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Podolsky's Generalized Fractional Order Model

Y. M. Alawaideh[1], B. M. Alkhamiseh[1], M. Slewa[2,*] and W. Al-Awaida[3]

[1]Department of Physics, The University of Jordan, Amman, Jordan
[2]Mechanical Engineering Department, Embry-Riddle Aeronautical University, Prescott, AZ, United Statas
[3]Department of Biology and Biotechnology, American University of Madaba, Madaba, Jordan

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Abstract: We study the electromagnetic field in this work because we are particularly interested in the gauge sector of Podolsky's generalized electrodynamics, where higher-order derivatives form. To represent Podolsky's generalized electrodynamics, canonical quantizations and a lower-order derivatives model are applied. We demonstrate that Podolsky's model is equivalent to one with reduced-order derivatives. The differential equations for both models should then be compared. After obtaining the Hamiltonian formulation, we applied this new formula to the Podolsky Generalized equation. This method is used to construct a combined Riemann-Liouville fractional derivative operator as well as a fractional variational theory. The fraction variational notion is utilized to build fractional Euler equations and fraction Hamilton equations. The Hamilton equations of motion are compatible with the Euler-Lagrange equations.

Keywords: Euler Lagrange Equations, Fractional Derivatives, Hamiltonian Formulation, Lagrangian Formulation, Reduced Order Model.

1 Introduction

Dirac's idea of restricted systems is significant in theoretical physics, and it is frequently utilized in modern basic particle physics [1,2,3,4]. By adding the constraints imposed by the Hessian matrix singularity to the conventional Hamiltonian and enabling the system to lose certain degrees of freedom, this approach determines the consistency requirements. Because first-class constraints result in gauge transformations that lead to gauge freedom, it's necessary to provide gauge fixing conditions for each one, which isn't always easy. Higher-ordered systems Lagrangians have piqued researchers' interest since they arise in a wide range of physical issues. Higher order regularization of quantum gauge field theories and so-called rigid strings [5, 6], rigid particles [7, 8], a relativistic particle with curvature and torsion in three-dimensional space-time [9], and the work of Podolsky [10] and Bopp [11], who independently proposed generalization of electrodynamics containing second order derivatives, are perhaps the most well-known examples.

Green [12,13] suggested an gauge invariant supersymmetric theories [14, 15] and the effective Lagrangian in gauge theories [16] are two more examples. Ostrogradski [17] was the first to tackle higher-order derivative theories, which led to the Euler-Lagrange equations and Hamilton equations of motion. [18] discusses how to quantify route integrals in higher-order derivative systems. Major Hamilton-Jacobi formalism applications, particularly for higher order derivatives, have been found [19, 20] .The action function is obtained by solving a set of Hamilton-Jacobi partial differential equations for both restricted and unconstrained systems, and it is then utilized to solve the equations of motion using the WKB approximation, as detailed in[21, 22]. Some authors investigated the Hamiltonian formulation of higher order dynamical systems utilizing Dirac's approach to restricted dynamics, where Hamiltonian formulation of regular higher order Lagrangians is created and Ostrogradski's traditional description of such systems is recovered [23, 24]. In reference [25] , a breakthrough was made for systems with higher order derivatives and degenerate coordinates.

A unique development of systems with higher order fractional derivatives has been reported in reference [26,27], which recovers the route integral quantization for both conservative and non-conservative systems. The discrete variational approach is used to derive discrete Euler-Lagrange equations for higher-order Lagrangians and the associated discrete Hamiltonian in reference[28].The essential features of the fractional derivative, on the other

*Corresponding author e-mail: Slewam@erau.edu

hand, are as follows [29,30,31].

- It imitates all of the ideas and features of an ordinary derivative, such as the quotient, product, and chain rules, that aren't satisfied by other common definitions.
- It has lately piqued the interest of a lot of academics, and several applications have been rebuilt using this definition.
- It solves ordinary differential equations and systems quickly and correctly.
- It's a technique for combining the Lagrange and Hamiltonian equations into system ordinary differential equations modeling.
- It enables the creation of new comparisons and applications.

As a consequence, we recreate the Podolsky field in this work using the fractional derivative, which the authors utilized in equation modeling, and then apply the Generalized fractional derivative to generate Hamiltonian equations for this system. In actuality, solving the new fractional derivatives equations is more difficult. As a result, we've proposed a new generalized solution approach for these equations. These formulae have been generalized in continuous systems to operate with second-order derivatives. This method is used to address the Podolsky problem in generalized electrodynamics.

The main goal of this paper is to introduce an auxiliary massive vector field B_{μ}, lower the order of the derivatives in Podolsky's generalized electrodynamics, and investigate quantization of the resulting model. The dynamical development is shown to occur in a phase space with two first-class and two second-class constraints, for a total of five degrees of freedom, which is the same as in a typical higher order model.

The remainder of the article is arranged as follows: The definitions of fractional derivatives are briefly discussed in Section 2. The fractional structure of the Euler-Lagrangian equation is discussed in Section 3. The equations of motion in terms of fractional Hamiltonian density are discussed in Section 4. One illustrative example is examined in Section 5. The Euler-Lagrange equations are then used to obtain fractional Podolsky equations in section 6. In Section 7, we show that Podolsky's model is equivalent to a reduced order derivatives one. The article ends with some final conclusions (Section8).

2 Definitions and properties of Fractional derivative of Riemann–Liouville

Some of the basic concepts used in this work are briefly presented in this section of the study. The fractional derivatives of the Riemann-Liouville left and right are defined as follows:

The fractional derivative left of Riemann-Liouville:

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau. \tag{1}$$

The right Riemann- Liouville fractional derivative

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_a^x (\tau-x)^{n-\alpha+1} f(\tau)d\tau. \tag{2}$$

where Γ denotes the Gamma function, and α is the order of the derivative such that $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n f(x) \tag{3}$$

$$ {}_aD_t^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n f(t) \qquad \alpha = 1,2,. \tag{4}$$

3 The Euler-Lagrangian Formula in fraction form with Podolsky's Generalized Equation.

In the Lagrangian, the dynamics of a physical system are encoded as a function of the positions and velocity of all the degrees of freedom comprising the system. One considers paths in the configuration space to extract the dynamics. The position and velocity at each time and also the value of the Lagrangian are determined for a given direction. The continuous structure of Lagrangian density denoted by the dynamic field variables, generalized coordinate $\boldsymbol{\psi_\rho}$ and its second-order derivatives, generalized velocities ${}_{x_\mu}D_b^{\beta}\, {}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)$ defined as

$$\mathcal{L} = \mathcal{L}\begin{bmatrix} \boldsymbol{\psi_\rho},\ {}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t),\ {}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t), \\ {}_aD_{x_\mu}^{\alpha}\, {}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t),\ {}_{x_\mu}D_b^{\beta}\, {}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} \tag{5}$$

For this Lagrangian density, the Euler-Lagrange equation can be given as a fractional form.

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}} + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} \\ + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^{\alpha}\, {}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^{\beta}\, {}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \tag{6}$$

Using the variational principle, we can write:

$$\delta S = \int \delta\mathcal{L}\ d^4x = 0 \tag{7}$$

Using Eq. (5), the variation of $\mathcal{L}$ is:

$$\delta\mathcal{L} = \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}}\delta\boldsymbol{\psi_\rho} \\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t)}\delta\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)}\delta_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t)}\delta\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t) + \\ \frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)}\delta_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} d^3x \quad (8)$$

Substituting Eq. (6) into Eq. (7), and using the following commutation relation

$$\begin{bmatrix} \delta\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t) = {}_aD^{\alpha}_{x_\mu}\delta\boldsymbol{\psi}_\rho(x,t) \\ \delta_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) = {}_{x_\mu}D^{\beta}_{b}\delta\boldsymbol{\psi}_\rho(x,t) \end{bmatrix} \quad (9)$$

$$\begin{bmatrix} \delta\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t) = {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\delta\boldsymbol{\psi}_\rho(x,t) \\ \delta_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) = {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\delta\boldsymbol{\psi}_\rho(x,t) \end{bmatrix} \quad (10)$$

we get,

$$\int \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}}\delta\boldsymbol{\psi_\rho} + \frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t)}\delta\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t) \\ +\frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)}\delta_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) \\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t)}\delta\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t) + \\ \frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\ \partial_\mu\boldsymbol{\psi}_\rho(x,t)}\delta_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} d^4x = 0 \quad (11)$$

Integrating by parts the second and the third and the fourth and the fifth terms in Eq. (11) , this lead to Euler – Lagrange equations.

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}} - {}_aD^{\alpha}_{x_\mu}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t)} - \\ {}_{x_\mu}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} \\ + {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t)} + \\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \quad (12)$$

As a special case, taking σ,μ= 0,i. the Euler – Lagrange reduce to the original relations like:

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}} - {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\boldsymbol{\psi_\rho}(x,t)} \\ - {}_tD^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_tD^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} - {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_i}\boldsymbol{\psi_\rho}(x,t)} - \\ {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_i}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} \\ + {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\boldsymbol{\psi_\rho}(x,t)} + \\ {}_tD^{\beta}_{b}\ {}_tD^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_tD^{\beta}_{b}\ {}_tD^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} \\ + {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{x_i}\boldsymbol{\psi_\rho}(x,t)} + \\ {}_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \quad (13)$$

Other motion equations are derived from the variables of the other fields ($\boldsymbol{\phi}$ and $\boldsymbol{A_i}$), using the Euler-Lagrange theorem, as follows:

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{\phi}} - {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\boldsymbol{\phi}} - {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}\boldsymbol{\phi}} \\ + {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\boldsymbol{\phi}} + {}_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\boldsymbol{\phi}} \end{bmatrix} = 0 \quad (14)$$

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{A_i}} - {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\boldsymbol{A_i}} - {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_i}\boldsymbol{A_i}} \\ + {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\boldsymbol{A_i}} + {}_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\boldsymbol{A_i}} \end{bmatrix} = 0 \quad (15)$$

4 New Derivatives with Fractional Hamiltonian Formulation Using Podolsky's Generalized Equation

The Lagrangian of the classical field containing fractional partial derivatives, is in the form

$$L = \mathcal{L}\begin{bmatrix} \boldsymbol{\psi_p},\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{\psi_\rho}(x,t), {}_{x_\mu}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) \\ ,\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{\psi_\rho}(x,t), \\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} \quad (16)$$

Eq. (16) can be written as follows: $(\mu,\sigma = 0,i)$, we get:

$$L = \mathcal{L}\begin{bmatrix} \boldsymbol{\psi_\rho},\ {}_aD^{\alpha}_{t}\boldsymbol{\psi_\rho},\ {}_aD^{\alpha}_{x_i}\boldsymbol{\psi_\rho},\ {}_aD^{2\alpha}_{t}\boldsymbol{\psi_\rho}, \\ {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\boldsymbol{\psi_\rho},\ {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{t}\boldsymbol{\psi_\rho},\ {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{x_i}\boldsymbol{\psi_\rho}, \end{bmatrix} \quad (17)$$

We introduce the generalized momenta as [17]:

$$\boldsymbol{\pi}_{\alpha}^{1}=\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}-{}_{a}D_{t}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}\right] \tag{18}$$

$$\boldsymbol{\pi}_{\alpha}^{2}=\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}, \tag{19}$$

In many cases, we take $\pi_{\beta}^{1}=0$ *and* $\pi_{\beta}^{2}=0$ because we define (in the Lagrangian density and the Hamiltonian density) the time derivative in the right side as ${}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}$ and ${}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}$ so that $\pi_{\alpha}^{1}=\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}-{}_{a}D_{t}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}\right]=0,$ *and* $\pi_{\alpha}^{2}=\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)}=0$, Therefore take $\pi_{\beta}^{1}=0$ *and* $\pi_{\beta}^{2}=0$ .Thus, the Hamiltonian reads as :

$$H=\pi_{\alpha}^{1}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)+\pi_{\alpha}^{2}\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}(x,t)-L\begin{bmatrix}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\end{bmatrix} \tag{20}$$

When we calculate the Hamiltonian's total differential, we get

$$dH=\begin{bmatrix}d\,\pi_{\alpha}^{1}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}+\ \boldsymbol{\pi}_{\alpha}^{1}d\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}+\\ d\pi_{\alpha}^{2}\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}+\boldsymbol{\pi}_{\alpha}^{2}d\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\psi_{\rho}\\ +d\boldsymbol{\pi}_{\alpha}^{3}\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\\ -\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\psi_{\rho}\\ -\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_r}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\end{bmatrix} \tag{21}$$

By substituting the conjugate momenta values, we get

$$dH=\begin{bmatrix}{}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\,d\,\boldsymbol{\pi}_{\alpha}^{1}+\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}d\boldsymbol{\pi}_{\alpha}^{2}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\boldsymbol{\psi}_{\boldsymbol{\rho}}\\ -\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_r}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\\ -\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\\ \frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}-\end{bmatrix} \tag{22}$$

This means that, the Hamiltonian is a function of the form

$$\mathcal{H}=\mathcal{H}\begin{bmatrix}\boldsymbol{\pi}_{\alpha}^{1},\boldsymbol{\pi}_{\alpha}^{2},\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_r}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}},\end{bmatrix} \tag{23}$$

The total differential of the Hamiltonian, therefore, takes the form of

$$d\mathcal{H}=\begin{bmatrix}\frac{\partial\mathcal{H}}{\partial\boldsymbol{\pi}_{\alpha}^{1}}d\boldsymbol{\pi}_{\alpha}^{1}+\frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{2}}d\boldsymbol{\pi}_{\alpha}^{2}+\frac{\partial\mathcal{H}}{\partial\psi_{\rho}}d\boldsymbol{\psi}_{\boldsymbol{\rho}}+\\ \frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}+\frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\\ +\frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}+\\ \frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\end{bmatrix} \tag{24}$$

We get the Hamilton equations of motion by comparing (22) and (24),

$$\begin{cases}\frac{\partial\mathcal{H}}{\partial\boldsymbol{\psi}_{\boldsymbol{\rho}}}=-\frac{\partial\mathcal{L}}{\partial\boldsymbol{\psi}_{\boldsymbol{\rho}}}\\ \frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{1}}=\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\\ \frac{\partial\mathcal{H}}{\partial\boldsymbol{\pi}_{\alpha}^{2}}=\ {}_{a}D_{t}^{2\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\end{cases} \tag{25}$$

The equation (25) is rewritable with the Euler Lagrange and this is the form of the equation.

$$\begin{cases}\frac{\partial\mathcal{H}}{\partial\boldsymbol{\psi}_{\boldsymbol{\rho}}}=-\pi_{\alpha}^{1}+\boldsymbol{\pi}_{\alpha}^{2}-\\ {}_{a}D_{x_i}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\left({}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\right)}\right]+\ {}_{a}D_{x_i}^{2\alpha}\left[\frac{\partial\mathcal{L}}{\partial\left({}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}\right)}\right]\\ \frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}=-\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{t}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}\\ \frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}=-\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{t}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}\\ \frac{\partial\mathcal{H}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}=-\frac{\partial\mathcal{L}}{\partial\ {}_{a}D_{x_i}^{\alpha}\ {}_{a}D_{x_i}^{\alpha}\boldsymbol{\psi}_{\boldsymbol{\rho}}}\end{cases} \tag{26}$$

We can re-write Eq(26) using the field variables $(\boldsymbol{A}_{\boldsymbol{0}},\boldsymbol{A}_{\boldsymbol{i}})$so that we get

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial \mathbf{A_0}} = - {}_aD_t^{\alpha}\pi_1^1 + {}_aD_t^{2\alpha}\boldsymbol{\pi_2^1} - {}_aD_{x_i}^{\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_0})}\right] + {}_aD_{x_i}^{2\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_0})}\right] \\ \frac{\partial \mathcal{H}}{\partial \mathbf{A_i}} = - {}_aD_t^{\alpha}\pi_1^2 + {}_aD_t^{2\alpha}\boldsymbol{\pi_2^2} - {}_aD_{x_i}^{\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_i})}\right] + {}_aD_{x_i}^{2\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_i})}\right] \\ \frac{\partial \mathcal{H}}{\partial \pi_1^1} = {}_aD_t^{\alpha}\boldsymbol{A_0} \\ \frac{\partial \mathcal{H}}{\partial \pi_1^2} = {}_aD_t^{\alpha}\boldsymbol{A_i} \\ \frac{\partial \mathcal{H}}{\partial \pi_2^1} = {}_aD_t^{2\alpha}\boldsymbol{A_0} \\ \frac{\partial \mathcal{H}}{\partial \boldsymbol{\pi_2^2}} = {}_aD_t^{2\alpha}\boldsymbol{A_i} \end{cases} \tag{27}$$

Expand $\pi_{\alpha}^1, \boldsymbol{\pi_{\alpha}^2}$, with the equation taking the form of $\left(\pi_1^1, \boldsymbol{\pi_1^2}, \pi_2^1, \boldsymbol{\pi_2^2}\right)$ respectively, we get:

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial \mathbf{A_0}} = - {}_aD_t^{\alpha}\pi_1^1 + {}_aD_t^{2\alpha}\boldsymbol{\pi_2^1} \\ - {}_aD_{x_i}^{\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_0})}\right] + {}_aD_{x_i}^{2\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_0})}\right] \\ \frac{\partial \mathcal{H}}{\partial \mathbf{A_i}} = - {}_aD_t^{\alpha}\pi_1^2 + {}_aD_t^{2\alpha}\boldsymbol{\pi_2^2} \\ - {}_aD_{x_i}^{\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_i})}\right] + {}_aD_{x_i}^{2\alpha}\left[\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\mathbf{A_i})}\right] \\ \frac{\partial \mathcal{H}}{\partial \pi_1^1} = {}_aD_t^{\alpha}\boldsymbol{A_0} \\ \frac{\partial \mathcal{H}}{\partial \pi_1^2} = {}_aD_t^{\alpha}\boldsymbol{A_i} \\ \frac{\partial \mathcal{H}}{\partial \pi_2^1} = {}_aD_t^{2\alpha}\boldsymbol{A_0} \\ \frac{\partial \mathcal{H}}{\partial \boldsymbol{\pi_2^2}} = {}_aD_t^{2\alpha}\boldsymbol{A_i} \end{cases} \tag{28}$$

The above equations describe the form of Hamiltonian equations, with regard to field variables$(\boldsymbol{A_0}, \boldsymbol{A_i})$ and fractional derivatives of Lagrangian. Equations (29) is the Hamiltonian equation using the fractionary derivatives of Riemann-Liouville. $\boldsymbol{\alpha} \rightarrow \mathbf{1}$ is taken, we obtain the classical integer case equations[32].

5 Illustrative Examples

Podolsky's Generalized Electrodynamics in fractional form

The most general form of Lagrangian density for a four-vector field is given by the so-called Podolsky Lagrangian density [33]

$$\mathcal{L}_{LW} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - a^2\partial_{\lambda}F^{\alpha\lambda}\partial^{\alpha}F_{\rho\alpha} \tag{29}$$

Where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{30}$$

and an is a variable-length argument. The Greek indices run from 0 to 3, and we use the Minkowski metric (+, −, −, −).This is a simple Lagrangian, with just the second-order derivative term proportional to the parameter serving as a roundabout. In reality, because Podolsky's term is entirely reliant on A_{μ} through the combination $F_{\mu\nu}$, the model exhibits the same gauge invariance as ordinary electrodynamics. That is, Eq.(29) remains constant during the transition.

$$\delta A_{\mu} = {}_aD_{x_{\nu}}^{\alpha}\Lambda \tag{31}$$

when it comes to a particular function (x). Cuzinatto and de Melo [33] demonstrated that if one requires invariance under Eq. (31) and allows for a second order derivatives Lagrangian with suitable assumptions, then Eq.(31) may be obtained by following Utiyama's program [34].

So, we can write the Lagrangian (29) as:

$$\mathcal{L} = -\frac{1}{2}F_{0i}F^{0i} - a^2[(\partial_i F^{0i})^2 - (\partial_0 F^{0i})^2] \tag{32}$$

In Riemann - Liouville fractional form we use these relations for rewriting the Podolsky Lagrangian density :

$$\begin{bmatrix} F_{\mu\nu} = {}_aD_{x_{\mu}}^{\alpha}A_{\nu} - {}_aD_{x_{\nu}}^{\alpha}A_{\mu} \\ F^{\mu\nu} = {}_aD_{x^{\mu}}^{\alpha}A^{\nu} - {}_aD_{x^{\nu}}^{\alpha}A^{\mu} \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} \partial_{\alpha} = {}_aD_{x_{\mu}}^{\alpha} = \left({}_aD_t^{\alpha}, {}_aD_{x_i}^{\alpha}\right) \\ \partial^{\alpha} = {}_aD_{x^{\mu}}^{\alpha} = \left({}_aD_t^{\alpha}, -{}_aD_{x^i}^{\alpha}\right) \end{bmatrix} \tag{34}$$

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_aD_{x_{\mu}}^{\alpha}A_{\nu}\,{}_aD_{x^{\mu}}^{\alpha}A^{\nu} - {}_aD_{x_{\mu}}^{\alpha}A_{\nu}\,{}_aD_{x^{\nu}}^{\alpha}A^{\mu}\right] \tag{35}$$

$$\begin{bmatrix} A^{\alpha} = (\phi, \vec{A}) \\ A_{\alpha} = (\phi, -\vec{A}) \end{bmatrix} \tag{36}$$

using definition of left Riemann – Liouville fractional derivative, the fractional electromagnetic lagrangian density formulation takes the form :

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} \left({}_aD_t^{\alpha}A_i\right)^2 - 2\,{}_aD_t^{\alpha}A_i\,{}_aD_{x_i}^{\alpha}\phi \\ +\left({}_aD_{x_i}^{\alpha}\phi\right)^2 \end{bmatrix} - a^2\begin{bmatrix} \left[{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i - {}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}\phi\right]^2 \\ -\left[{}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}A_i - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\phi\right]^2 \end{bmatrix} \tag{37}$$

Equations of motion, using Euler- Lagrange Eq.(14), by taking the derivative with respect to ϕ , we get:

$$\left[1 + 2a^2\left({}_aD_{x_i}^{\alpha}\right)^2\right]{}_aD_{x_i}^{\alpha}\left[{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i - {}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}\phi\right] = 0 \tag{38}$$

Now use the general formula (15) for the other field variable A^i. to get other motion equations.

$$\left[1 + 2a^2\left({}_aD_t^{\alpha}\right)^2\right]{}_aD_t^{\alpha}\left[{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i - {}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}\phi\right] = 0 \tag{39}$$

Adding equations (35) and (36) to get

$$\begin{bmatrix} 1+2a^2\left(\left({}_aD_t^{\alpha}\right)^2-\left({}_aD_{x_i}^{\alpha}\right)^2\right) \\ \left[{}_aD_t^{\alpha}+{}_aD_{x_i}^{\alpha}\right]\left[{}_aD_{x_i}^{\alpha}\phi-{}_aD_t^{\alpha}A_i\right]\end{bmatrix}=0 \qquad (40)$$

This is the second non-homogeneous fractional equation.
If α goes to 1,then Eq. (38) and (39) become:

$$[1+2a^2(\partial_0^2-\partial_i^2)][\partial_0+\partial_i]F_{i0}=0 \qquad (41)$$

or equivalently

$$\left[1+a^2\partial_\mu\partial^\mu\right]\partial_\mu F_{\mu\nu}=0 \qquad (42)$$

The sign $\partial_\mu\partial^\mu$ represents the D'Alembertian operator, which is ⌋ in this case. It's worth noting that the equations of motion are invariant under a variety of circumstances Eq.(31). Podolsky's expanded version of Maxwell equations, incorporating Gauss' rule for v = 0 and the Maxwell-Ampere equation for spatial values of the index, may be thought of as Eq.(42). Because the Maxwell equations are definitional rather than kinematic equations, the second half of the equations remains the same Eq.(30). As can be shown, the duality symmetry between electric and magnetic forces is violated for Eq.(42). As indicated in Section 1, a detailed account is available. The subject is covered in [35].

The model (29) clearly represents a well-defined generic situation. In the limit a⇢ 0, electrodynamics reproduces conventional Maxwell's theory. As a result, the gauge field obtains a new 1/a mass in a gauge invariant fashion, in addition to the normal massless mode, as can be shown from the equations of motion. However, we point out that the limit a ⇢ 0, is not continuous in terms of the model's degrees of freedom because it involves an abrupt shift in the theory's constraint structure. We are obliged to deal with a vector particle with two potential mass excitations corresponding to the zero and 1/a poles of the propagator if we fix a gauge and directly quantize model (29). An alternate view, which will be presented in the next section, is to think of these two enormous models as belonging to two different fields.

Now we want to derive Eq.(32) using the Hamiltonian density equations of motion. First we determine π_1^1, π_1^2,π_1^3 and π_2^1 using Eqs.(28):

$$\begin{cases} \pi_1^1=\dfrac{\partial L}{\partial({}_aD_t^{\alpha}\phi)}-\dfrac{d}{dt}\left[\dfrac{\partial L}{\partial({}_aD_t^{2\alpha}\phi)}\right]=0 \\ \pi_1^2=\dfrac{\partial L}{\partial({}_aD_t^{\alpha}A_i)}-\dfrac{d}{dt}\left[\dfrac{\partial L}{\partial({}_aD_t^{2\alpha}A_i)}\right] \\ \quad={}_aD_t^{\alpha}A_i-{}_aD_t^{\alpha}A_0 \\ \pi_2^1=\dfrac{\partial L}{\partial({}_aD_t^{2\alpha}\phi)}=0 \\ \pi_2^2=\dfrac{\partial L}{\partial({}_aD_t^{2\alpha}A_i)}=-2a^2\,{}_aD_t^{\alpha}F_{i0} \\ \quad=-2a^2\left[{}_aD_t^{2\alpha}A_i-{}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}A_0\right]\end{cases} \qquad (42)$$

The primary constraints are:

$$\pi_1^1=0 \qquad (43)$$

$$\pi_2^1=0 \qquad (44)$$

Then, using Eq.(20), the Hamiltonian density can be written as:

$$H=\begin{bmatrix}(\pi_1^2)^2+\pi_1^2\,{}_aD_{x_i}^{\alpha}A_0+\dfrac{1}{4a^2}(\pi_2^2)^2 \\ -a^2\left[{}_aD_{x_i}^{\alpha}\pi_1^2\,{}_aD_{x_k}^{\alpha}\pi_1^2-{}_aD_{x_i}^{\alpha}F^{ik}\,{}_aD_{x_k}^{\alpha}F^{kj}\right] \\ +\dfrac{\pi_2^2}{2}\left[2\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\phi+{}_aD_{x_k}^{\alpha}F^{kj}-{}_aD_{x_i}^{\alpha}F^{ij}\right]\end{bmatrix}+\frac{1}{4}F_{ik}F^{ik} \qquad (45)$$

By taking the derivative with respect to$(\pi_1^2,\pi_2^2,\mathbf{A_0},\mathbf{A_i})$, using the Hamiltonian Eq. (28), we get:

$${}_aD_t^{\alpha}A_0=\frac{\partial\mathcal{H}}{\partial\pi_1^2}=\pi_1^2+{}_aD_{x_i}^{\alpha}A_0 \qquad (46)$$

$${}_aD_t^{2\alpha}A_i=\frac{\partial\mathcal{H}}{\partial\boldsymbol{\pi}_2^2}=\frac{1}{2a^2}(\pi_2^2)^2+{}_aD_t^{\alpha}\,{}_aD_{x_i}^{\alpha}\phi \qquad (47)$$

$$\frac{\partial\mathcal{H}}{\partial\mathbf{A_0}}=-{}_a\mathrm{D}_t^{\alpha}\pi_1^1+{}_a\mathrm{D}_t^{2\alpha}\boldsymbol{\pi}_2^1-{}_a\mathrm{D}_{x_i}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial({}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_0})}\right]+{}_a\mathrm{D}_{x_i}^{2\alpha}\left[\frac{\partial\mathcal{L}}{\partial({}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_0})}\right]={}_aD_{x_i}^{\alpha}F_{i0}+2a^2\,{}_a\mathrm{D}_{x_i}^{3\alpha}F_{i0}=0 \qquad (48)$$

$$\frac{\partial\mathcal{H}}{\partial\mathbf{A_i}}=-{}_a\mathrm{D}_t^{\alpha}\pi_1^2+{}_a\mathrm{D}_t^{2\alpha}\boldsymbol{\pi}_2^2-{}_a\mathrm{D}_{x_i}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial({}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_i})}\right]+{}_a\mathrm{D}_{x_i}^{2\alpha}\left[\frac{\partial\mathcal{L}}{\partial({}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_i})}\right] \qquad (49)$$

$$=-{}_a\mathrm{D}_t^{\alpha}\left[{}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_0}-{}_a\mathrm{D}_t^{\alpha}\mathbf{A_i}\right]-2a^2\,{}_a\mathrm{D}_t^{3\alpha}\left[{}_a\mathrm{D}_{x_i}^{\alpha}\mathbf{A_0}-{}_a\mathrm{D}_t^{\alpha}\mathbf{A_i}\right]=0 \qquad (50)$$

If α = 1, then Eq. (48) and Eq. (50) respectively give:

$$[1+2a^2\,\partial_i^2]\partial_i F_{i0}=0$$

$[1 + 2a^2\,\partial_0^2]\partial_0 F_{i0} = 0$

Or

$$\left[1 + 2a^2\left(\left({}_aD_t^\alpha\right)^2 - \left({}_aD_{x_i}^\alpha\right)^2\right)\right]\left[{}_aD_t^\alpha + {}_aD_{x_i}^\alpha\right]\left[{}_aD_{x_i}^\alpha\phi - {}_aD_t^\alpha A_i\right] = 0 \quad (51)$$

The equation above is exactly the same as the equation obtained in fractional form by Eq. (51). By considering α = 1 , we get:

$$[1 + a^2\partial_\mu\partial^\mu]\partial_\mu F_{\mu\nu} = 0 \quad (52)$$

The fourth-order motion equations are as follows. The result in Eq.(52) is identical to the result in[32, 33].

7 Reduced-Orders Fractional Podolsk Model

To investigate the model's temporal evolution and continue quantization, we must first write down the appropriate Hamiltonian in phase space. The Lagrangian Eq.(29) is unique due to gauge invariance, resulting in a restricted Hamiltonian system. A rudimentary constraint structure analysis was conducted in [36], and it was revisited in [38], which needed canonical momenta conjugated to both A_μ and its time derivative ${}_aD_t^\alpha A_\mu$. We offer an alternate method of lowering the derivatives order of Eq.(29) first by adding an auxiliary vector field B_μ, which avoids the requirement to consider A_μ and ${}_aD_t^\alpha A_\mu$ as independent fields. we write down the Lagrangian density

$$\mathcal{L} = \begin{bmatrix} -\frac{1}{4}\left[{}_aD_{x_\mu}^\alpha A_\nu\,{}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\mu}^\alpha A_\nu\,{}_aD_{x^\nu}^\alpha A^\mu\right] - \\ \frac{a^2}{2}[B_\mu B^\mu] + a^2\left[{}_aD_{x_\mu}^\alpha B_\nu\,{}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu\right] \end{bmatrix} \quad (53)$$

Consider the reduced-order action $\int d^4x\,\mathcal{L}$ that corresponds to it. The reduced action is gauge invariant in the same way that (7) is.

$$\boldsymbol{\delta} A_\mu = {}_aD_{x_\nu}^\alpha \Lambda \qquad \delta B_\mu = 0 \quad (54)$$

The modification of this reduced-order action with regard to the fields A_μ and B_μ gives rise to the coupled equations of motion.

$$\mathbf{0} = \frac{\boldsymbol{\delta S}}{\boldsymbol{\delta A_\mu}} = \left[{}_aD_{x_\mu}^\alpha\,{}_aD_{x^\mu}^\alpha\eta^{\mu\nu} - {}_aD_{x^\mu}^\alpha\,{}_aD_{x^\nu}^\alpha\right][A_\nu - a^2B_\nu] \quad (55)$$

And

$$\mathbf{0} = \frac{\boldsymbol{\delta S}}{\boldsymbol{\delta B_\mu}} = a^2\left[{}_aD_{x_\nu}^\alpha\left({}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu\right) - B_\mu\right] \quad (56)$$

This is an analogous system of eight second-order linear partial differential equations on fields A_μ and B_μ to the four fourth-order ones Eq.(6) on the field A_μ. If we replace the relation $B^\mu = {}_aD_{x_\nu}^\alpha\left({}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu\right)$ from the second set above into the first set (9) to reconstruct the basic system, we can see this immediately.

Another straight forward way of looking at it is as follows: Begin by defining the auxiliary field $\left[{}_aD_{x_\mu}^\alpha\,{}_aD_{x^\mu}^\alpha\eta^{\mu\nu} - {}_aD_{x^\mu}^\alpha\,{}_aD_{x^\nu}^\alpha\right]A_\nu$. Eq.(52). On the one hand, the fourth-order set (6) is clearly analogous to the second-order set.

$$\left[1 + a^2\,{}_aD_{x_\mu}^\alpha\,{}_aD_{x^\mu}^\alpha\right]B^\mu = 0 \quad (57)$$

And

$$-B^\mu = \left[{}_aD_{x_\mu}^\alpha\,{}_aD_{x^\mu}^\alpha\eta^{\mu\nu} - {}_aD_{x^\mu}^\alpha\,{}_aD_{x^\nu}^\alpha\right]A_\nu \quad (58)$$

On the other hand, Eq. (58) is nothing more than Eq.(56) when Eq.(57) is subtracted from Eq.(57) and ${}_aD_{x_\mu}^\alpha B^\mu = 0$ is assumed Eq.(55).

As a result, the two partial differential equations systems are equal. Furthermore, we see that the auxiliary field B is divergenceless and obeys the Klein-Gordon equation Eq.(57) with mass 1/a from a physical standpoint. While the 1/a mass vector excitations have been transferred to the B field, we can confidently assert that the A field remains massless. It's worth noting that a duality relationship between the Podolsky and Proca models including changes in the mass term sign was recently discovered in [37].

8 Conclusions

Finding new formalisms in the realm of restricted systems allows us to evaluate an issue using a variety of approaches and select the most suited one. After reducing the order of the derivatives in Podolsky's generalized electrodynamics, we quantized the resulting model. The dynamical evolution was shown to occur in a phase space with two first-class and two second-class constraints, similar to a higher-order model.

We investigated the Podolsky electrodynamics system for continuous systems with second order derivatives in depth using the innovative formalism presented in form fractional. We used fractional derivatives from Riemann-Liouville and we presented Hamilton's equations to construct Lagrangian and Hamiltonic formulations for second order continuous systems. An essential example is given to illustrate the new formula. The results of the Euler-Lagrange formula are similar. For **n ⟶ 1**, our results would be similar to those obtained in [20, 32, 36].

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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