Applied Mathematics & Information Sciences **5**(2) (2011), 192-204
– An International Journal

Boehmians and pseudoquotients

Piotr Mikusiński

Department of Mathematics, University of Central Florida, Orlando, Florida, USA

Email Address: piotrm@mail.ucf.edu.edu

Received Dec. 23, 2010; Accepted Jan 3, 2011

We consider the algebraic constructions of Boehmians and pseudoquotients, explain the motivation for these ideas, describe early developments and later modifications, and indicate some of their applications. While the majority of applications are in the area of generalized functions and generalized integral transforms, some recent developments suggest that Boehmians and pseudoquotients can provide useful and versatile tools for constructing new objects in other areas of mathematics.

Keywords: Mikusiński operators, regular operators, Boehmians, pseudoquotients, tempered distributions, Bochner Theorem.

1 Introduction

This paper describes an evolution of an idea. In its simplest form, it can be traced to the construction of rational numbers. In this case we consider ordered pairs (p, q) where $p \in \mathbb{Z}$, the set of all integers, and $q \in \mathbb{N}$, the set of all positive integers. Two such pairs (p, q) and (r, s) are called equivalent, denoted by $(p, q) \sim (r, s)$, if $sp = rq$. The rational numbers are defined as equivalence classes of these pairs and we write

$$[(p, q)] = \frac{p}{q}$$

in spite the fact that this notation is formally incorrect.

This construction generalizes to an integral domain. In the next section we consider one example of such a construction, namely the operational calculus of Jan Mikusiński. Work in this area lead T. K. Boehme to the idea of regular operators, which in turn were generalized to Boehmians. In Section 3 we describe the original construction of Boehmians and then its abstract generalization. A special class of abstract Boehmians is the class of pseudoquotients (or generalized quotients) which are discussed in Section 4. Pseudoquotients are simpler than general Boehmians and have desirable properties. We give some examples of spaces of pseudoquotients in Section 5.

It is our hope that this paper shows that Boehmians and pseudoquotients are natural concepts that have good properties and interesting applications. They are worth studying in their abstract form as well as in specific applications.

2 Operational Calculus

The main idea of the Operational Calculus of Jan Mikusiński appeared in a little booklet entitled *Hyperliczby* (Hypernumbers) published in Poland in 1944. The work was handwritten by Jan Mikusiński on X-ray film and printed with homemade ink. Only seven copies were made. A monograph on the subject was first published in Polish in 1953 [25]. The first English translation was published in 1959 [26].

Consider the space $\mathcal{C}([0,\infty))$ of continuous complex-valued functions with addition and convolution defined by

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy.$$

Since this is an integral domain, we can construct a quotient field $\mathbb{M}$. Elements of $\mathbb{M}$ are called *Mikusiński operators*.

Mikusiński operators are global objects. In particular, they do not have a well-defined support and it is not possible to define equality of two operators on an open subset of $[0,\infty)$. To address these problems, T. K. Boehme [6] identified a class of operators, called regular operators, with desirable local properties. The key idea was to use delta sequences (also known as approximate identities). A sequence $\varphi_n \in \mathcal{C}([0,\infty))$ is called a *delta sequence* if

1. $\varphi_n \geq 0$ for every $n \in \mathbb{N}$,
2. $\int_0^\infty \varphi_n(t)dt = 1$ for every $n \in \mathbb{N}$,
3. $\operatorname{supp} \varphi_n \subseteq [0, \varepsilon_n]$ for some $\varepsilon_n \to 0$.

An operator $\frac{f}{g} \in \mathbb{M}$ is called a *regular operator* if there exists a delta sequence (φ_n) and a sequence of functions $f_n \in \mathcal{C}([0,\infty))$ such that

$$\frac{f}{g} = \frac{f_n}{\varphi_n} \quad \text{for all } n \in \mathbb{N}.$$

Regular operators constitute a large subclass of $\mathbb{M}$ and are sufficient for most applications. Since regular operators are elements of $\mathbb{M}$, they are restricted to $[0,\infty)$. It turns out that this restriction is not essential. In 1981, Jan Mikusiński and Piotr Mikusiński [27], and independently Józef Burzyk (personal communication), proposed an extension of regular operators to the real line $\mathbb{R}$. The construction was no longer an example of the construction of the field of quotients from an integral domain, but there were some obvious similarities. In fact, it can be considered a generalization of the construction of the field of quotients.

3 Boehmians

The definition of delta sequences requires minor modifications: A sequence $\varphi_n \in \mathcal{C}^\infty(\mathbb{R})$ is called a *delta sequence* if

1. $\varphi_n \geq 0$ for every $n \in \mathbb{N}$,
2. $\int_{-\infty}^{\infty} \varphi_n(t)dt = 1$ for every $n \in \mathbb{N}$,
3. $\operatorname{supp} \varphi_n \subseteq [-\varepsilon_n, \varepsilon_n]$ for some $\varepsilon_n \to 0$.

Now we define a set $\mathcal{A}$ of all pairs of sequences $((f_n), (\varphi_n))$ such that

1. $f_n \in \mathcal{C}(\mathbb{R})$,
2. (φ_n) is a delta sequence,
3. $\varphi_n * f_m = \varphi_m * f_n$ for all $m, n \in \mathbb{N}$,

where
$$(\varphi * f)(t) = \int_{-\infty}^{\infty} \varphi(s) f(t-s) ds.$$
For $((f_n), (\varphi_n)), ((g_n), (\psi_n)) \in \mathcal{A}$ we define a relation
$$((f_n), (\varphi_n)) \sim ((g_n), (\psi_n)) \quad \text{if} \quad \varphi_n * g_m = \psi_m * f_n \;\; \text{for all} \;\; m, n \in \mathbb{N}.$$
From properties of delta sequences it follows that this is an equivalence relation. This allows us to define a new space $\mathcal{B} = \mathcal{A}/\sim$, that is, the space of equivalence classes of pairs of sequences from $\mathcal{A}$. Elements of $\mathcal{B}$ are called *Boehmians* and denoted
$$[((f_n), (\varphi_n))] = \frac{f_n}{\varphi_n}.$$
Note that it would be more appropriate to use $\frac{(f_n)}{(\varphi_n)}$, but we choose not to do it for the sake of simplicity.

The space $\mathcal{B}$ is a vector space with the operations defined as follows
$$\lambda \frac{f_n}{\varphi_n} = \frac{\lambda f_n}{\varphi_n} \quad \text{and} \quad \frac{f_n}{\varphi_n} + \frac{g_n}{\psi_n} = \frac{\psi_n f_n + \varphi_n g_n}{\varphi_n \psi_n}.$$
Elements of $\mathcal{C}(\mathbb{R})$ can be identified with Boehmians via the map
$$f \mapsto \frac{\varphi_n * f}{\varphi_n},$$
where (φ_n) is an arbitrary delta sequence. If ψ is an integrable function with compact support, we can define the convolution of ψ and an arbitrary Boehmian:
$$\psi * \frac{f_n}{\varphi_n} = \frac{\psi * f_n}{\varphi_n}.$$

According to this definition and the identification mentioned above we have

$$\varphi_m * \frac{f_n}{\varphi_n} = f_m.$$

Differentiation is also well-defined in $\mathcal{B}$:

$$D_j \frac{f_n}{\varphi_n} = D_j \frac{\varphi_n * f_n}{\varphi_n * \varphi_n} = \frac{D_j \varphi_n * f_n}{\varphi_n * \varphi_n}.$$

The space of Boehmians can be equipped with a complete metric topology that has desirable properties [28], [11]. For example, the embedding of $\mathcal{C}(\mathbb{R}^N)$ into $\mathcal{B}$, convolution with an integrable function with compact domain, and differentiation are continuous. The space of Boehmians $\mathcal{B}$ is a large space of generalized functions that contains Schwartz distributions and ultradistributions of Beurling and of Roumieu type [29].

The original construction of Boehmians produced a concrete space of generalized functions. The components in that construction were the space of functions $\mathcal{C}(\mathbb{R})$, a class of delta sequences, and the operation of convolution. It was soon recognized that similar constructions are possible with different components. In fact, it is a very general method for constructing extensions of spaces. Since the paper introducing Boehmians was published in 1981, many spaces of Boehmians were defined. In the references we list selected examples of papers introducing different spaces of Boehmians. One of the main motivations for introducing different spaces was generalization of integral transform. This idea requires a proper choice of a space of functions for which a given integral transform is well-defined, a choice of a class of delta sequences that is transformed by that integral transform to a well-behaved class of approximate identities, and finally a "convolution product" that behaves well under the transform. If these conditions are met, the transform usually has a extension to the constructed space of Boehmains and the extension has desirable properties [31]. The range of such an extension could be a space of functions, or a space of distributions, or another space of Boehmians.

Spaces of Boehmians can be studied as abstract structures. The general framework for such a construction requires the following components:

1. A nonempty set X,
2. A commutative semigroup G acting on X,
3. A nonempty index set I,
4. A set $\Delta \subset G^I$ such that
   (a) $\alpha : X \to X^I$ is injective for every $\alpha \in \Delta$,
   (b) $\alpha, \beta \in \Delta$ implies $\alpha\beta \in \Delta$.

The general construction resembles the original case. First we define

$$\mathcal{A} = \{(\xi, \alpha) : \xi \in X^I, \alpha \in \Delta, \text{ and } \alpha(i)\xi(j) = \alpha(j)\xi(i) \text{ for all } i, j \in I\}$$

and then an equivalence relation

$$(\xi, \alpha) \sim (\zeta, \beta) \quad \text{if} \quad \alpha(i)\zeta(j) = \beta(j)\xi(i) \ \text{ for all } \ i, j \in I.$$

The space of Boehmians is defined as the space of equivalence classes of elements of $\mathcal{A}$, that is, $\mathcal{B}(\mathcal{X}, \Delta) = \mathcal{A}/\sim$.

Note that in this abstract generalization no structure of X is assumed. In particular, elements of Δ are not assumed to be formed from elements of X.

One does not expect to be able to prove much about such a general structure, but two basic properties always hold. First, X can be identified with a subspace of $\mathcal{B}(\mathcal{X}, \Delta)$ via the map

$$\iota(x) = \frac{\alpha x}{\alpha},$$

where $\alpha \in \Delta$ is arbitrary. Second, the action of G can be extended to $\mathcal{B}(\mathcal{X}, \Delta)$. For $g \in G$ and $\frac{\xi}{\alpha} \in \mathcal{B}(\mathcal{X}, \Delta)$ we define

$$g\frac{\xi}{\alpha} = \frac{g\xi}{\alpha}.$$

It is of interest to study what structures on X and what properties of those structures are inherited by $\mathcal{B}(\mathcal{X}, \Delta)$. Not much has been done in that direction, except for the case when the index set I is reduced to a single element. In this case $\Delta \subseteq G$, or simply $\Delta = G$, and elements of $\mathcal{B}(\mathcal{X}, \Delta)$ take the form $\frac{x}{\varphi}$ and are called pseudoquotients.

4 Pseudoquotients

Assume that G is a commutative semigroup acting on X injectively. For $(x, \varphi), (y, \psi) \in X \times G$ the equivalence relation takes the form

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \ \ \psi x = \varphi y.$$

The space of equivalence classes is denoted by $\mathcal{B}(X, G) = (X \times G)/\sim$ and its elements are called *generalized quotients* or *pseudoquotients* [10], [32]. As before, we use the notation $[(x, \varphi)] = \frac{x}{\varphi}$. In spite of the simplicity of this construction, the space $\mathcal{B}(X, G)$ has some interesting properties. As in the general case, elements of X can be identified with elements of $\mathcal{B}(X, G)$ via the embedding $\iota : X \to \mathcal{B}(X, G)$ defined by $\iota(x) = \frac{\varphi x}{\varphi}$ and the action of G can be extended to $\mathcal{B}(X, G)$ via $\varphi\frac{x}{\psi} = \frac{\varphi x}{\psi}$. Note that

$$\varphi\frac{x}{\varphi} = \frac{\varphi x}{\varphi} = x.$$

It turns out that elements of G, as maps from $\mathcal{B}(X,G)$ to $\mathcal{B}(X,G)$, are invertible. In fact, for any $\varphi \in G$ we can define

$$\varphi^{-1}\frac{x}{\psi} = \frac{x}{\varphi\psi}$$

and then extend G to a group.

Theorem 4.1. *$\widehat{G} = \{\varphi^{-1}\psi : \varphi, \psi \in G\}$ is a commutative group acting on $\mathcal{B}(X,G)$ bijectively.*

The construction of pseudoquotients can thus be viewed as an extension of a pair (X,G), of a set X with a commutative semigroup acting on X injectively, to a pair $(\mathcal{B}(X,G),\widehat{G})$, of a set $\mathcal{B}(X,G)$, that contains X as a subset, and a commutative group acting on $\mathcal{B}(X,G)$ bijectively, that contains G as a subsemigroup.

If we assume that X has an algebraic structure and elements of G preserves that structure, then $\mathcal{B}(X,G)$ inherits the structure of G. For example, we have the following two theorems.

Theorem 4.2. *If $(X,\odot)$ is a (commutative) group and G is a commutative semigroup of injective homomorphisms on X, then $\mathcal{B}(X,G)$ is a (commutative) group with the group operation defined by*

$$\frac{x}{\varphi} \odot \frac{y}{\psi} = \frac{\psi x \odot \varphi y}{\varphi\psi}.$$

Theorem 4.3. *If X is a vector space and G is a commutative semigroup of injective linear mappings from X into X, then $\mathcal{B}(X,G)$ is a vector space with the operations defined by*

$$\frac{x}{\varphi} + \frac{y}{\psi} = \frac{\psi x + \varphi y}{\varphi\psi} \quad \textit{and} \quad \lambda\frac{x}{\varphi} = \frac{\lambda x}{\varphi}.$$

There are similar theorems for other structures. One can also prove general theorems about extensions of transformations on X.

Theorem 4.4. *Let $T : X \to X$. Then*

$$\widetilde{T}\frac{x}{\varphi} = \frac{Tx}{\varphi}$$

is a well-defined extension of T to $\widetilde{T} : \mathcal{B}(X,G) \to \mathcal{B}(X,G)$ if and only if T commutes with G.

The following theorem was motivated by specific applications. One such application will be described later in this note.

Theorem 4.5. *Let $(X,+)$ be a commutative group and let G be a commutative semigroup of injective homomorphisms on X. If $T : X \to X$ is a homomorphism such that $T\varphi - \varphi T$ commutes with G for every $\varphi \in G$, then*

$$\widetilde{T}\frac{x}{\varphi} = \frac{2Tx}{\varphi} - \frac{T\varphi x}{\varphi^2}$$

is an extension of T to a map $\widetilde{T} : \mathcal{B}(X, G) \to \mathcal{B}(X, G)$.

Here is an example of a general theorem about transformations between two spaces. It describes a situation one often deals with when extending integral transforms.

Theorem 4.6. *Let X and Y be nonempty sets and let G and H be commutative semigroups of injections on X and Y, respectively. If $T : X \to Y$ and $\eta : G \to H$ is a semigroup homomorphism such that $T(\varphi x) = \eta(\varphi)Tx$ for all $x \in X$ and all $f \in G$. Then*

$$\widetilde{T}\frac{x}{\varphi} = \frac{Tx}{\eta(\varphi)}$$

defines an extension of T to a map $\widetilde{T} : \mathcal{B}(X, G) \to \mathcal{B}(Y, H)$.

When X is a topological space and G is a commutative semigroup of continuous maps acting on X equipped with its own topology (usually the discrete topology), we can define the product topology on $X \times G$ and then the quotient topology on $\mathcal{B}(X, G) = (X \times G)/_{\sim}$. This is the standard topology on $\mathcal{B}(X, G)$. It is easy to show that the embedding $\iota : X \to \mathcal{B}(X, G)$ is continuous. Moreover, the map $\frac{x}{\psi} \mapsto \frac{\varphi x}{\psi}$ as well as the map $\frac{x}{\psi} \mapsto \frac{x}{\varphi\psi}$ are continuous for every $\varphi \in G$. In fact, we have the following theorem.

Theorem 4.7. $\widehat{G} = \{\varphi^{-1}\psi : \varphi, \psi \in G\}$ *is a commutative group of homeomorphisms on* $\mathcal{B}(X, G)$.

Studying which topological properties of X are inherited by the topology of pseudoquotients is of basic importance. While there are some results in that direction (see [16], [7], [24], [8]), many questions remain open. The theorems presented in this sections are meant to provide an indication that the construction of pseudoquotients is natural and has good properties. In the next section we present some specific examples of spaces of pseudoquotients which show that pseudoquotients can be useful in applications.

5 Examples of Applications of Pseudoquotients

In this section we present three examples of applications of pseudoquotients to generalized functions, functional analysis, and abstract harmonic analysis. The first example is a construction of a specific space of generalized functions, namely tempered distributions. In the second example pseudoquotients provide a method for extending an inner product space with some special properties. In the third example pseudoquotients allow us to extend an important theorem in abstract harmonic analysis.

5.1 Tempered Distributions

A function $f : \mathbb{R}^N \to \mathbb{C}$ is called a *moderate function* if $f = pg$ for some polynomial p and some $g \in L^2$. The space of all moderate functions will be denoted by $\mathcal{M}$. Note that for $f \in \mathcal{M}$ and $g \in L^1$ the convolution

$$f * g(x) = \int_{\mathbb{R}^N} f(y)g(x-y)dy$$

is well-defined. Now we let

$$E(x) = e^{-(|x_1|+|x_2|+\cdots+|x_N|)}$$

and

$$\mathcal{G} = \{E^n : n = 1, 2, 3, \dots\},$$

where E^n denotes the n-fold convolution, that is, $E^2 = E * E$ and $E^{n+1} = E * E^n$. Then $\mathcal{G}$ is a commutative semigroup acting on $\mathcal{M}$ injectively. It turns out that the space of pseudoquotients $\mathcal{B}(\mathcal{M}, \mathcal{G})$ is isomorphic to the space of tempered Schwartz distributions $\mathcal{S}'$.

Let $M_j : \mathcal{M} \to \mathcal{M}$ denote the multiplication operator defined by $(M_j f)(x) = x_j f(x)$, $j = 1, 2, \dots, N$, where $x = (x_1, x_2, \dots, x_N)$. By Theorem 4.5, the formula

$$M_j \frac{f}{E^k} = \frac{2M_j f}{E^k} - \frac{M_j(E^k * f)}{E^{2k}}$$

defines an extension of M_j to a linear map $M_j : \mathcal{B}(\mathcal{M}, \mathcal{G}) \to \mathcal{B}(\mathcal{M}, \mathcal{G})$. Consequently, multiplication by polynomials is well-defined in $\mathcal{B}(\mathcal{M}, \mathcal{G})$.

To define differentiation in $\mathcal{B}(\mathcal{M}, \mathcal{G})$ we first introduce an auxiliary function $C : \mathbb{R} \to \mathbb{R}$ defined by

$$C(t) = \begin{cases} -e^{-t} & \text{if } t \geq 0 \\ e^t & \text{if } t < 0 \end{cases}$$

and then, for each $j = 1, \dots, N$, a function $C_j : \mathbb{R}^N \to \mathbb{R}$, defined by

$$C_j(x) = e^{|x_j|} C(x_j) E(x).$$

It is easy to see that $C_j \in \mathcal{M}$ and $D_j E = C_j$, where D_j denotes the partial derivative with respect to x_j. Now, for an arbitrary $\frac{f}{E^k} \in \mathcal{B}(\mathcal{M}, \mathcal{G})$, we define

$$D_j \frac{f}{E^k} = \frac{C_j}{E} * \frac{f}{E^k}.$$

It can be shown that the above definition extends differentiation to $\mathcal{B}(\mathcal{M}, \mathcal{G})$. Note that the partial differential operator D_j can be identified with an element of $\mathcal{B}(\mathcal{M}, \mathcal{G})$.

From the general theory of pseudoquotients we know that E, as a map from $\mathcal{B}(\mathcal{M}, \mathcal{G})$ to $\mathcal{B}(\mathcal{M}, \mathcal{G})$, is invertible and

$$E^{-1} \frac{f}{E^k} = \frac{f}{E^{k+1}}.$$

In this case it can be represented as a differential operator, namely

$$E^{-1} = \Box = \frac{1}{2^N} \prod_{j=1}^{N} \left(I - D_j^2 \right),$$

where I denotes the identity operator.

Theorem 5.1. *The map*

$$\varphi\left(\frac{f}{E^n}\right) = \Box^n f$$

is an isomorphism of $\mathcal{B}(\mathcal{M}, \mathcal{G})$ *and* $\mathcal{S}'$.

In the above theorem, the differentiation on the right-hand side is understood in the distributional sense. Note that, since a moderate function can be identified with a tempered distribution, $\Box^n f$ is well-defined in $\mathcal{S}'$.

The Fourier transform provides another isomorphism between $\mathcal{B}(\mathcal{M}, \mathcal{G})$ and $\mathcal{S}'$. Indeed, the formula

$$\mathcal{F}\left(\frac{pf}{E^k}\right) = \frac{1}{2^{kN}} \prod_{j=1}^{N} \left(1 + M_j^2\right)^k p(iD)\widehat{f},$$

where $\widehat{f}$ denotes the Fourier transform in $L^2(\mathbb{R}^N)$ and the differentiation $p(iD)\widehat{f}$ is interpreted in the distributional sense, defines the Fourier transform from $\mathcal{B}(\mathcal{M}, \mathcal{G})$ to $\mathcal{S}'$. It can be shown that it is a homeomorphism between these spaces.

The Fourier transform can be also defined as an operation in $\mathcal{B}(\mathcal{M}, \mathcal{G})$. Let

$$s = \frac{1}{2^N} \prod_{j=1}^{N} \left(1 + M_j^2\right).$$

Here s is defined as a multiplication operator, but it can also be interpreted as a polynomial. It is not difficult to show that every moderate function has a representation in the form $s^k f$, where $k \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^N)$. Using such a representation we can define the Fourier transform as a map from $\mathcal{B}(\mathcal{M}, \mathcal{G})$ to $\mathcal{B}(\mathcal{M}, \mathcal{G})$. It takes the following simple and elegant form:

$$\mathcal{F}\left(\frac{s^k f}{E^n}\right) = s^n \frac{\hat{f}}{E^k}.$$

For more detail of the construction described in this section and proofs of the quoted theorems see [4].

5.2 Isometric Operators

In this section we consider the space of pseudoquotients $\mathcal{B}(E, G)$, where E is an inner product space, $T : E \to E$ is an isometric operator on E, and $G = \{T^n : n = 0, 1, 2, \dots\}$. It turns out that $\mathcal{B}(E, G)$ is an inner product space with the inner product defined by

$$\left\langle \frac{x}{T^n}, \frac{y}{T^m} \right\rangle = \langle T^m x, T^n y \rangle.$$

Indeed, if $\frac{w}{T^j} = \frac{x}{T^k}$ and $\frac{y}{T^m} = \frac{z}{T^n}$, then

$$\begin{aligned}\left\langle \frac{w}{T^j}, \frac{y}{T^m} \right\rangle &= \langle T^m w, T^j y\rangle \\ &= \langle T^{m+k+n} w, T^{j+k+n} y\rangle = \langle T^{m+j+n} x, T^{j+k+m} z\rangle \\ &= \langle T^n x, T^k z\rangle = \left\langle \frac{x}{T^k}, \frac{z}{T^n} \right\rangle,\end{aligned}$$

which shows that the inner product is well-defined on $\mathcal{B}(E,G)$.

Since $\left\langle \frac{Tx}{T}, \frac{Ty}{T} \right\rangle = \langle x, y\rangle$, E can be identified with a subspace of the inner product space $\mathcal{B}(E,G)$. Note that $T : \mathcal{B}(E,G) \to \mathcal{B}(E,G)$ is a unitary operator.

5.3 The Generalized Bochner Theorem

Now we consider an application of pseudoquotients presented in [3]. Let X be a locally compact abelian group. A continuous function $f : X \to \mathbb{C}$ is called *positive definite* if

$$\sum_{k,l=1}^{n} c_k \overline{c_l} f(x_l^{-1} x_k) \geq 0$$

for all $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in X$. The cone of all continuous positive definite functions on a locally compact group X will be denoted by $\mathcal{P}_+(X)$.

The well-known Bochner Theorem states that $f \in \mathcal{P}_+(X)$ if and only if there exists a unique bounded positive Radon measure μ_f on $\widehat{X}$, the dual group of X, such that

$$f(x) = \int_{\widehat{X}} \langle \xi, x\rangle d\mu_f(\xi),$$

where $\langle \xi, x\rangle$ denotes the action of the character ξ on x.

Now assume that X is a locally compact group such that $\widehat{X}$ is σ-compact and define

$$\mathcal{G} = \left\{ \varphi \in L^1(X) : \widehat{\varphi}(\xi) > 0 \text{ for all } \xi \in \widehat{X} \right\},$$

where $\widehat{\varphi}$ denotes the Fourier transform on φ. It can be shown that $\mathcal{G}$ is a commutative semigroup acting injectively on $\mathcal{P}_+(X)$ by convolution. The constructed space of pseudoquotients $\mathcal{B}(\mathcal{P}_+(X), \mathcal{G})$ allows us to generalize the Bochner Theorem.

Theorem 5.2 (Generalized Bochner Theorem). *$\mathcal{B}(\mathcal{P}_+(X), \mathcal{G})$ is isomorphic to the space of all positive Radon measure on $\widehat{X}$.*

References

[1] D. Atanasiu and P. Mikusiński, On the Fourier Transform, Boehmians, and Distributions, *Colloq. Math.* **108** (2007), 263–276.

[2] D. Atanasiu and P. Mikusiński, The Fourier transform of Levy measures on a semigroup, *Integ. Trans. Spec. Funct.* **19** (2008), 537–543.

[3] D. Atanasiu and P. Mikusiński, Fourier transform of Radon measures on locally compact groups, *Integ. Trans. Spec. Funct.* **21** (2010), 815-821.

[4] D. Atanasiu, P. Mikusiński, and D. Nemzer, An algebraic approach to tempered distributions, submitted.

[5] J. J. Betancor, M. Linares, and J. M. R. Mendez, The Hankel transform of integrable Boehmians, *Appl. Anal.* **58** (1995), 367–382.

[6] T. K. Boehme, The support of Mikusiński operators, *Trans. Amer. Math. Soc.* **176** (1973), 319–334.

[7] H. Boustique, P. Mikusiński, and G. Richardson, Convergence Semigroup Actions: Generalized Quotients, *Appl. Gen. Topol.* **10** (2009), 173–186.

[8] H. Boustique, P. Mikusiński, and G. Richardson, Convergence Semigroup Categories, *Appl. Gen. Topol.*, to appear.

[9] J. Burzyk, C. Ferens, and P. Mikusiński, On the Topology of Generalized Quotients, *Appl. Gen. Topol.* **9** (2008), 205–212.

[10] J. Burzyk and P. Mikusiński, A generalization of the construction of a field of quotients with applications in analysis, *Int. J. Math. Sci.* **2** (2003), 229–236.

[11] J. Burzyk, P. Mikusiński, and D. Nemzer, Remarks on topological properties of Boehmians, *Rocky Mountain J. Math.* **35** (2005), 727–740.

[12] E. R. Dill and P. Mikusiński, Strong Boehmians, *Proc. Amer. Math. Soc.* **119** (1993), 885–888.

[13] S. B. Gaikwad and M. S. Chaudhary, Fractional Fourier transforms of ultra Boehmians, *J. Indian Math. Soc. (N.S.)* **73** (2006), 53–64.

[14] C. Ganesan, Weighted Ultra Distributions and Boehmians, *Int. J. Math. Analysis* **4** (2010), 703–712.

[15] N. V. Kalpakam and S. Ponnusamy, Convolution transform for Boehmians, *Rocky Mountain J. Math.* **33** (2003), 1353–1378.

[16] V. Karunakaran and C. Ganesan, Topology and the notion of convergence on generalized quotient spaces, *Int. J. Pure Appl. Math.* **44** (2008), 797–808.

[17] V. Karunakaran and N. V. Kalpakam, Weierstrass Transform For Boehmians, *Int. J. Math. Game Theory Algebra* **11** (2001), 47–65.

[18] V. Karunakaran and R. Roopkumar, Boehmians and their Hilbert Transforms, *Integral Transform. Spec. Funct.* **13** (2002), 131–141.

[19] V. Karunakaran and R. Roopkumar, Ultra Boehmians and their Fourier transforms, *Fract. Calc. Appl. Anal.* **5** (2002), 181–194.

[20] V. Karunakaran and R. Roopkumar, Periodic Beurling Boehmians and Analytic Boehmians, *Methods Appl. Anal.* **10** (2003), 137–150.

[21] V. Karunakaran and R. Roopkumar, Operational Calculus and Fourier Transform on Boehmians, *Colloquium Mathematicum* **102** (2005) 21–32.

[22] V. Karunakaran and R. Vembu, Hilbert transform on periodic Boehmians, *Houston J. Math.* **29** (2003), 437–452.

[23] V. Karunakaran and A. Vijayan, Laplace transforms of generalized functions, *Houston J. Math.* **19** (1993), 75–88.
[24] H. J. Kim and P. Mikusiński, On Convergence and Metrizability of Pseudoquotients, *Int. J. Mod. Math.* **5** (2010), 285–298.
[25] J. Mikusiński, *Rachunek operatorów*, Monografie Matematyczne 30, PWN, Warszawa, 1953.
[26] J. Mikusiński, *Operational Calculus*, Pergamon Press, Oxford, 1959.
[27] J. Mikusiński and P. Mikusiński, Quotients de suites et leurs applications dans lanalyse fonctionnelle, *C. R. Acad. Sci. Paris Sr. I Math.* **293** (1981), 463–464.
[28] P. Mikusiński, Convergence of Boehmians, *Japan. J. Math. (N.S.)* **9** (1983), 159–179.
[29] P. Mikusiński, Boehmians and generalized functions, *Acta Math. Hungar.* **51** (1988), 271–281.
[30] P. Mikusiński, Tempered Boehmians and ultradistributions, *Proc. Amer. Math. Soc.* **123** (1995), 813–817.
[31] P. Mikusiński, Transforms of Boehmians, *Dissertationes Math.* **340** (1995), 201–206.
[32] P. Mikusiński, Generalized Quotients With Applications In Analysis, *Methods Appl. Anal.* **10** (2004), 377–386.
[33] P. Mikusiński and M. Morimoto, Boehmians on the sphere and their spherical harmonic expansions, *Fract. Calc. Appl. Anal.* **4** (2001), 25–35.
[34] P. Mikusiński, A. Morse, and D. Nemzer, The two-sided Laplace transform for Boehmians, *Integral Transform. Spec. Funct.* **2** (1994), 219–230.
[35] P. Mikusiński and A. Zayed, The Radon transform of Boehmians, *Proc. Amer. Math. Soc.* **118** (1993), 561–570.
[36] M. Morimoto, *Two definitions of Boehmians on the sphere*, in: Progress in analysis, World Sci. Publishing, Berlin, (2001), 143–148.
[37] D. Nemzer, Periodic generalized functions, *Rocky Mountain J. Math.* **20** (1990), 657–669.
[38] D. Nemzer, Lacunary Boehmians, *Integral Transforms Spec. Funct.* **16** (2005), 451–459.
[39] D. Nemzer, Boehmians on the Torus, *Bull. Korean Math. Soc.* **43** (2006), 831–839.
[40] R. Roopkumar, Wavelet analysis on a Boehmian space, *Int. J. Math. Math. Sci.* **15** (2003), 917–926.
[41] R. Roopkumar, Generalized Radon Transform, *Rocky Mountain J. Math.* **36** (2006), 1375–1390.
[42] R. Roopkumar, Stieltjes transform for Boehmians, *Integ. Trans. Spec. Funct.* **18** (2007), 819–827.
[43] R. Roopkumar, An Extension of Distributional Wavelet Transform, *Coll. Math.* **115** (2009), 195–206.
[44] R. Roopkumar, Mellin Transform for Boehmians, *Bull. Inst. Math. Acad. Sinica New Series* **4** (2009), 75–96.
[45] R. Roopkumar and E. R. Negrin, Poisson Transform on Boehmians, *Appl. Math. Comput.* **216**, (2010), 2740–2748.

[46] R. Roopkumar and E. R. Negrin, Exchange Formula For Generalized Lambert Transform And Its Extension To Boehmians, *Bull. Math. Anal. Appl.* **2** (2010), 34–41.

Piotr Mikusiński is a Professor of Mathematics and Chair of the Department of Mathematics at the University of Central Florida. He is an author or a coauthor of over 70 research articles and three textbooks in mathematical analysis. He serves on the editorial Board of Fractional Calculus and Applied Analysis: An International Journal for Theory and Applications. His research interests include convergence structures, generalized functions, doubly stochastic measures, measure and integration, and abstract harmonic analysis. He is one of the leading researchers studying Boehmians. In recent years he has been interested in abstract constructions motivated by Boehmians and their applications in mathematical analysis, functional analyisis, and abstract harmonic analysis.