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Recurrence Relations for Moments of Progressively Type-II Censored from lindley Weibull Distribution

D. M. A. El-Agoz[1,*], *M. A. W. Mahmoud*[2] *and N. M. Yhiea*[1]

[1]Department of Mathematics, Faculty of Science, Suez Canal University, Ismailia, Egypt
[2]Department of Mathematics, Faculty of Science Al-Azhar University Naser City (11884), Cairo, Egypt

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Abstract: In this paper, we establish recurrence relations for single and product moments based on progressively Type-II censored from three parameters lindley Weibull distribution (LWD) and doubly truncated LWD. Also, we can use it to compute all the means, variances and covariances of lindley Weibull progressive Type-II censored order statistics.

Keywords: lindley Weibull distribution, progressively Type-II censored, recurrence relations.

1 Introduction

The most common right censoring schemes are Type-I and Type-II censoring, but the traditional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the final point of the experiment. So, a more general censoring scheme called progressive Type-II right censoring is suggested. Under this general censoring scheme, n units are placed on test. only m are completely observed until failure.At the time of the first failure, R_1 of the $n-1$ surviving units are randomly removed (or censored)from the test. At the time of the next failure, R_2 of the $n-2-R_1$ surviving units are removed from the test at censored, and so on. Finally, at the time of the m-th failure, all the remaining $R_m = n-R_1-R_2-\ldots-R_{m-1}-m$ surviving units are censored. The m ordered observed failure times denoted by $X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ are called progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1,\ldots,R_m)$ It is clear that $n = m+\sum_{i=1}^{m} R_i$. The special case when $R_1 = R_2 = \ldots = R_{m-1} = 0$ so that $R_m = n-m$ is the case of conventional Type-II right censored sampling. Also when $R_1 = R_2 = \ldots = R_m = 0$, so that $m = n$, so in this case, the progressively Type-II right censoring scheme reduces to the case of no censoring (ordinary order statistics). Many authors have discussed inference under progressive Type-II censored using different lifetime distributions, see for example, [1,2,3,4,5,6,7,8,9] and [10]. A thorough overview of the subject of progressive censoring and the excellent review article is given in [11]. [12] developed an algorithm to simulate general, progressively Type-II censored samples from the uniform or any other continuous distribution. The joint probability density function for progressively Type-II censored sample of size m from a sample of size n is given by, for details see [13]

$$f_{x_{1:m:n},\ldots,x_{m:m:n}}(x_1,\ldots,x_m) = C_{n,R_{m-1}} \prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \tag{1}$$
$$-\infty < x_1 < x_2 < \ldots < x_m < \infty,$$

where

$$C_{n,R_{m-1}} = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-R_1-R_2-\ldots-R_{m-1}-m+1), \tag{2}$$

* Corresponding author e-mail: dina_mohamed@science.suez.edu.eg

The probability density function (PDF), cumulative distribution function (CDF), reliability function S(x), and hazard rate function h(x) of the LWD are given respectively, by

$$f(x) = \frac{\beta\theta^2}{\theta+1}[\alpha^\beta x^{\beta-1} + \alpha^{2\beta}x^{2\beta-1}]exp[-\theta(\alpha x)^\beta], \tag{3}$$

$$F(x) = 1 - exp[-\theta(\alpha x)^\beta][1+\frac{\theta}{\theta+1}(\alpha x)^\beta], \tag{4}$$

$$S(x) = exp[-\theta(\alpha x)^\beta][1+\frac{\theta}{\theta+1}(\alpha x)^\beta], \tag{5}$$

and

$$h(x) = \frac{\frac{\beta\theta^2}{\theta+1}[\alpha^\beta x^{\beta-1} + \alpha^{2\beta}x^{2\beta-1}]}{[1+\frac{\theta}{\theta+1}(\alpha x)^\beta]}, \tag{6}$$

also, the characterizing differential equation given by

$$f(x) = \frac{\beta\theta^2[\alpha^\beta x^{\beta-1} + \alpha^{2\beta}x^{2\beta-1}][1-F(x)]}{\theta+1+\theta(\alpha x)^\beta}, \tag{7}$$

From Eq. (2), we see that if α = 1 and β = 1; then LWD reduces to lindley distribution. with parameter θ.
This paper is organized as follows: In Sections (2,3) the recurrence relations for single and product moments of progressive Type-II censored order statistics from LWD are derived. Finaly, in section 4 is devoted to get recurrence relations for the single and product moments of progressive Type-II right censored order statistics from the doubly truncated LWD. In this paper, we derive new recurrence relations satisfied by the single and product moments of the progressively Type-II censored order statistics from the LWD and doubly truncated LWD. Several authors have been developed new recurrence relations satisfied by the single and product moments for different distributions. [14] have established recurrence relations for moments of generalized order statistics within a class of doubly truncated distributions. [15] have established recurrence relations for single and product moments of order statistics from a generalized logistic distribution. [16] have established recurrence relations for moments of progressively censored order statistics from logistic distribution. [17] have established recurrence relations and identities for moments of order statistics.

2 Recurrence Relations for the Single Moments

Let $\underline{X} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ be the progressively Type-II censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \ldots R_m)$ taken from the lindley Weibull distribution whose PDF and CDF are given by (3) and (4) .The single moments of the progressively Type-II can be written as

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)(k)} &= C_{n,R_{m-1}} \iint_{0<x_1<x_2<\ldots<x_m<\infty} \int x_i^k f(x_1)[1-F(x_1)]^{R_1} \\ &\times f(x_2)[1-F(x_2)]^{R_2} \ldots . f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots . . dxm.\end{aligned} \tag{8}$$

where

$C_{n,R_{m-1}}$ is defined in (2).

The following recurrence relations gives the single moments of progressively Type-II censored are derived on the basis of the Lindley Weibull distribution .

Theorem 1. For $2 \le m \le n, \quad k, \beta > 0$ and $\theta > 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+\beta)} &= \frac{-(\theta+1)(\beta\theta)}{[\frac{R_1+1}{k+\beta}(\theta+1)\beta\theta+1]}[\frac{n-R_1-1}{k+\beta}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(k+\beta)} \\ &+ \alpha^\beta\frac{n-R_1-1}{k+2\beta}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(k+2\beta)} + \alpha^\beta\frac{R_1+1}{k+2\beta}\mu_{1:m:n}^{(R_1,R_2,R_3,\ldots,R_m)(k+2\beta)}].\end{aligned} \tag{9}$$

Proof. From (8), we have

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k)} &= C_{n,R_{m-1}} \iint_{0<x_1<x_2<\dots<x_m<\infty} \int x_1^k f(x_1)[1-F(x_1)]^{R_1} \\ &\quad \times f(x_2)[1-F(x_2)]^{R_2} \dots . f(x_m)[1-F(x_m)]^{R_m} dx_1 \dots . dxm.\end{aligned} \tag{10}$$

By using (7) we get,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k)} &= C_{n,R_{m-1}} \iint_{0<x_1<x_2<\dots<x_m<\infty} \int_0^{x_2} I(x_2) f(x_2) \\ &\quad \times [1-F(x_2)]^{R_2} \dots . f(x_m)[1-F(x_m)]^{R_m} dx_2 \dots . dxm,\end{aligned} \tag{11}$$

where

$$I(x_2) = \int_0^{x_2} x_1^k \beta\theta^2 [\alpha^\beta x^{\beta-1} + \alpha^{2\beta} x^{2\beta-1}][1-F(x_1)]^{R_1+1} dx_1.$$

Integrating by parts, we get

$$\begin{aligned}\int_0^{x_2} x_1^k \beta\theta^2 [\alpha^\beta x^{\beta-1} + \alpha^{2\beta} x^{2\beta-1}][1-F(x_1)]^{R_1+1} dx_1 &= \frac{\beta\theta^2\alpha^\beta}{k+\beta} x_2^{k+\beta} [1-F(x_2)]^{1+R_1} \\ &+ \frac{\beta\theta^2\alpha^\beta (R_1+1)}{k+\beta} \int_0^{x_2} x_1^{k+\beta} f(x_1)[1-F(x_1)]^{R_1} dx_1 \\ &+ \frac{\beta\theta^2\alpha^{2\beta}}{k+2\beta} x_2^{k+2\beta} [1-F(x_2)]^{1+R_1} \\ &+ \frac{\beta\theta^2\alpha^{2\beta} (R_1+1)}{k+2\beta} \int_0^{x_2} x_1^{k+2\beta} f(x_1)[1-F(x_1)]^{R_1} dx_1.\end{aligned} \tag{12}$$

Substituting the above expression into (11) we get (9) .

Theorem 2. For $2 \le i \le m-1, \quad k,\beta > 0$ and $\theta > 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)} = \frac{-(\theta+1)(\beta\theta)}{[\frac{R_i+1}{k+\beta}(\theta+1)\beta\theta+1]} &[\frac{(n-R_1-R_2-\dots-R_i-i)}{k+\beta} \mu_{i:m-1:n}^{(R_1,\dots.,R_i+R_{i+1}+1,\dots,R_m)(k+\beta)} \\ &- \frac{(n-R_1-R_2-\dots-R_{i-1}-i+1)}{k+\beta} \mu_{i-1:m-1:n}^{(R_1,\dots.,R_{i-1}+R_i+1,\dots,R_m)(k+\beta)} \\ &+ \alpha^\beta \frac{(n-R_1-R_2-\dots-R_i-i)}{k+2\beta} \mu_{i:m-1:n}^{(R_1,\dots.,R_i+R_{i+1}+1,\dots,R_m)(k+2\beta)} \\ &- \alpha^\beta \frac{(n-R_1-R_2-\dots-R_{i-1}-i+1)}{k+2\beta} \mu_{i-1:m-1:n}^{(R_1,\dots.,R_{i-1}+R_i+1,\dots,R_m)(k+2\beta)} \\ &+ \alpha^\beta \frac{(R_i+1)}{k+2\beta} \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+2\beta)}].\end{aligned} \tag{13}$$

Proof. Making use of (7) and (8), yields

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k)} = C_{n,R_{m-1}} \iint_{0<x_1<x_2<\dots<x_{i-1}<x_{i+1}<\dots<\infty} \int I(x_{i-1},x_{i+1}) \times f(x_1)[1-F(x_1)]^{R_1} \dots \\ f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}} \\ \times \dots f(x_m)[1-F(x_m)]^{R_m} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_m, \tag{14}$$

where

$$I(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k \beta\theta^2 [\alpha^\beta x^{\beta-1} + \alpha^{2\beta} x^{2\beta-1}][1-F(x_i)]^{R_i+1} dx_i.$$

Integrating by parts, we get

$$I(x_{i-1},x_{i+1}) = \frac{\beta\theta^2\alpha^\beta}{k+\beta}[x_{i+1}^{k+\beta}[1-F(x_{i+1})]^{1+R_i} - x_{i-1}^{k+\beta}[1-F(x_{i-1})]^{1+R_i}] \\ + \frac{\beta\theta^2\alpha^\beta(R_i+1)}{k+\beta}\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta} f(x_i)[1-F(x_i)]^{R_i} dx_i \\ + \frac{\beta\theta^2\alpha^{2\beta}}{k+2\beta}[x_{i+1}^{k+2\beta}[1-F(x_{i+1})]^{1+R_i} - x_{i-1}^{k+2\beta}[1-F(x_{i-1})]^{1+R_i}] \\ + \frac{\beta\theta^2\alpha^{2\beta}(R_i+1)}{k+2\beta}\int_{x_{i-1}}^{x_{i+1}} x_i^{k+2\beta} f(x_i)[1-F(x_i)]^{R_i} dx_i. \tag{15}$$

Substituting the above expression into (14) we get (13) .

Theorem 3. For $2 \le m \le n, \quad k,\beta > 0$ and $\theta > 0$,

$$\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)} = \frac{-(\theta+1)(\beta\theta)}{[\frac{R_m+1}{k+\beta}(\theta+1)\beta\theta+1]}[\frac{-(n-R_1-R_2-\dots-R_{m-1}-m+1)}{k+\beta}\mu_{m-1:m-1:n}^{(R_1,\dots,R_{m-1}+R_m+1)(k+\beta)} \\ -\alpha^\beta\frac{(n-R_1-R_2-\dots-R_{m-1}-m+1)}{k+2\beta}\mu_{m-1:m-1:n}^{(R_1,\dots,R_{m-1}+R_m+1)(k+2\beta)} \\ +\alpha^\beta\frac{(R_m+1)}{k+2\beta}\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)(k+2\beta)}] \tag{16}$$

Proof. Using (7) and (8), we can write

$$\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)(k)} = C_{n,R_{m-1}} \iint_{0<x_1<x_2<\dots<x_{m-1}<\infty} \int I(x_{m-1}) f(x_1)[1-F(x_1)]^{R_1} \times f(x_2)[1-F(x_2)]^{R_2} \dots \\ \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1 dx_2 \dots dx_{m-1}. \tag{17}$$

where

$$I(x_{m-1}) = \int_{x_{m-1}}^{\infty} x_m^k \beta\theta^2[\alpha^\beta x^{\beta-1} + \alpha^{2\beta} x^{2\beta-1}][1-F(x_m)]^{R_m+1} dx_m.$$

Integrating by parts, we obtain

$$
\begin{aligned}
I(x_{m-1}) = &-\frac{\beta\theta^2\alpha^\beta}{k+\beta}x_{m-1}^{k+\beta}[1-F(x_{m-1})]^{1+R_m} \\
&+\frac{\beta\theta^2\alpha^\beta(R_m+1)}{k+\beta}\int_{x_{m-1}}^{\infty} x_m^{k+\beta}f(x_m)[1-F(x_m)]^{R_m}dx_m \\
&-\frac{\beta\theta^2\alpha^{2\beta}}{k+2\beta}x_{m-1}^{k+2\beta}[1-F(x_{m-1})]^{1+R_m} \\
&+\frac{\beta\theta^2\alpha^{2\beta}(R_m+1)}{k+2\beta}\int_{x_{m-1}}^{\infty} x_m^{k+2\beta}f(x_m)[1-F(x_m)]^{R_m}dx_m.
\end{aligned}
\tag{18}
$$

Substituting the above expression into (17) we get (16) .

3 Recurrence Relations for the Product Moments

For any continuous distribution, we can write the (i,j)-th product moment of progressively Type-II censored order statistics are derived on the basis of the Lindley Weibull distribution

$$
\begin{aligned}
\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)} &= E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}X_{j:m:n}^{(R_1,R_2,\ldots,R_m)}] \\
&= C_{n,R_{m-1}}\iint_{0<x_1<x_2<\ldots<x_m<\infty}\int x_i x_j f(x_1)[1-F(x_1)]^{R_1} \\
&\times f(x_2)[1-F(x_2)]^{R_2}\ldots.f(x_m)[1-F(x_m)]^{R_m}dx_1\ldots..dxm.
\end{aligned}
\tag{19}
$$

Theorem 1. For $1 \le i < j \le m-1, \quad m \le n, \quad \beta,\theta > 0$,

$$
\begin{aligned}
\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)} = &\frac{-(\theta+1)(\beta\theta)}{[1+\theta(\theta+1)(1+R_j)]}[\frac{n-R_1-R_2-\ldots-R_j-j}{\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_j+R_{j+1}+1,..,R_m)(\beta)} \\
&-\frac{n-R_1-R_2-\ldots-R_{j-1}-j+1}{\beta}\mu_{i,j-1:m-1:n}^{(R_1,R_2,\ldots,R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)(\beta)} \\
&+\alpha^\beta\frac{n-R_1-R_2-\ldots-R_j-j}{2\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)(2\beta)} \\
&-\alpha^\beta\frac{n-R_1-R_2-\ldots-R_{j-1}-j+1}{2\beta}\mu_{i,j-1:m-1:n}^{(R_1,R_2,\ldots,R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)(2\beta)} \\
&+\alpha^\beta\frac{1+R_j}{2\beta}\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(2\beta)}]
\end{aligned}
\tag{20}
$$

Proof

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,...,R_m)} &= E[X_{i:m:n}^{(R_1,R_2,...R_m)}\{X_{j:m:n}^{(R_1,R_2,...R_m)}\}^0] \\ &= C_{n,R_{m-1}} \iint_{0<x_1<x_2<...<x_{j-1}<x_{j+1}<...<x_m<\infty} \int x_i I(x_{j-1},x_{j+1}) \\ &\times f(x_1)[1-F(x_1)]^{R_1}.....f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}} \\ &\times ....f(x_m)[1-F(x_m)]^{R_m} dx_1....dx_{j-1}dx_{j+1}.....dx_m,\end{aligned} \tag{21}$$

where

$$I(x_{j-1},x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^0 \beta\theta^2(\alpha^\beta x_j^{\beta-1} + \alpha^{2\beta} x_j^{2\beta-1})[1-F(x_j)]^{R_j+1} dx_j.$$

Integrating by parts, we get

$$\begin{aligned}I(x_{j-1},x_{j+1}) &= \theta^2\alpha^\beta[x_{j+1}^\beta[1-F(x_{j+1})]^{1+R_j} - x_{j-1}^\beta[1-F(x_{j-1})]^{1+R_j}] \\ &+ \theta^2\alpha^\beta(R_j+1)\int_{x_{j-1}}^{x_{j+1}} x_j^\beta f(x_j)[1-F(x_j)]^{R_j} dx_j \\ &+ \frac{\theta^2\alpha^{2\beta}}{2}[x_{j+1}^{2\beta}[1-F(x_{j+1})]^{1+R_j} - x_{j-1}^{2\beta}[1-F(x_{j-1})]^{1+R_j}] \\ &+ \frac{\theta^2\alpha^{2\beta}(R_j+1)}{2}\int_{x_{j-1}}^{x_{j+1}} x_j^{2\beta} f(x_j)[1-F(x_j)]^{R_j} dx_j.\end{aligned} \tag{22}$$

Substituting the above expression into (21) we get (20) .

Theorem 2. For $1 \le i \le m-1, \quad m \le n, \quad \beta,\theta > 0$,

$$\begin{aligned}\mu_{i,m:m:n}^{(R_1,R_2,...,R_m)} &= \frac{-(\theta+1)(\beta\theta)}{[1+\theta(\theta+1)(1+R_m)]}[\frac{-(n-R_1-R_2-...-R_{m-1}-m+1)}{\beta}\mu_{i,m-1:m-1:n}^{(R_1,R_2,...,R_{m-1}+R_m+1)(\beta)} \\ &- \alpha^\beta\frac{n-R_1-R_2-...-R_{m-1}-m+1}{2\beta}\mu_{i,m-1:m-1:n}^{(R_1,R_2,...,R_{m-1}+R_m+1)(2\beta)} \\ &+ \alpha^\beta\frac{1+R_m}{2\beta}\mu_{i,m:m:n}^{(R_1,R_2,...,R_m)(2\beta)}]\end{aligned} \tag{23}$$

Proof.

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,...,R_m)} &= E[X_{i:m:n}^{(R_1,R_2,...,R_m)}\{X_{m:m:n}^{(R_1,R_2,...,R_m)}\}^0] \\ &= C_{n,R_{m-1}}\iint_{0<x_1<...<x_{m_1}<\infty}\int x_i I(x_{m-1}) \\ &\times f(x_1)[1-F(x_1)]^{R_1}.....f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1dx_2.....dx_{m-1},\end{aligned} \tag{24}$$

where

$$I(x_{m-1})=\int_{x_{m-1}}^{\infty}x_m^0\beta\theta^2(\alpha^{\beta}x_m^{\beta-1}+\alpha^{2\beta}x_m^{2\beta-1})[1-F(x_m)]^{R_m+1}dx_m.$$

Integrating by parts, we get

$$\begin{aligned}I(x_{m-1})&=-\theta^2\alpha^{\beta}[x_{m-1}^{\beta}[1-F(x_{m-1})]^{1+R_m}]\\&+\theta^2\alpha^{\beta}(R_m+1)\int_{x_{m-1}}^{\infty}x_m^{\beta}f(x_m)[1-F(x_m)]^{R_m}dx_m\\&-\frac{\theta^2\alpha^{2\beta}}{2}[x_{m-1}^{2\beta}[1-F(x_{m-1})]^{1+R_m}\\&+\frac{\theta^2\alpha^{2\beta}(R_m+1)}{2}\int_{x_{m-1}}^{\infty}x_m^{2\beta}f(x_m)[1-F(x_m)]^{R_m}dx_m.\end{aligned}\tag{25}$$

Substituting the above expression into (24) we get (23) .

Theorem 3. For $1\le i<j\le m-1,\quad m\le n,\quad \beta,\theta>0,$

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(r,s+\beta)}=\frac{-(\theta+1)(\beta\theta)}{[1+\frac{\theta(\theta+1)}{s+\beta}\beta(1+R_j)]}&[\frac{n-R_1-R_2-\ldots-R_j-j}{s+\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_j+R_{j+1}+1,\ldots,R_m)(r,s+\beta)}\\&-\frac{n-R_1-R_2-\ldots-R_{j-1}-j+1}{s+\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)(r,s+\beta)}\\&+\alpha^{\beta}\frac{n-R_1-R_2-\ldots-R_j-j}{s+2\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)(r,s+2\beta)}\\&-\alpha^{\beta}\frac{n-R_1-R_2-\ldots-R_{j-1}-j+1}{s+2\beta}\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)(r,s+2\beta)}\\&+\alpha^{\beta}\frac{1+R_j}{s+2\beta}\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(r,s+2\beta)}]\end{aligned}\tag{26}$$

Proof

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(r,s)}&=E[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)(r)}\{X_{j:m:n}^{(R_1,R_2,\ldots,R_m)(s)}\\&=C_{n,R_{m-1}}\iint_{0<x_1<x_2<\ldots<x_{j-1}<x_{j+1}<\ldots<x_m<\infty}\int x_i^{r}I(x_{j-1},x_{j+1})\\&\times f(x_1)[1-F(x_1)]^{R_1}\ldots\ldots f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}}\\&\times f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}}\ldots\ldots f(x_m)[1-F(x_m)]^{R_m}dx_1dx_2\ldots dx_{j-1}dx_{j+1}\ldots dx_m,\end{aligned}\tag{27}$$

where

$$I(x_{j-1},x_{j+1})=\int_{x_{j-1}}^{x_{j+1}}x_j^s\beta\theta^2(\alpha^\beta x_j^{\beta-1}+\alpha^{2\beta}x_j^{2\beta-1})[1-F(x_j)]^{R_j+1}dx_j$$

Integrating by parts, we get

$$\begin{aligned}I(x_{j-1},x_{j+1})=&\frac{\beta\theta^2\alpha^\beta}{s+\beta}[x_{j+1}^{s+\beta}[1-F(x_{j+1})]^{1+R_j}-x_{j-1}^{s+\beta}[1-F(x_{j-1})]^{1+R_j}]\\&+\frac{\beta\theta^2\alpha^\beta(R_j+1)}{s+\beta}\int_{x_{j-1}}^{x_{j+1}}x_j^{s+\beta}f(x_j)[1-F(x_j)]^{R_j}dx_j\\&+\frac{\beta\theta^2\alpha^{2\beta}}{s+2\beta}[x_{j+1}^{s+2\beta}[1-F(x_{j+1})]^{1+R_j}-x_{j-1}^{s+2\beta}[1-F(x_{j-1})]^{1+R_j}]\\&+\frac{\beta\theta^2\alpha^{2\beta}(R_j+1)}{s+2\beta}\int_{x_{j-1}}^{x_{j+1}}x_j^{s+2\beta}f(x_j)[1-F(x_j)]^{R_j}dx_j.\end{aligned}\tag{28}$$

Substituting the above expression into (27) we get (26) .

4 The Doubly Truncated lindley Weibull Distribution

In this section, we present recurrence relations for the single and product moments of progressively type-II censored order statistics from the doubly truncated lindley weibull distribution. PDF of the doubly truncated LWD is given by:

$$f_t(x)=\frac{\beta\theta^2}{(P-Q)(1+\theta)}[\alpha^\beta x^{\beta-1}+\alpha^{2\beta}x^{2\beta-1}]exp[-\theta(\alpha x)^\beta],\tag{29}$$

$$0<Q_1<x<P_1,\quad \alpha,\theta,\beta>0,\quad x\geq 0,$$

where

$$\begin{aligned}Q=F_t(Q_1)=1-exp[-\theta(\alpha Q_1)^\beta][1+\frac{\theta}{\theta+1}(\alpha Q_1)^\beta],\\P=F_t(P_1)=1-exp[-\theta(\alpha P_1)^\beta][1+\frac{\theta}{\theta+1}(\alpha P_1)^\beta].\end{aligned}\tag{30}$$

Here, 1-P is the proportion of right truncation on the LWD and Q is the propotion of the left truncation.Thus CDF of doubly truncated LWD can be put in the form

$$F_t(x)=\frac{1}{P-Q}\{exp[-\theta(\alpha Q_1)^\beta][1+\frac{\theta}{\theta+1}(\alpha Q_1)^\beta]-exp[-\theta(\alpha x)^\beta][1+\frac{\theta}{1+\theta}(\alpha x)^\beta]\}.\tag{31}$$

So, the characterizing differential equation for this distribution is given by

$$f_t(x)=\frac{\beta\theta^2}{\theta+1+\theta(\alpha x)^\beta}[\alpha^\beta x^{\beta-1}+\alpha^{2\beta}x^{2\beta-1}][\frac{1-P}{P-Q}+[1-F_t(x)]].\tag{32}$$

Thus, we can conclude new recurrence relations for the single and product moments of progressively type-II censored order statistics from the doubly truncated lindley Weibull distribution.

4.1 Recurrence Relations for the Single Moments based on Truncated Lindley Weibull Distribution

Relation 1. For $2 \le m \le n-1$ and $\beta,\theta > 0$

$$\begin{aligned}
\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)} = \frac{-(\theta+1)(\beta\theta)}{[1+\frac{\theta(\theta+1)}{k+\beta}\beta(1+R_1)]}[(\frac{1-P}{P-Q})(\frac{n-R_1-1}{k+\beta}\mu_{1:m-1:n-1}^{(R_1+R_2,\dots,R_m)(k+\beta)} - n\frac{Q_1^{k+\beta}}{k+\beta} \\
+\frac{R_1}{k+\beta}\mu_{1:m:n-1}^{(R_1-1,R_2,\dots,R_m)(k+\beta)} + \alpha^{\beta}\frac{n-R_1-1}{k+2\beta}\mu_{1:m-1:n-1}^{(R_1+R_2,\dots,R_m)(k+2\beta)} - \alpha^{\beta}\frac{nQ_1^{k+2\beta}}{k+2\beta} \\
+\alpha^{\beta}\frac{R_1}{k+2\beta}\mu_{1:m:n-1}^{(R_1-1,R_2,\dots,R_m)(k+2\beta)}) + \frac{n-R_1-1}{k+\beta}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)(k+\beta)} \\
-\frac{nQ_1^{k+\beta}}{k+\beta} + \alpha^{\beta}\frac{n-R_1-1}{k+2\beta}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)(k+2\beta)} - \alpha^{\beta}\frac{nQ_1^{k+2\beta}}{k+2\beta} \\
+\alpha^{\beta}\frac{(1+R_1)}{k+2\beta}\mu_{1:m:n}^{(R_1,R_2,R_3,\dots,R_m)(k+2\beta)}]
\end{aligned} \tag{33}$$

Relation 2. For $2 \le i \le m-1$ and $\beta,\theta > 0$

$$\begin{aligned}
\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)} = \frac{-(\theta+1)(\beta\theta)}{[1+\frac{\theta(\theta+1)}{k+\beta}\beta(1+R_i)]}\{(\frac{1-P}{P-Q})(\frac{n-R_1-\dots-R_i-i}{k+\beta}\mu_{i:m-1:n-1}^{(R_1,\dots,R_i+R_{i+1},\dots,R_m)(k+\beta)} \\
-\frac{n-R_1-\dots-R_{i-1}-i+1}{k+\beta}\mu_{i-1:m-1:n-1}^{(R_1,\dots,R_{i-1}+R_i,R_{i+1},\dots,R_m)(k+\beta)} + \frac{R_i}{k+\beta}\mu_{i:m:n-1}^{(R_1,\dots,R_{i-1},\dots,R_m)(k+\beta)} \\
+\alpha^{\beta}\frac{n-R_1-\dots-R_i-i}{k+2\beta}\mu_{i:m-1:n-1}^{(R_1,\dots,R_i+R_{i+1},\dots,R_m)(k+2\beta)} \\
-\alpha^{\beta}\frac{n-R_1-\dots-R_{i-1}-i+1}{k+2\beta}\mu_{i-1:m-1:n-1}^{(R_1,\dots,R_{i-1}+R_i,R_{i+1},\dots,R_m)(k+2\beta)} \\
+\alpha^{\beta}\frac{R_i}{k+2\beta}\mu_{i:m:n-1}^{(R_1,\dots R_{i-1},\dots,R_m)(k+2\beta)}) + \frac{n-R_1-\dots-R_i-i}{k+\beta}\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+\beta)} \\
-\frac{n-R_1-\dots-R_{i-1}-i+1}{k+\beta}\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,R_{i+1},\dots,R_m)(k+\beta)} \\
+\alpha^{\beta}\frac{n-R_1-\dots-R_i-i}{k+2\beta}\mu_{i:m-1:n}^{(R_1,\dots,R_i+R_{i+1}+1,\dots,R_m)(k+2\beta)} \\
-\alpha^{\beta}\frac{n-R_1-\dots-R_{i-1}-i+1}{k+2\beta}\mu_{i-1:m-1:n}^{(R_1,\dots,R_{i-1}+R_i+1,R_{i+1}\dots,R_m)(k+2\beta)} + \alpha^{\beta}\frac{1+R_i}{k+2\beta}\mu_{i:m:n}^{(R_1,\dots,R_m)(k+2\beta)}\}
\end{aligned} \tag{34}$$

Relation 3. For $2 \le m \le n-1$ and $\beta, \theta > 0$

$$\begin{aligned}
\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)(k+\beta)} = & \frac{-(\theta+1)(\beta\theta)}{[1+\frac{\theta(\theta+1)}{k+\beta}\beta(1+R_m)]}\{-(\frac{1-P}{P-Q})[\frac{n-R_1-\ldots-R_{m-1}-m+1}{k+\beta}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1}+R_m)(k+\beta)} \\
& -\frac{R_m}{k+\beta}\mu_{m:m:n-1}^{(R_1,\ldots,R_m-1)(k+\beta)} + \alpha^{\beta}\frac{n-R_1-\ldots-R_{m-1}-m+1}{k+2\beta}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1}+R_m)(k+\beta)} \\
& -\alpha^{\beta}\frac{R_m}{k+2\beta}\mu_{m:m:n-1}^{(R_1,\ldots,R_m-1)(k+2\beta)}] - \frac{n-R_1-\ldots-R_{m-1}-m+1}{k+\beta}\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)(k+\beta)} \\
& -\alpha^{\beta}\frac{n-R_1-\ldots-R_{m-1}-m+1}{k+2\beta}\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)(k+2\beta)} \\
& +\alpha^{\beta}\frac{1+R_m}{k+2\beta}\mu_{m:m:n}^{(R_1,\ldots,R_m)(k+2\beta)}\}.
\end{aligned} \tag{35}$$

4.2 Recurrence Relations for the Product Moments based on Truncated Lindley Weibull Distribution

Relation 1. For $1 \le i < j \le m-1, m \le n$ and $\beta, \theta > 0$

$$\begin{aligned}
\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(\beta)} = & \frac{-(\theta+1)(\beta\theta)}{[1+\theta(\theta+1)(1+R_j)]}\{(\frac{1-P}{P-Q})[\frac{n-R_1-\ldots-R_j-j}{\beta}\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_j+R_{j+1},\ldots,R_m)(\beta)} \\
& -\frac{n-R_1-\ldots-R_{j-1}-j+1}{\beta}\mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-1}+R_j,R_{j+1},\ldots,R_m)(\beta)} + \frac{R_j}{\beta}\mu_{i,j:m:n-1}^{(R_1,\ldots,R_{j-1},\ldots,R_m)(\beta)} \\
& +\alpha^{\beta}\frac{n-R_1-\ldots-R_j-j}{2\beta}\mu_{i,j:m-1:n-1}^{(R_1,\ldots,R_j+R_{j+1},\ldots,R_m)(2\beta)} \\
& -\alpha^{\beta}\frac{n-R_1-\ldots-R_{j-1}-j+1}{2\beta}\mu_{i,j-1:m-1:n-1}^{(R_1,\ldots,R_{j-1}+R_j,R_{j+1},\ldots,R_m)(2\beta)} \\
& +\alpha^{\beta}\frac{R_j}{2\beta}\mu_{i,j:m:n-1}^{(R_1,\ldots R_{j-1},\ldots,R_m)(2\beta)}] + \frac{n-R_1-\ldots-R_j-j}{\beta}\mu_{i,j:m-1:n}^{(R_1,\ldots,R_j+R_{j+1}+1,\ldots,R_m)(\beta)} \\
& -\frac{n-R_1-\ldots-R_{j-1}-j+1}{\beta}\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)(\beta)} \\
& +\alpha^{\beta}\frac{n-R_1-\ldots-R_j-j}{2\beta}\mu_{i,j:m-1:n}^{(R_1,\ldots,R_j+R_{j+1}+1,\ldots,R_m)(2\beta)} \\
& -\alpha^{\beta}\frac{n-R_1-\ldots-R_{j-1}-j+1}{2\beta}\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-1}+R_j+1,R_{j+1}\ldots,R_m)(2\beta)} + \alpha^{\beta}\frac{1+R_j}{2\beta}\mu_{i,j:m:n}^{(R_1,\ldots,R_m)(2\beta)}\}
\end{aligned} \tag{36}$$

Relation 2. For $1 \leq i \leq m-1, m \leq n$ and $\beta, \theta > 0$

$$\mu_{i,m:m:n}^{(R_1,R_2,\ldots,R_m)(\beta)} = \frac{-(\theta+1)(\beta\theta)}{[1+\theta(\theta+1)(1+R_m)]}\{-(\frac{1-P}{P-Q})[\frac{n-R_1-\ldots-R_{m-1}-m+1}{\beta}\mu_{i,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1}+R_m)(\beta)}$$

$$-\frac{R_m}{\beta}\mu_{i,m:m:n-1}^{(R_1,\ldots,R_m-1)(\beta)} + \alpha^{\beta}\frac{n-R_1-\ldots-R_{m-1}-m+1}{2\beta}\mu_{i,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-1}+R_m)(2\beta)}$$

$$-\alpha^{\beta}\frac{R_m}{2\beta}\mu_{i,m:m:n-1}^{(R_1,\ldots,R_m-1)(2\beta)}] - \frac{n-R_1-\ldots-R_{m-1}-m+1}{\beta}\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)(\beta)} \quad (37)$$

$$-\alpha^{\beta}\frac{n-R_1-\ldots-R_{m-1}-m+1}{2\beta}\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)(2\beta)}$$

$$+\alpha^{\beta}\frac{1+R_m}{2\beta}\mu_{i,m:m:n}^{(R_1,\ldots,R_m)(2\beta)}\}.$$

Remark 1. Setting P= 1 and Q= 0 in Relations (33),(34),(35),(36),(37),we obtain theorems (9),(13),(16),(20),(23).

Remark 2. Setting $R_1 = R_2 = \ldots. = R_m = 0$, so that m = n in which the case of the progressively Type-II censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, the relations established for the LWD reduce to the corresponding recurrence relations based on the usual order statistics.

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