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On Geometrical Properties of Some Banach Spaces

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Abstract: In this work, we prove that modular spaces $V\rho(\lambda;p)$ defined in [26] have the k-nearly uniform convexity($k-NUC$ property) when they are endowed with the Luxemburg norm. We also prove that these spaces have the uniform Opial property with the Luxemburg norm. The above investigated geometric properties will enable us to obtain some fixed point results in modular spaces.

Keywords: de la Vallée-Poussin means, Luxemburg norm, the uniform Opial property, k-nearly uniform convexity.

1. Introduction

In summability theory, de la Vallée-Poussin's mean is first used to define the (V,λ)-summability by Leindler [14]. Malkowsky and Savaş [17] introduced and studied some sequence spaces which arise from the notion of generalized de la Vallée-Poussin mean. Also the (V,λ)-summable sequence spaces have been studied by many authors including [7] and [23].

There are many papers on the geometrical properties of some Banach spaces in literature. Some of them are as follows: In [20] Opial defined the Opial property with his name mentioned and he proved that ℓ_p space $(1<p<\infty)$ satisfies this property but the spaces $L_p[0,2\pi]$ $(p\neq 2,$ $(1<p<\infty)$ do not. Franchetti [9] has shown that any infinite dimensional Banach space has an equivalent norm that satisfies the Opial property. Later on, Prus [21] introduced and investigated uniform Opial property for Banach spaces. In [11], the notion of nearly uniform convexity for Banach spaces was introduced by Huff. Also Huff [11] proved that every nearly uniformly convex Banach spaces is reflexive and it has the uniformly Kadec-Klee property. Moreover, Kutzarova [13] defined and studied k-nearly uniformly Banach spaces.

Recently, there are a lot of interest in investigating geometric properties of several sequence spaces. Some of the recent work on sequence spaces and their geometrical properties is given in the sequel: Shue [24] first defined the Cesáro sequence spaces with a norm. In [6], it is shown that the Cesáro sequence spaces ces_p $(1\leq p<\infty)$ have k-nearly uniformly convex and uniform Opial properties. Şimşek and Karakaya [25] studied the uniform Opial property and some other geometric properties of generalized modular spaces of Cesáro type defined by weighted means. In addition, some related papers on this topic can be found in [1], [12], [18], [19] and[22].

Quite recently, Şimşek et al. [26] introduced a new modular sequence space defined by de la Vallée-Poussin's mean and investigated some topological and geometric properties as Kadec-Klee and Banach-Saks of type p. Moreover, the sequence space involving de la Vallée-Poussin's mean is more general than Cesáro sequence space defined by Shue [24] and investigated by Cui and Hudzik [5].

In this paper we consider modular sequence spaces $V_\rho(\lambda;p)$ defined in [26], where ρ is a countably orthogonally additive modular satisfying the strong Δ_2-condition ($\rho\in\Delta_2^s$ for short) and we establish that the space $V_\rho(\lambda;p)$ equipped with the Luxemburg norm satisfy $k-nearly\ uniformly\ convex$ property and $uniform\ Opial$ property whence we deduce some fixed point results in modular spaces.

2. Preliminaries, Background and Notation

For a Banach space X, let $(X,||\cdot||)$ (for the brevity $X=(X,||\cdot||)$) be a normed linear space and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. unit sphere) of X. Let ℓ^0 denote the space of all real sequences and $\mathbb{N}$ denote the set of all natural numbers. For any sequence $\{x_n\}$ in X, we denote by $conv(\{x_n\})$ the convex hull of the elements of $\{x_n\}$ (see [2]).

A Banach space X is called $uniformly\ convex$ (UC) if for each $\varepsilon>0$, there is $\delta>0$ such that for $x,y\in S(X)$, the

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inequality $\|x-y\|>\varepsilon$ implies that

$$\left\|\frac{1}{2}(x+y)\right\|<1-\delta.$$

Recall that for a number $\varepsilon>0$ a sequence $\{x_n\}$ in X is said to be an $\varepsilon-seperated\ sequence$ if

$$sep\left(\{x_n\}\right)=\inf\{\|x_n-x_m\|\,,\ n\neq m\}>\varepsilon.$$

A Banach space X is said to have the $Kadec-Klee\ property$ $(H-property)$ if every weakly convergent sequence on the unit sphere is convergent in norm.

A Banach space X is said to have the $uniform\ Kadec-Klee\ property\ (UKK)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that if x is the weak limit of a normalized ε-separated sequence, then $\|x\|<1-\delta$ (see; Huff [11]). We have that every (UKK) Banach space have the Kadec-Klee property.

A Banach space X is said to be the $nearly\ uniformly\ convex\ (NUC)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for every sequence $\{x_n\}\subset B(X)$ with $sep\left(\{x_n\}\right)>\varepsilon$, we have

$$conv(\{x_n\})\cap(1-\delta)B(X)\neq\varnothing.$$

Let $k\geq 2$ be an integer. A Banach space X is said to be $k-nearly\ uniformly\ convex\ (k-NUC)$ if for any $\varepsilon>0$ there exists $\delta>0$ such that for every sequence $\{x_n\}\subset B(X)$ with $sep\left(\{x_n\}\right)>\varepsilon$, there are $n_1,n_2,...,n_k\in\mathbb{N}$ such that

$$\left\|\frac{x_{n_1}+x_{n_2}+...+x_{n_k}}{k}\right\|<1-\delta.$$

A Banach space X is (NUC) whenever it is $(k-NUC)$ for some integer $k\geq 2$. Clearly, $(k-NUC)$ Banach spaces are (NUC) but the opposite implication does not hold in general (see [13]).

A point $x\in S(X)$ is called an $extreme\ point$ if for any $y,z\in B(X)$ the equality $2x=y+z$ implies $y=z$.

A Banach space X is said to be *rotund* (abbreviated as (R)) if every point of $S(X)$ is an extreme point.

A Banach space X is said to be $fully\ k-rotund$ (write kR) (see [8]) if for every sequence $\{x_n\}\subset B(X)$, the condition

$$\|x_{n_1}+x_{n_2}+...+x_{n_k}\|\to k\quad\text{as}\quad n_1,n_2,...,n_k\to\infty$$

implies that $\{x_n\}$ is convergent.

It is well known that (UC) implies (kR) and (kR) implies $((k+1)R)$, and (kR) spaces are reflexive and rotund, and it is easy to see that $(k-NUC)$ implies (kR).

A Banach space $X\subset\ell^0$ is said to be a *Köthe sequence space* if there is a sequence $x=(x(i))_{i=1}^{\infty}\in X$ with all $x(i)\neq 0$ with for every $x\in\ell^0$ and $y\in X$ with $|x(i)|\leq|y(i)|$ for all $i\in\mathbb{N}$, there holds $x\in X$ and $\|x\|\leq\|y\|$.

An element $x\in X$ is said to be *absolutely continuous* if

$$\lim_{n\to\infty}\|0,0,0,...,0,x(n+1),x(n+2),...\|=0.$$

The set of all absolutely continuous elements in X is denoted by X_a and it is a subspace of X. We say that X is absolutely continuous if $X_a=X$.

A Banach space X is said to have the *Opial property* (see [20]) if for any weakly null sequence (x_n) and every $x\neq 0$ in X there holds

$$\liminf_{n\to\infty}\|x_n\|<\liminf_{n\to\infty}\|x_n+x\|\,.$$

A Banach space X is said to have the *uniform Opial property* (see [21]) if for each $\varepsilon>0$ there exists $\tau>0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x\in X$ with $\|x\|\geq\varepsilon$ there holds

$$1+\tau\leq\liminf_{n\to\infty}\|x_n+x\|\,.$$

Let β be the *ball measure of noncompactness* in X i.e.

$$\beta(A)=\inf\{\varepsilon>0: A\text{ can be covered by a finite family of balls of diameter}\leq\varepsilon\}$$

for any bounded set $A\subset X$.

A Banach space X is said to *have property* (L) if $\lim_{\varepsilon\to 1^-}\Delta(\varepsilon)=1$, where (see [10])

$$\Delta(\varepsilon)=\inf\left\{1-\inf\left(\|x\|:x\in A\right)\right\},$$

where the first infimum is taken over all closed sets A in the unit ball $B(X)$ of X with $\beta(A)\geq\varepsilon$.

The function Δ is called the *modulus of noncompact convexity* (see [10]). It has been proved in [21] that property (L) is useful to study the fixed point property and that a Banach space X has property (L) if and only if it is reflexive and has the uniform Opial property.

Throughout the paper, the sequence $p=(p_k)$ is a bounded sequence of positive real numbers with

$$\liminf_{k\to\infty}p_k>1. \tag{1}$$

Besides, we will need the following inequalities in the sequel;

$$|a_k+b_k|^{p_k}\leq K\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{2}$$

where $K=\max\{1,2^{H-1}\}$ with $H=\sup_k p_k$.

We start with a brief collection of basic concepts and facts of the theory of modular spaces.

Let X be an arbitrary real vector space.

(a) A functional $\rho:X\to[0,\infty]$ is called *modular* if for arbitrary x,y in X,

(i) $\rho(x)=0$ if and only if $x=0$,

(ii) $\rho(\alpha x)=\rho(x)$ for every scalar α with $|\alpha|=1$,

(iii) $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ if $\alpha+\beta=1$ and $\alpha\geq 0,\beta\geq 0$,

(b) if (iii) is replaced by

(iii)′ $\rho(\alpha x+\beta y)\leq\alpha\rho(x)+\beta\rho(y)$ if $\alpha+\beta=1$ and $\alpha\geq 0$, $\beta\geq 0$, we say that ρ is a *convex modular*.

(c) A modular ρ defines a corresponding modular space, i.e the vector space X_ρ given by

$$X_\rho=\left\{x\in X:\rho(\lambda x)\to 0\text{ as }\lambda\to 0^+\right\}.$$

In general, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an $F-norm$. Recall that a functional $\|\cdot\|:X\to[0,\infty]$ defines an F-*norm* if and only if

1) $\|x\|=0\Leftrightarrow x=0$,

2) $\|\alpha x\|=\|x\|$ whenever $|\alpha|=1$,

3) $\|x+y\|\leq\|x\|+\|y\|$

4) $\|\alpha_n x_n-\alpha x\|\to 0$ if $\alpha_n\to\alpha$ and $\|x_n-x\|\to 0$.

An F-norm defines a distance on X by

$$d(x,y)=\|x-y\|\,.$$

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The linear metric space (X, d) is called an *F-space* if it is a complete metric space.

The modular space X_ρ can be equipped with the F-norm defined by

$$\|x\| = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \le \alpha\right\}.$$

When ρ is convex the formula

$$\|x\| = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \le 1\right\}$$

defines a norm in X_ρ which is frequently called *Luxemburg norm* and signed by $\|x\|$.

A modular m is said to satisfy the Δ_2-condition ($\rho \in \Delta_2$ for short) if for any $\varepsilon > 0$ there exist constants $K \ge 2$ and $a > 0$ such that

$$\rho(2x) \le K\rho(x) + \varepsilon$$

for all $x \in X_\rho$ with $\rho(x) \le a$.

If ρ satisfies the Δ_2-condition for any $a > 0$ with $K \ge 2$ dependent on a, we say that ρ satisfies the strong Δ_2-condition ($\rho \in \Delta_2^s$ for short).

Now we begin the consideration of a new sequence space.

Let $\Lambda = (\lambda_k)$ be a nondecreasing sequence of positive real numbers tending to infinity and let $\lambda_1 = 1$ and $\lambda_{k+1} \le \lambda_k + 1$.

The generalized de la Vallée-Poussin means of a sequence $x = \{x_k\}$ are defined as follows:

$$t_k(x) = \frac{1}{\lambda_k}\sum_{j\in I_k} x_k \ \text{ where } \ I_k = [k-\lambda_k+1, k] \text{ for } \ k = 1, 2, ... \ .$$

We write

$$[V,\lambda]_0 = \left\{x \in \ell^0 : \lim_{k\to\infty}\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j| = 0\right\},$$

$$[V,\lambda] = \left\{x \in \ell^0 : x - le \in [V,\lambda]_0, \text{ for some } l \in \mathbb{C}\right\}$$

and

$$[V,\lambda]_\infty = \left\{x \in \ell^0 : \sup_k\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j| < \infty\right\}$$

for the sequence spaces that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method, resp.(see [14]). In the special case where $\lambda_k = k$ for $k = 1, 2, ...$ the spaces $[V,\lambda]_0$, $[V,\lambda]$ and $[V,\lambda]_\infty$ reduce to the spaces w_0, w and w_∞ introduced by Maddox [16].

In [26], it is defined a generalized modular sequence space $V_\rho(\lambda; p)$ by

$$V_\rho(\lambda; p) := \left\{x \in \ell^0 : \rho(\tau x) < \infty, \text{ for some } \tau > 0\right\},$$

where

$$\rho(x) = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k}.$$

If we take $p_k = p$ for all $k \in N$, we obtain the space $V_\rho(\lambda)$ studied by Şimşek [27].

Note that the Luxemburg norm on the sequence space $V_\rho(\lambda; p)$ is defined as follows:

$$||x|| = \inf\left\{\tau > 0 : \rho(\frac{x}{\tau}) \le 1\right\}, \qquad \text{for all } x \in V_\rho(\lambda; p)$$

or equivalently

$$||x|| = \inf\left\{\tau > 0 : \rho(\frac{x}{\tau}) = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|\frac{x_j}{\tau}|\right)^{p_k} \le 1\right\}.$$

The details of the modular sequence spaces mentioned above can be found in [26].

3. Modular Space $V_\rho(\lambda; p)$ with k-NUC property

The uniform Opial condition and the *k-NUC* property are geometric properties which are strongly connected with the existence of fixed points for some kinds of space.

In this section we consider the Banach space $V_\rho(\lambda; p)$ endowed with the Luxemburg norm and we prove that it satisfies the *k-NUC* property.

For this, we need some the following results which are important for our consideration:

Lemma 1. *If $\rho \in \Delta_2^s$, then for any $x \in X_\rho$, $\|x\| = 1$ if and only if $\rho(x) = 1$.*

Proof. (See [4], Cor. 2.2)

Lemma 2. *If $\rho \in \Delta_2^s$, then for any sequence (x_n) in X_ρ, $\|x_n\| \to 0$ if and only if $\rho(x_n) \to 0$.*

Proof. (See [4], Lemma 2.3)

Lemma 3. *If $\rho \in \Delta_2^s$, then for any $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $\rho(x) \le 1 - \varepsilon$ implies $\|x\| \le 1 - \delta$.*

Proof. Suppose that the lemma does not hold. Then there exist $\varepsilon > 0$ and $x_n \in X_\rho$ such that $\rho(x_n) < 1 - \varepsilon$ and $\frac{1}{2} \le \|x_n\| \to 1$. Let $a_n = \frac{1}{\|x_n\|-1}$. Then $a_n \to 0$ as $n \to \infty$. Let $L = \sup\{\rho(2x_n); n \in \mathbb{N}\}$. Since $\rho \in \Delta_2^s$, there exists $K \ge 2$ such that

$$\rho(2u) \le K\rho(u) + 1 \tag{3}$$

for every $u \in X_\rho$ with $\rho(u) < 1$.

By using (3), we have $\rho(2x_n) \le K\rho(x_n) + 1 < K + 1$ for all $n \in \mathbb{N}$. Hence, $0 < L < \infty$. By Lemma 1, we have

$$\begin{aligned}1 = \rho(\frac{x_n}{\|x_n\|}) &= \rho\left(2a_nx_n + (1 - a_n)x_n\right)\\ &\le a_n\rho(2x_n) + (1 - a_n)\rho(x_n)\\ &\le a_nL + (1 - \varepsilon) \to 1 - \varepsilon,\end{aligned}$$

which is a contradiction. The proof is complete.

Before proving the main result of this section, firstly we need the following lemma proved by Cui and Hudzik in [4].

Lemma 4. *If $\rho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\rho(x + y) - \rho(x)| < \varepsilon$$

whenever $x, y \in V_\rho(\lambda; p)$ with $\rho(x) \le L$ and $\rho(y) \le \delta$.

Theorem 1. *The Banach space $V_\rho(\lambda; p)$ is $k - NUC$ for any integer $k \ge 2$.*

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Proof. Let $\varepsilon > 0$ and $(x_n) \subset B(V_\rho(\lambda;p))$ with $sep(\{x_n\}) > \varepsilon$. Let $x_n^m = (0,0,...,x_n(m), x_n(m+1),...)$ for each $m \in \mathbb{N}$. Since for each $i \in \mathbb{N}$, $\{x_{n_k}(i)\}_{i=1}^\infty$ is bounded, therefore using the diagonal method one can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence $\{x_{n_k}(i)\}$ converges for each $i \in \mathbb{N}$. Therefore, there exists an increasing sequence of positive integer (k_m) such that $sep(\{x_{n_k}^m\}_{k>k_m}) \geq \varepsilon$. Hence there is a sequence of positive integers $(n_m)_{m=1}^\infty$ with $n_1 < n_2 < n_3 < ...$ such that

$$\|x_{n_m}^m\| \geq \frac{\varepsilon}{2} \tag{4}$$

for all $m \in \mathbb{N}$. Then by Lemma 2, we may assume that there exists $\mu > 0$ such that

$$\rho(x_{n_m}^m) \geq \mu$$

for all $m \in \mathbb{N}$.

Let $\beta > 0$ be such that $1 < \beta < \liminf_{n\to\infty} p_n$. For fixed integer $k \geq 2$, let $\varepsilon_1 = \frac{k^{\beta-1}-1}{2k^\beta(k-1)}\left(\frac{\mu}{2}\right)$.

Write $\rho(x) = \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x(i)|\right)^{p_n}$. By using the Lemma 4, then there exists $\delta > 0$ such that

$$|\rho(x+y) - \rho(x)| < \varepsilon_1 \tag{5}$$

whenever $\rho(x) \leq 1$ and $\rho(y) \leq \delta$.

There exists $m_1 \in \mathbb{N}$ such that $\rho(x_1^{m_1}) \leq \delta$. Next there exists $m_2 > m_1$ such that $\rho(x_2^{m_2}) \leq \delta$. In such a way, there exists $m_2 < m_3 < ... < m_{k-1}$ such that $\rho(x_j^{m_j}) \leq \delta$ and $\beta \leq p_j$ for all $j \geq m_{k-1}$. Define $m_k = m_{k-1}+1$. By condition (4), there exists $n_k \in \mathbb{N}$ such that $\rho(x_{n_k}^{m_k}) \geq \mu$. Put $n_i = i$ for $1 \leq i \leq k-1$. Then in virtue of (4), (5) and convexity of the functions $f_i(u) = |u|^{p_i}$, we have

$$\rho\left(\frac{x_{n_1}+x_{n_2}+...+x_{n_k}}{k}\right) = \sum_{n=1}^{m_1}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}\left|\frac{x_{n_1}(i)+x_{n_2}(i)+...+x_{n_k}(i)}{k}\right|\right)^{p_n}$$
$$+\sum_{n=m_1+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}\left|\frac{x_{n_1}(i)+...+x_{n_k}(i)}{k}\right|\right)^{p_n}$$
$$\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_1+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}\left|\frac{x_{n_2}(i)+...+x_{n_k}(i)}{k}\right|\right)^{p_n} + \varepsilon_1$$
$$\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_2+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}\left|\frac{x_{n_3}(i)+...+x_{n_k}(i)}{k}\right|\right)^{p_n} + 2\varepsilon_1$$
$$\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_2+1}^{m_3}\frac{1}{k}\sum_{j=3}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+...+\sum_{n=m_{k-1}}^{m_k}\frac{1}{k}\sum_{j=k-1}^{k}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n}$$
$$+\sum_{n=m_k+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_j}(i)|\right)^{p_n} + (k-1)\varepsilon_1$$
$$\leq \frac{\rho(x_{n_1})+...+\rho(x_{n_{k-1}})}{k}$$
$$+\frac{1}{k}\sum_{n=1}^{m_{k-1}}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_k}(i)|\right)^{p_k}$$
$$+\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}\left|\frac{x_{n_k}(i)}{k}\right|\right)^{p_k} + (k-1)\varepsilon_1$$
$$\leq \frac{k-1}{k} + \frac{1}{k}\sum_{n=1}^{m_{k-1}}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_k}(i)|\right)^{p_k}$$
$$+\frac{1}{k^\beta}\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_k}(i)|\right)^{p_k} + (k-1)\varepsilon_1$$

$$\leq 1 + (k-1)\varepsilon_1$$
$$-\left(\frac{k^{\beta-1}-1}{k^\beta}\right)\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i\in I_n}|x_{n_k}(i)|\right)^{p_n}$$
$$\leq 1 + (k-1)\frac{k^{\beta-1}-1}{2k^\beta(k-1)}\mu. - \left(\frac{k^{\beta-1}-1}{k^\beta}\right)\mu$$
$$\leq 1 - \left(\frac{k^{\beta-1}-1}{k^\beta}\right)\left(\frac{\mu}{2}\right).$$

By Lemma 3, there exist $\eta > 0$ such that

$\left\|\frac{x_{n_1}+x_{n_2}+...+x_{n_k}}{k}\right\| < 1-\eta$. Therefore $V_\rho(\lambda;p)$ is $(k-NUC)$ for any integer $k \geq 2$.

From Theorem 1, we get that $V_\rho(\lambda;p)$ is $(k-NUC)$. Clearly $(k-NUC)$ Banach spaces are (NUC), and (NUC) implies property (H) and reflexivity, (see, Huff [11]). Also, in [11], Huff proved that X is (NUC) if and only if X is reflexive and (UKK). On the other hand, it is well known that

$$(UC) \Rightarrow (kR) \Rightarrow (k+1)R$$

and (kR) spaces are reflexive and rotund, and it is easy to see that

$$(k-NUC) \Rightarrow (kR).$$

By the facts presented in the introduction and the results just proved above, we get the following corollaries:

Corollary 1. *Let* $\liminf_{k\to\infty} p_k > 1$. *The space* $V_\rho(\lambda;p)$ *is* (NUC) *and then it is reflexive.*

Corollary 2. *The space* $V_\rho(\lambda;p)$ *is* (UKK).

Corollary 3. *The space* $V_\rho(\lambda;p)$ *is* (kR).

Corollary 4. *The space* $V_\rho(\lambda;p)$ *is rotund.*

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4. Modular Space $V_\rho(\lambda; p)$ with uniform Opial Property

In this section, we will show that the modular space $V_\rho(\lambda; p)$ has the uniform Opial property. The Opial property is important because Banach spaces with this property have the weak fixed point property (see [21]).

Theorem 2. *Assume that $\rho \in \Delta_2^s$, ρ is countably orthogonally additive and the modular sequence space $V_\rho(\lambda; p)$ is a Banach space. Then $V_\rho(\lambda; p)$ has the uniform Opial property.*

Proof. Let $\varepsilon > 0$ be given. There is $\varepsilon > 0$ such that (see 2) $\rho(x) \geq \varepsilon$ whenever $\|x\| \geq \varepsilon$. Since $\rho \in \Delta_2^s$, by Lemma 4, there is $\delta \in (0, \frac{\varepsilon}{4})$ such that

$$|\rho(x+y) - \rho(x)| < \frac{\varepsilon}{8}$$

whenever $\rho(x) \leq 1$ and $\rho(y) \leq \delta$. We can write

$$\rho_{i_0}(x) = \sum_{n=1}^{i_0} \left(\frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^{p_n}$$

and

$$\rho_{N-i_0}(x) = \sum_{n=i_0+1}^{\infty} \left(\frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^{p_n}.$$

By the countable orthogonal additivity of ρ, there is $i_0 \in \mathbb{N}$ such that

$$\rho_{N-i_0}\left(\sum_{i=i_0+1}^{\infty} x(i) e_i \right) \leq \frac{\varepsilon}{8}.$$

Let (x_n) be a weakly null sequence in $S(X)$. It is obvious that $x_n \to 0$ coordinatewise. Hence, there is $n_0 \in \mathbb{N}$ such that

$$\rho_{i_0}\left(\sum_{i=1}^{i_0} x_n(i) e_i \right) \leq \frac{\varepsilon}{8} \qquad (\forall n \geq n_0).$$

Therefore

$$\begin{aligned}
\rho(x_n + x) &= \rho_{i_0}\left(\sum_{i=1}^{i_0} (x_n(i) + x(i))\, e_i \right) \\
&\quad + \rho_{N-i_0}\left(\sum_{i=i_0+1}^{\infty} (x_n(i) + x(i))\, e_i \right) \\
&\geq \rho_{i_0}\left(\sum_{i=1}^{i_0} x(i) e_i \right) - \frac{\varepsilon}{8} \\
&\quad + \rho_{N-i_0}\left(\sum_{i=i_0+1}^{\infty} x_n(i) e_i \right) - \frac{\varepsilon}{8} \\
&\geq \frac{3}{4}\varepsilon - \frac{\varepsilon}{8} + \rho(x_n) - \frac{\varepsilon}{8} \\
&\geq 1 - \frac{\varepsilon}{4} + \frac{2}{4}\varepsilon \\
&= 1 + \frac{\varepsilon}{4}
\end{aligned}$$

for $n \geq n_0$. By Lemma 3, there is $\varepsilon_1 > 0$ that depends only on ε and such that $\|x_n + x\| > 1 + \varepsilon_1$ whenever $n \geq n_0$. This means that $V_\rho(\lambda; p)$ has the uniform Opial property.

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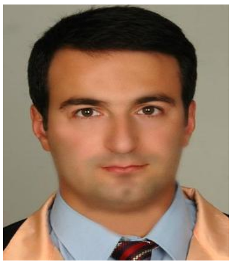

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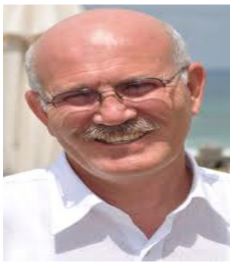

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