribution as well as inverse moments, entropy and incomplete moments, conditional moments, mean residual life and mean inactivity time are explicitly derived. The analysis findings indicate that the suggested model is capable of representing increasing, decreasing, and bathtub-shaped hazard rate behavior and this shows that the model is appropriate in modeling more complex lifetime effects observed in engineering, medical and reliability practice. A large-scale simulation experiment was conducted to measure the inferential accuracy of seventeen parameter estimation procedures, taking a variety of accuracy measures including Bias, Mean Squared Error (MSE), Mean Relative Error (MRE), maximum absolute difference, average absolute difference and average squared absolute error. It is clear that the estimators are consistent based on the result of the simulation as all measures of performance change monotonically as the sample size increases. Further, spacing-based estimators especially the MSADE, MSSDE and MSLNDE estimators are more often better than the classical maximum likelihood estimation in stability and robustness, particularly when the sample size is small and moderate.

The results obtained can serve as a solid empirical suggestion to the practitioners when fitting the DUS-GPQF model, in terms of estimator selection. The viability of the suggested form of distribution was also confirmed by the study of two real-life datasets on aircraft windshield service time and carbon fiber rupturing stress. The DUS-GPQF distribution in both instances has routinely attained the lowest possible values of information criteria (AIC, CAIC, BIC and HQIC), as well as better values of goodness-of-fit, and desirable Kolmogorov-Smirnov statistics, over a number of well-known competing models. The graphical diagnostics, fitted PDFs, CDFs, and Q-Q plots also support well the high fitting power of the suggested model. These empirical findings prove that DUS-GPQF distribution is a statistically reasonable and practically efficient structure in modeling complex lifetime data.

Although the current research confirms that the DUS-GPQF distribution is a strong and versatile lifetime model, there are multiple research opportunities that can be pursued in the future. To improve the inferential strength further, first, Bayesian inference procedures of the parameters of the DUS-GPQF with informative and non-informative priors, and Markov chain Monte Carlo algorithms, may be established. Second, future studies on the suggested model should include extensions to censored, truncated or progressively censored data structures to increase its relevance in survival and reliability research. Third, a set of extensions of the DUS-GPQF distribution to regression models could be considered in order to include covariate data, thus making it applicable to accelerated failure time and proportional hazard models. Lastly, generalizations using multivariate and competing-risks based on DUS transformation is an area that has a lot of potential in terms of future methodological enhancement.

To conclude, it can be said that the DUS-GPQF distribution presents a significant addition to the statistical literature of lifetime modeling with both theoretical and practical strengths. It is flexible, has high inferential stability and a good empirical record, which makes it a useful addition to the arsenal of applied statisticians and reliability analysts.

Conflicts of Interest Statement

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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Appendix

Survival, cumulative hazard, and hazard functions. Let $F(t; \theta)$ denote the DUS-QH CDF in (6). The survival (reliability) function is

$$S(t; \theta) = 1 - F(t; \theta). \quad t > 0. \quad (43)$$

The cumulative hazard is $H(t; \theta) = -\log S(t; \theta)$, and the hazard is $h(t; \theta) = f(t; \theta)/S(t; \theta)$. When the DUS transformation is written as

$$F(t; \theta) = T_\lambda(G_0(t; a; b; c)). \quad (44)$$

where G_0 is the baseline GPQFR CDF and $T_\lambda(\cdot)$ is the monotone DUS map in (7), then

$$S(t; \theta) = 1 - T_\lambda(G_0(t)). \quad H(t; \theta) = -\log(1 - T_\lambda(G_0(t))). \quad (45)$$

Quantile function and random variate generation. For $u \in (0, 1)$, the quantile $Q(u)$ solves $F(Q(u); \theta) = u$. Because T_λ is strictly increasing, we can invert in two steps:

$$G_0(Q(u); a; b; c) = T_\lambda^{-1}(u). \quad \text{hence} \quad Q(u) = G_0^{-1}(T_\lambda^{-1}(u)). \quad (46)$$

For the quadratic-hazard baseline, $G_0^{-1}(\cdot)$ has no closed form because the cumulative hazard is cubic in t ; therefore $Q(u)$ is obtained numerically (e.g., NewtonRaphson or bisection). This inversion also yields a direct simulation scheme: generate $U \sim \text{Unif}(0, 1)$, compute $v = T_\lambda^{-1}(U)$, then set $X = G_0^{-1}(v)$. Moments and numerical evaluation. The r -th raw moment ($r > 0$) is

$$\mu'_r = E[X^r] = \int_0^\infty t^r f(t; \theta) dt \quad (47)$$

Closed-form expressions are generally unavailable because $f(t; \theta)$ involves the exponential of a cubic polynomial through the baseline hazard; however, moments are well-defined under the condition stated in (15). Using the series representation in Section 2.6, write

$$f(t; \theta) = \sum_{m \geq 0} w_m(\lambda) f_0(t; a; b; c) [G_0(t; a; b; c)]^m \quad (48)$$

where f_0 is the GPQFR density and $\{w_m(\lambda)\}$ are expansion coefficients induced by $T_\lambda(\cdot)$. Then, by dominated convergence,

$$\mu'_r = \sum_{m \geq 0} w_m(\lambda) \int_0^\infty t^r f_0(t) [G_0(t)]^m dt \quad (49)$$

which can be evaluated accurately via one-dimensional quadrature (Gauss Laguerre or adaptive Simpson) for each $(a; b; c; \lambda)$. In applications, the mean and variance are computed numerically from μ'_1 and $\mu'_2 - (\mu'_1)^2$, and higher moments can be used to obtain skewness and kurtosis.