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Energy and Wiener Index of Unit Graphs

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Abstract: In this paper, we are giving MATLAB program to find the energy and Wiener index of unit graphs. Our program demonstrates an intrinsic relationship between the elements of ring and structural properties of graphs. The unit graph $G(\mathbb{Z}_n)$ turns out to be strongly regular when $n = 2^k$. Several new directions for further research are also indicated by means of raising problems.

Keywords: commutative ring, energy, unit, wiener index.

1 Introduction

For standard terminology and notation in abstract algebra and graph theory, not specifically mentioned or defined in this paper, we refer the reader to the standard text-books [9] and [8], respectively. However, unless mentioned otherwise, we shall consider the ring $R = \mathbb{Z}_n$ the ring of integers modulo n.

Given an abstract algebraic ring R, an element $a \in R$ is called a *unit* of R if there exists a nonzero element $b \in R$ such that $a \cdot b := ab = e = ba$; where 'e' is the multiplicative identity of ring R or in other words, any element in R is known as *unit* if it has multiplicative inverse. Clearly, in any ring R the element '1' is a unit; it is specifically called the *trivial unit* of R. Thus, the set $U(R)$ of units of ring R is nonempty.

Proposition 1.1. [6] The total number of units in the ring $\mathbb{Z}_n$ are $\phi(n)$, where ϕ is Euler's totient function.

Proposition 1.2. [6] The set of units of ring $\mathbb{Z}_n$, denoted by $U(n)$ forms a group under multiplication modulo n, which is known as the group of units.

2 Units and Unit Graphs

In this section our aim is to collect different results, regarding units with the help of MATLAB program. The units of the ring $\mathbb{Z}_n$ using MATLAB 2009 can be obtained as follows:

Program to obtain the units of ring $\mathbb{Z}_n$:

```
function unit =Calculateunit (p)
n=p;
unit=[];
for i=0:n-1
      for j=0:n-1
     if gcd(i,p)==1
         unit=[unit,i];
          break;
      end
   end
end
```

The above program calculates the numbers which are relatively prime to n by determining the greatest common divisor (gcd) of each element of the ring $\mathbb{Z}_n$ with n and store them in a row matrix named as units of $\mathbb{Z}_n$.

It is easy to verify that the above program enumerates all the units of ring $\mathbb{Z}_n, 1 < n < \infty$. See for instance, the units of $\mathbb{Z}_4$ are '1' and '3'.

Remark 2.1. The sum of units need not be a unit. For example, units of the ring $\mathbb{Z}_6$ are '1' and '5', whence the sum $1+5 = 0(\text{mod } 6)$ which is not a unit in $\mathbb{Z}_6$.

Lemma 2.2. The sum of all the units of ring $\mathbb{Z}_n$, $n > 2$ is always divisible by n.

Proof. We shall prove this lemma with the help of the above program, if we add only two codes

```
Sum = sum(unit);
Remainder = rem(Sum, n);
```

Clearly, remainder is always zero. Therefore, the sum of

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all the units of $\mathbb{Z}_n$ is always divisible by n. $\square$

Indeed, the above program will be useful to draw the unit graphs [7] as well as useful in some related results.

The concept of *unit graph* denoted by $G(\mathbb{Z}_n)$ was introduced by Grimaldi [7], based on the elements and units of $\mathbb{Z}_n$. The vertex set of $G(\mathbb{Z}_n)$ is the elements of ring $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x+y$ is an unit of $\mathbb{Z}_n$. This investigation was then continued by Ashrafi et al. [1], his aim was to generalize unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for arbitrary ring R. Afterwards, a relatively number of papers being published in the literature for more details (see [1], [7], [10]).

Program to draw the unit graphs $G(\mathbb{Z}_n)$:

```
function gra =DrawGra(n,Z,p)
for i=0:n-1
          axes(i+1,:)=[cos(2*pi*i/n),sin(2*pi*i/n)];
end
Gz=zeros(n);
hold on
for i=1:n
    plot(axes(i,1),axes(i,2),'*')
        for z = 1:size(Z,2)
          if mod(i+j,p)== Z(:,z)
          Gz(i,i)=1;
         plot(axes(i,1),axes(i,2),'rO')
       end
     end
end
for i = 1:n-1
          for j = i+1:n
            for z = 1:size(Z,2)
            if mod(i+j,p)== Z(:,z)
        Gz(i,j)=1;Gz(j,i)=1;
        plot(axes([i,j],1),axes([i,j],2));
          end
       end
     end
end
hold off
```

To draw the unit graphs the first step in the program is to plot the vertices (elements of $\mathbb{Z}_n$) and then it joins only those vertices for which $\mathrm{mod}(i+j,n) \in$ unit.

Few examples of the unit graph are shown in Figures 1 and 2, which are obtained by above program.

Remark 2.3. If we replace $\mathrm{mod}(i+j,p)$ with $\mathrm{mod}(i-j,p)$ everywhere in the above unit graph program, then we get a program which draws the unitary Cayley graph $X_n = Cay(\mathbb{Z}_n,U_n)$ [3].

Remark 2.4. The unitary addition Cayley graph [14] is a particular case of unit graph $G(R)$ when the ring R is just $\mathbb{Z}_n$, i.e., $Cay^+(\mathbb{Z}_n,U_n) \cong G(\mathbb{Z}_n)$, therefore the above program also works for unitary addition Cayley graph $G_n = Cay^+(\mathbb{Z}_n,U_n)$.

Towards attempting the problem to find energy and Wiener index program it would be necessary to have a

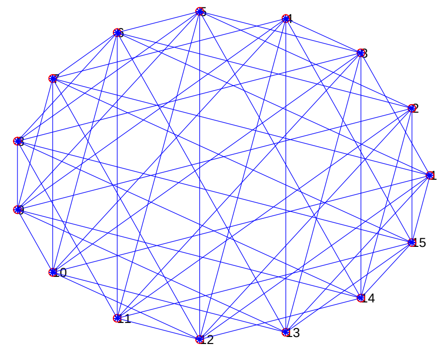

Fig. 1: $G(\mathbb{Z}_{15})$

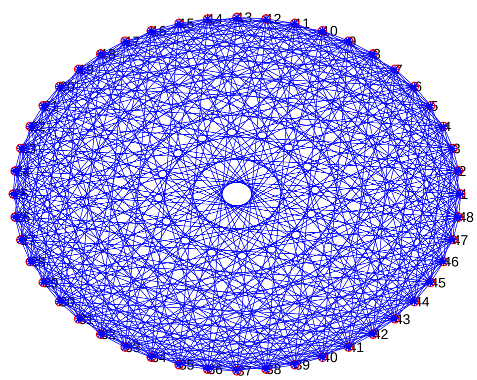

Fig. 2: $G(\mathbb{Z}_{48})$

program for the adjacency matrix, associated with unit graphs. The following is a well known definition of the adjacency matrix.

Let $G=(V,E)$ be graph with n vertices and m edges. Then *adjacency matrix* is a $n\times n$ matrix defined as $A=[a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix for the unit graphs can be obtained using the program given below.

Program to find the adjacency matrix of unit graph $G(\mathbb{Z}_n)$:

```
    function adjm = CalculateAd(Z,p,n)
for i = 0:n-1
   for j = 0:n-1
      for z = 1:size(Z,2)
         if mod(i+j,p)== Z(:,z)
             adjm(i+1,j+1)=1;
           adjm(i+1,i+1)=0;
          end
```

```
    end
  end
end
```

A graph G is called $k-$*regular* if degree of each vertex is k. A regular graph is said to be *strongly regular graph* with parameters (n,k,λ,μ) if it is $k-$regular graph on n vertices satisfying the following properties :

i) any two adjacent vertices have exactly λ common neighbors;
ii) any two non-adjacent vertices ave exactly μ common neighbors.

Definition 2.5. The spectrum of a graph G is the set of numbers that are eigenvalues of G, together with their multiplicities. We denote the *spectrum* of a graph G by Spec(G). Suppose $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$ be distinct eigenvalues of a graph G with their respective multiplicities $m_1, m_2, m_3, \ldots, m_n$. Then the spectrum of the graph G is written as

$$\mathrm{Spec(G)} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

Theorem 2.6. The unit graph $G(\mathbb{Z}_n)$ is strongly regular, when $n = 2^k,\ k > 1$.

Proof. The unit graph $G(\mathbb{Z}_{2^k})$ is a 2^{k-1}-regular graph and the spectrum of unit graph $G(\mathbb{Z}_{2^k}),\ k > 1$ is given by

$$\mathrm{Spec}G(\mathbb{Z}_{2^k}) = \begin{pmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 1 & 2^k - 2 & 1 \end{pmatrix}.$$

Now it is easy to observe that unit graph $G(\mathbb{Z}_{2^k})$ has exactly three distinct eigenvalues, namely, $-2^{k-1}, 2^{k-1}$, and 0. The result follows due to the fact that, if G is a connected regular graph with exactly three distinct eigenvalues, then G is strongly regular (Lemma 2, [16]). $\square$

3 Wiener Index and Energy of $G(\mathbb{Z}_n)$

In this section, we give a program to determine the energy and Wiener index. Since we already have a program to find the adjacency matrix of unit graph $G(\mathbb{Z}_n)$, so we are ready to discuss about Wiener index and energy of the unit graphs.

The *Wiener index* of a graph was the first reported topological index based on the graph distance [15]. This index is defined as the sum of all the distance between every pair of vertices of a graph, for more information (see [4]). If we denote the length of shortest path between x and y by $d(x,y)$ where $x,y \in V(G(\mathbb{Z}_n))$, then the *Wiener index* is defined as

$$W(G)) = \sum_{x,y \in V(G)} d(x,y).$$

In mathematical chemistry, there is a numerous mathematical and chemical literature on the Wiener index, see [11], [12], [13], [15].

The following program calculates the Wiener index of unit graphs $G(\mathbb{Z}_n)$.

Program to obtain the Wiener index of unit graphs $G(\mathbb{Z}_n)$:

```
function Wiener=CalculateWiener(adjm)
adjm(adjm==0)=inf;
A=triu(adjm,1)+tril(adjm,-1);
m=length(A);
adjm=zeros(m);
j=1;
while j¡=m
  for i=1:m
    for k=1:m
        adjm(i,k)=min(A(i,k),A(i,j)+A(j,k));
      end
    end
  A=adjm;
 j=j+1;
end
Wiener=sum(sum(A))/2;
```

With the help of above program we found that Wiener index of the unit graphs $G(\mathbb{Z}_2), G(\mathbb{Z}_3), G(\mathbb{Z}_5), G(\mathbb{Z}_{15})$, and $G(\mathbb{Z}_{48})$ are $1, 4, 12, 52$, and 2064 respectively.

Proposition 3.1. The Wiener index of unit graph $G(\mathbb{Z}_p)$ is always a multiple of $p-1$, where p is an odd prime.

Now, we give the program for the energy of unit graphs. The concept of graph energy introduced in [5] as follows:

The *energy* $E = E(G)$ of a graph G of order n is defined as the sum of the absolute values of the eigenvalues of G; that is,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where λ_i's are eigenvalues of graph G.

Recently there has been a tremendous research activity in the areas like hyperenergetic graphs, maximum energy graphs, equienergetic graphs. The graph G of order n whose energy satisfies $E(G) > 2(n-1)$ is called *hyperenergetic* and graph with energy $E(G) \leq 2(n-1)$ is called *non-hyperenergetic* for more details (see [5]).

Program to determine the energy of unit graphs $G(\mathbb{Z}_n)$:

function energy= CalculateEnergy(adjm)
$R = rank(adjm);$
$[eigValue] = eig(adjm);$
$A = abs(eigValue);$
$energy = sum(A);$

Theorem 3.2. For the unit graph $G(\mathbb{Z}_n)$ the energy $E(G(\mathbb{Z}_n)) = n$, when $n = 2^k,\ k > 1$.

Proof. In the light of Theorem 2.6, the spectrum of unit graph $G(\mathbb{Z}_{2^k}),\ k > 1$ is given by

$$\mathrm{Spec}G(\mathbb{Z}_{2^k}) = \begin{pmatrix} -2^{k-1} & 0 & 2^{k-1} \\ 1 & 2^k - 2 & 1 \end{pmatrix}.$$

The energy of $G(\mathbb{Z}_{2^k}),\ k \neq 1$ can be determined easily by spectrum,

$$|-2^{k-1}| + |2^{k-1}| = 2(2^{k-1}) = 2^k.$$

Consequently, $E(G(\mathbb{Z}_n)) = n$. $\Box$

Lemma 3.3. The energy of unit graph $G(\mathbb{Z}_p)$ is never an integer, where p is either prime or square of a prime, $p > 3$.

Proof. It is a straightforward observation that unit graphs $G(\mathbb{Z}_p),\ p > 3$ are connected. Also, for any real symmetric matrix A, rank is the total multiplicity of nonzero eigenvalues. The rank of the adjacency matrix of unit graph $G(\mathbb{Z}_p),\ p > 3$ is $n - \frac{p-1}{2}$, as multiplicity of 0 is always $\frac{p-1}{2}$ and remaining others eigenvalues are -2, $-(1+a)$ and $(p-2+a)$ having multiplicity $\frac{p-3}{2}$, 1 and 1 respectively, where a is different for each p and $0 < a < \frac{1}{2}$. Thus, the spectrum

$$\mathrm{Spec}G(\mathbb{Z}_p) = \begin{pmatrix} p-2+a & 0 & -(1+a) & -2 \\ 1 & \frac{p-1}{2} & 1 & \frac{p-3}{2} \end{pmatrix}.$$

In view of definition of energy and spectrum, it is easy to verify that energy of $G(\mathbb{Z}_p),\ p > 3$ is equal to

$$|p-2+a| + |-(1+a)| + |-2|\frac{p-3}{2} = 2(p-2+a).$$

Clearly, which is not an integer as $0 < a < \frac{1}{2}$.
Similarly, the spectrum of unit graphs $G(\mathbb{Z}_{p^2})$ is given as

$$\begin{pmatrix} (p-1)(p-1+a) & 0 & -(p+a) & -1 \\ 1 & p-1 & 1 & (p-1)^2 \end{pmatrix},$$

whence the energy of unit graphs $G(\mathbb{Z}_{p^2})$ is equal to $|(p-1)(p-1+a)| + |-(p+a)| + |-1|(p-1)^2$, which is again not an integer. Hence the result. $\Box$

Theorem 3.4. For an odd prime p, the energy of unit graph $G(\mathbb{Z}_{2p})$ is $E(G(\mathbb{Z}_{2p})) = 2^2\phi(p)$.

Proof. Since the unit graph $G(\mathbb{Z}_{2p})$ is $(p-1)$-regular, $(p-1)$ must be eigenvalue of $G(\mathbb{Z}_{2p})$ (cf: [2]). Moreover, the unit graph $G(\mathbb{Z}_{2p})$ is connected as well, so the multiplicity of $(p-1)$ is 1 (cf: [2]). On the other hand the unit graphs $G(\mathbb{Z}_{2p})$ is bipartite, therefore, $-(p-1)$ is also an eigenvalue with same multiplicity, i.e., '1'. Now it is easy to compute the spectrum of unit graph $G(\mathbb{Z}_{2p})$ which is as follows:

$$\mathrm{Spec}G(\mathbb{Z}_{2p}) = \begin{pmatrix} -(p-1) & -1 & 1 & (p-1) \\ 1 & (p-1) & (p-1) & 1 \end{pmatrix}.$$

It follows that energy of $G(\mathbb{Z}_{2p})$ is equal to

$$|-(p-1)| + (p-1)|-1| + (p-1) + (p-1) = 4(p-1)$$
$$= 2^2(p-1).$$

Consequently, $E(G(\mathbb{Z}_{2p})) = 2^2\phi(p)$. $\Box$

Theorem 3.5. The energy of unit graph $G(\mathbb{Z}_n)$ is never an integer, where $n = p^k,\ k > 1$ and p is an odd prime.

Proof. The proof of this theorem is given by the arguments analogues to those used in the Lemma 3.3. $\Box$

4 Conclusion and scope

In this paper a MATLAB program for the unit graph, energy and Wiener index has given. An important outcome of this paper is that unit graph $G(\mathbb{Z}_n),\ n = 2^k,\ k > 1$ provides a class of strongly regular graphs. Further, there is a scope to get the program for unit graph associated with arbitrary ring R, for ring of Gaussian integer $\mathbb{Z}_n[i]$, and also one can generalize the results to the direct product of finite commutative rings $R = \mathbb{Z}_m \times \mathbb{Z}_n$.

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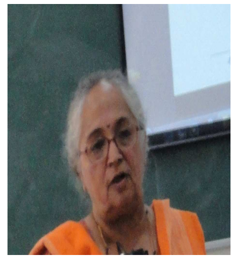

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